Talk given at KU

Nature of errors

Systematic codes.

single error correction

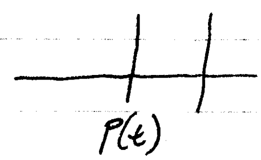
geometric model.

I. <u>Nature of errors.</u>

Transmission on noisy channel -

1. Intermittant — — – — (self curing)
2. permanent ——— —— burned out

Probability of ~~[illegible]~~ failures

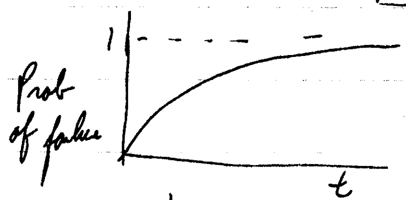


$P(\Delta t) = 1 - \frac{\Delta t}{t_0}$ prob. of surviving

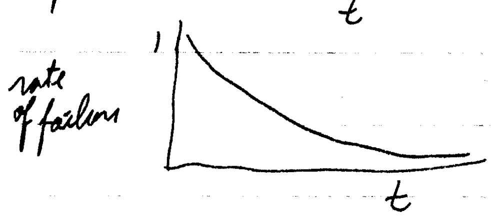
t_0 = average time between failures

$$P(t+\Delta t) = P(t)\left(1 - \frac{\Delta t}{t_0}\right)$$

$$\underline{P(t+\Delta t) - P(t)} = -\frac{\Delta t}{t_0} P(t)$$



also fixing time



Order of Magnitudes.   1000 hours = 3.6×10^6 sec.

$$P = \sum_i^N p = Np$$

(~~4~~2 days)

100 reg. position ~10 components each

$N = 10^3$ $p = 1$ per hour

R. W. Hamming "Error Detecting & Error Correcting Codes" Bell System Tech J. 29, no 2 147-160 (Apr '50)

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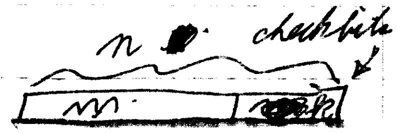
2 out of 5 codes - Relay computer
3 out of 7 codes - radio telegraphy
word count on telegrams.



Systematic codes.

n = no. of bits m = no. of information bits
$k = n - m$ = no. of check bits (error detection & correction)
$R = n/m$ "redundancy"

Typical situations:
(1) unattended operation
(2) large - interrelated systems where single failure incapacitates system.
(3) signalling in presence of noise.

Single error detecting codes:

parity check: $n-1$ info., 1 parity bit. (odd-even)

Redundancy $R = \frac{n}{n-1} = 1 + \frac{1}{n-1}$

Prob. of single error $= p << 1$, n cannot be as large as $\frac{1}{p}$
or double errors are too probable.

Single Correcting Code

checking number: k parity bits

write 0's when agree 1's when disagree

$m+k+1$ different things must be described
(error in each bit + 0 for no error)

→ $\underline{2^k \geq m+k+1}$ ← condition on k.

$\xleftrightarrow{n}$ $\boxed{m \mid k}$

since $n = m+k$ total no.

$$2^m \leq \frac{2^n}{n+1}$$

table of max m for given n

n	m	k
1	0	1
6	3	3
16	11	5

Example:

	1	2	3	4	5	6	7	sample:
	0	1	1	1	1	0	0	
1	—		—		—		—	0
2		—	—			—	—	0
4				—	—	—	—	1
error	0	1	1	1	0	0	0	
1	—		—		—		—	1
2		—	—			—	—	0
4				—	—	—	—	1

001
100
diff 101

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	—		—		—		—		—		—		—		—
2		—	—			—	—			—	—			—	—
4				—	—	—	—					—	—	—	—
8								—	—	—	—	—	—	—	—

1 3 5 7
0 0 0
0 1 0 1

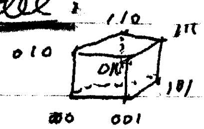
← no. itself is included if even parity.

Single Error Correction:

add one more bit — even parity check, on others.

Geometrical model: 2^n points = vertices of n-dim cube.

$n = 3$ 000



8 vertices

metric: $D(x, y)$ = no. of coords for which x & y are different. {least no. of edges.

single error changes one coord, two errors...2 coords, d more...d coords

$$D(x,y) = 0 \quad \text{iff } x = y$$

$$D(x,y) = D(y,x) > 0 \quad \text{if } x \neq y$$

$$D(x,y) + D(y,z) \geq D(x,z) \quad \text{triangle inequality.}$$

Two units apart: 001
010
100
111

"Sphere" of radius r all points ⊃ any of these can be center

for $r=2$, any $r=1$ is not a code point.

for $r=3$, any $r=1$ is closer to one or another & can be corrected.

Min distance	
1	unique
2	single det.
3	single correct. or double det
4	single ~~det~~ corr. + double det. or triple det
5	double corr. — or triple det. + single corr quadruple det.

— bits — data transmission "noisy channel"

— error what rate of failure would one expect?

1. — intermittent error - - - - -
2. permanent failure —— (blown-out tube)

me per component 1 per million operations

1 per 1000 hours

704 2×10^4 oper/sec

3.6×10^3 sec/hour $= 3.6 \times 10^6$ seconds

$3(36) = 108$ positions

however each position of 704 reg. $3(36) = 108$ pos.

Probability of failure

$\xrightarrow{(\Delta x)}$

$P(x+\Delta x)$

$P(x+\Delta x) = P(x)(1 - C\Delta x)$

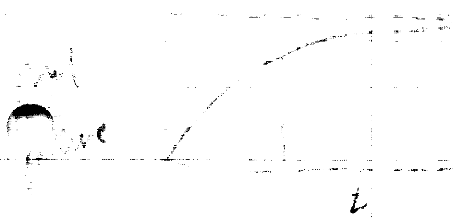
$\frac{P(x+\Delta x) - P(x)}{\Delta x} = P(x) C$

$\frac{dP}{P} = \ldots dx$

$\ln P = -cx \ldots$

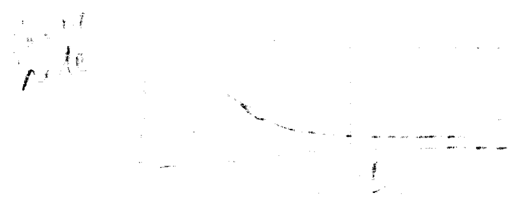
$P = P_0 e^{-cx}$

$P = P_0 e^{-t/t_0}$



prob. of "survival"

$1 - P$ = prob. of failure



Prob. of Failure of system $= \sum_i^N p_i = Np$ (if all same)

if $p = 10^{-6}$ per sec ~(300 hrs)

10^{-3} sec = 15 min.

108 pos. each with 10 components $Np = 10^{-3}$ per sec
only 15 min.

actual $P = 0.1$ %/1000 hours $= \frac{0.001}{3.6 \times 10^6 \text{ sec}} = 2.8 \times 10^{-10}$ sec^{-1}

equivalence: if by finite no of steps can be transformed from one to other.

1. Interchange of any 2 positions
2. complementing of the values in any position.

(equiv rotations & reflections.) get equivalence classes

Theorem

packing: To prove: max no. of points n-dim cube 2 units apart is 2^{n-1} and each is equivalent.

proof of max: A points in orig cube

vertices of n-dim. cube is composed of two (n-1)-dim. cubes.

one of (n-1) dim cubes has at least $A/2$

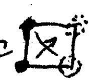
decompose this into (n-2) dim. , one at least has $A/2^2$

⋮

until 2 dim cube having $A/2^{n-2}$ points

a square can have at most 2 points sep. by 2 units

hence the orig. n-dim cube has at most 2^{n-1} not less than 2 units

or  only 2 max.)

proof of equivalence: each half has half of points

- also each have half 0's & half 1's p. 157 — Vahan

no distinction between data & check bits.

Systematic code: symbol lengths are equal

1. positions checked are indep. of info. in the symbol.
2. checks are indep of each other
3. we use parity checks.

Single Error Corr. Codes ...

at least 3 units from each other } each point can have sphere of radius 1 about it.

each sphere has center & n points around it. ($n+1$ points)

max is then $\frac{2^n}{n+1}$ points

Single Error Corr. plus double error det. codes. Theorem page 158

- add a single row. n-th coord fixed by even parity check of n ~~[illegible]~~ positions

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Misc:

can surround each code point by a sphere of radius 2

no. points on sphere $= 1 + C(n,1) + C(n,2)$

binomial coefs.