Numerical Analysis Seminar :

Continued Fraction : Sullivan Campbell Oct 24, 57

$$F = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_1}{b_2 + \cdots + \cfrac{a_n}{b_n + \cdots}}}$$

Truncated at n'th step,

If $\sum |b_i| \to$ converges

Then $\frac{1}{b_1 + \frac{1}{b_2 \cdots}}$ diverges

Rational Fn :

$$\frac{P^n}{P^{n-1}} = \frac{a_0 X^n + \cdots a_n}{b_0 X^{n-1} + \cdots b_n} \left(\begin{matrix}\text{always} \\ \text{can be} \\ \text{written}\end{matrix}\right) = \cfrac{1}{r_1 X + \delta_1 + \cfrac{1}{r_2 X + \delta_2 \cdots \cfrac{1}{r_n X + \delta_n}}}$$

(can get rid of Mpys)

very powerful in physical problems.

To evaluate however this is usually bad to compute.

→ (Divide Speed on machine is important)

Continued fractions converge often where power series do not.

→ also takes only n steps instead of 2n steps in power series (quite commonly so.)

power series	cont. fraction
2n-1 mpy	n divides
2n-1 adds.	2n adds.

if divide is no more than 2 times mpy. cont. fractions are better.

$$A_{-1} = 1, \quad A_0 = b_0, \quad \begin{cases} A_n = b_n A_{n-1} + a_n A_{n-2} \\ B_n = b_n B_{n-1} + a_n B_{n-2} \end{cases}$$
$$B_{-1} = 0, \quad B_0 = 1,$$

Recursion Relations.

$F_n = \dfrac{A_n}{B_n}$ convergence.

one can stop at convergence, but takes more op's,

but can transfer to case where all of a's are 1's or b's are 1's,
decreases amount of storage, but still costly in time. cuts out 1 mpy

- also need not evaluate F_n each time, (e.g. every 5 times)

→ Error is much greater in recursive than taking fixed no. of terms.

Householder, Makin theory of Errors.

$$\Delta = \begin{vmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{vmatrix} = \begin{vmatrix} b_n A_{n-1} + a_n A_{n-2} & A_{n-1} \\ b_n B_{n-1} + a_n B_{n-2} & B_{n-1} \end{vmatrix} = \begin{vmatrix} a_n A_{n-2} & A_{n-1} \\ a_n B_{n-2} & B_{n-1} \end{vmatrix}$$

$$= a_n \Delta_{n-1} = (-1)^{n-1} a_1 a_2 \cdots a_n$$

$$F_{n+1} - F_n = \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \frac{\Delta_n}{B_n B_{n-1}}$$ Small Correction term.

each calc must calc new B Mpy + Add + M + M (for A) + div

$$|F - F_n| \le |F_{n+1} - F_n| = \left|\frac{\prod a_i}{B_n B_{n+1}}\right|$$ Tells the no of terms needed to given convergence. for terms with same sign.

Error Types:

$$[f^*(x^*)]^* - f(x) = [f^*(x^*)]^* - f^*(x^*) + [f^*(x^*)] - f(x^*) + f(x^*) - f(x)$$

Truncated series, | E_r residual error. | E_g generated (depends on machine + algorithm used) | E_p propagated a property of f itself

$$G = b_0 + \cfrac{c_1 a_1}{c_1 b_1 + \cfrac{c_1 c_2 a_2}{c_2 b_2 + \cfrac{c_2 c_3 a_3}{\ddots}}}$$

can use any scale fraction in any way.

$$\frac{C_n}{D_n} = \frac{c_1 c_2 \cdots c_n A_n}{c_1 c_2 \cdots c_n B_n}$$

can use to get rid of xtra mpys. $(r=1)$, $r, x+s$

Let $F = b_0 + \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{\ddots}}}$ can always get in this form. by above

$$= b_0 + \cfrac{a_1}{1 + a_2 - \cfrac{a_2 a_3}{1 + a_3 + a_4 - \cfrac{a_4 a_5}{1 + a_5 + a_6 \cdots}}}$$

even terms. "contracted fractions" gives $2n$ convergence in n terms.

can be done in some special cases.

Ref Wall p. 348

e^x expansion $= 1 + \cfrac{x}{1 - \cfrac{x}{2 + \cfrac{x}{3 - \cfrac{x}{2 + \cfrac{x}{5 - \cfrac{x}{7 + \cdots}}}}}}$

$b_0 = 1$

$a_n = (-1)^{n-1} x \quad \geq 1$ $\qquad c_{2n} = \frac{1}{2} \quad n \geq 1$

$b_n \quad = 1$ $\qquad c_{2n+1} = \frac{1}{2n+1} \quad n \geq 0$

get $e^x = 1 + \cfrac{x}{1 - \cfrac{x/2}{1 + \cfrac{x/2n}{1 - \cdots}}}$

Now contract to half as many terms.

$$e^x = 1 + \cfrac{x}{1 - \frac{x}{2} + \cfrac{\frac{x^2}{4\cdot 3}}{1 + \frac{x^2}{4\cdot 5} \dots}} \quad ?$$

or $$e^x = 1 + \frac{x}{1 - \frac{x}{2} + F}$$

can get in form where one has div-add seqs

Note also:

$$\sinh x = \frac{x(1+F)}{(1+F)^2 - \frac{x^2}{4}} \qquad \tanh x = \cfrac{x}{1 + \cfrac{x^2/3}{1 + \frac{x^2}{15} \dots}}$$

$$\cosh x = \frac{x^2/2}{(\quad '' \quad)}$$

either can be done faster than sinh + cosh

much better than $e^x - e^{-x}$ form for significance.

$$F = \cfrac{x^2/4\cdot 4}{\frac{3}{4} + \cfrac{x^2/4\cdot 4\cdot 8}{5/8 + \cfrac{x^2/4\cdot 8\cdot 8}{7/8 + \cfrac{x^2/4\cdot 8\cdot 6}{9/16 + \cfrac{x^2/4\cdot 16\cdot 16}{11/16 + \dots}}}}}$$

get no error in F because it is shifted off on each stage.
get equiv. to double precision

---

another Form (Garwin?) of

$$f = b_0 + \cfrac{a_1/b_1}{1 + \cfrac{a_2/b_1 b_2}{1 + a_3/b_2 b_3 \dots}}$$

$\mu_0 = b_0$
$\mu_1 = a_1/b_1$
$\mu_n = a_n/b_{n-1} b_n$

$$\begin{cases} A_{m+1} = A_m + \mu_{m+1} A_{m-1} \\ B_{m+1} = B_m + \mu_{m+1} B_{m-1} \end{cases}$$

Recursion $$\begin{cases} \sigma_{m+1} = \dfrac{1}{1 + \mu_{m+1}\sigma_m} \\ \delta_{m+1} = (\sigma_{m+1} - 1)\delta_m \end{cases}$$

where $f_0 = b_0$
$\delta_1 = a_1/b_1, \quad \sigma_1 = 1$
$\frac{A_m}{B_m} = b_0 + \sum_{i=1}^{m} \delta_i$ ← test on size