MAINTENANCE DEVELOPMENT MEMO #10

September 23, 1957

SUBJECT: Measuring the Increase in Operation Time by use of Error Correction

INTRODUCTION:

To show how much the operation time is increased when single bit error correction is added to a system which has single bit error detection, general expressions for the probability of errors are used to derive the gain obtained. The effect of a 'solid' fault on a channel is discussed in Part VI, but since many of what are considered 'solid' faults are really intermittent errors with a high probability, first consider only intermittent errors arising from 1) components which have deteriorated to marginal operation, 2) noise, or 3) intermittent faults like cold solder joints. Each channel is considered to be independent and that all words occur with equal probability.

CONCLUSIONS:

With error correction applied to a system, the increase in operation time if the circuitry necessary to correct the error is neglected, is very great. These large gains are off-set by the errors in the error correction as shown by equation 52. Usually the error detection circuits have a higher degree of fail-safeness than does the error correction circuitry, which also must be considered when evaluating the use of error correction over error detection.

Part I

If the number of operations for each error for a unit is an average of 100, then it is said that the probability of error for the unit, p, is .01 or

$$P = \frac{1}{T} \tag{1}$$

where T is expressed in operation cycles. Also the probability of correct operation, q, is .99. It is assumed that these probabilities are constant, and that the operation is either correct or incorrect so that

$$p + q = 1 \tag{2}$$

Now let P (f) be the probability of unsuccessful operation of the combination of units and P (s) be the probability of successful operation. Again,

$$P(f) + P(s) = 1 \tag{3}$$

The system under study has N information channels to which there was added a parity channel for single bit error detection. Using 'D' as a subscript for this system, general expression for P_D (f), P_D (s), and T_D must be found in terms of p, q, and $N + 1$.

There is only one way in which can be no error in the system. The probability for this is the product of probabilities of no error in each channel or

$$F_D\,(O) = (q)^{N+1} \tag{4}$$

There are $N + 1$ ways in which one channel can be in error and the probability for an error in a selected channel is the product of the probability of error in the selected channel and the probabilities of no error in each of the other channels or

$$F_D\,(1) = (N + 1)\,(p)\,(q)^{N} \tag{5}$$

Equations 4 and 5, will be recognized as the first and second terms of the binomial expansion of $(q + p)^{N+1}$. Stated in another way, the probability of exactly k errors is given by the $k + 1$st term of the expansion of $(q + p)^{N+1}$ for a $N + 1$ channel system.

Part II

A system with single error detection operates successfully only if no errors have occurred; that is k = o, and thus

$$P_D\,(s) = (q)^{N+1} \tag{6}$$

$$P_D\,(f) = 1 - P_D\,(s) = 1 - (q)^{N+1} = 1 - (1-p)^{N+1} \tag{7}$$

$$T_D = \frac{1}{P_D(f)} \tag{8}$$

When p is very small, q is very, very close to 1. If the form of equation 7 that uses $(q)^{N+1}$ is calculated a great number of decimal places must be retained or the significant digits in P_D (f) will be lost. By use of the binomial expansion of (1-p), which is

$$(1-p)^{N+1} = 1 - (N+1)\,p + \frac{(N+1)\,(N)}{2}\,p^2 - \frac{(N+1)\,(N)\,(N-1)}{6}\,p^3 + \ldots \tag{9}$$

equation 7, can be approximately expressed by

$$P_D\,(f) = (N + 1)\,p \tag{10}$$

to three significant places when

$$p < L_N = \frac{1}{50N} \tag{11}$$

Values of L_N appear in Table I.

Part III

To achieve single error correction and double error detection, 'V' more channels must be added. Table I gives the number of extra channels that are required for various N. Using 'C' as a subscript for this system, general expressions for P_c (f), P_c (s), and T_c must be found. Now, the probability of k errors is given in terms of the expansion of $(q + p)^{N + V}$ or

$$F_c(O) = (q)^{N + V} \qquad (12)$$

and

$$F_c(1) = (N + V)(p)(q)^{N + V - 1} = (N + V)(p)(1-p)^{N + V - 1} \qquad (13)$$

The occurrence of single bit errors is expressed by equation 13. These are now correctable and the sum of 12 and 13 is therefore P_c (s). Then,

$$P_c(f) = 1 - (q)^{N + V} - (N + V)(p)(q)^{N + V - 1} \qquad (14)$$

and

$$T_c = \frac{1}{P_c(f)} \qquad (15)$$

The gain will be defined as the ratio of T_c to T_d or

$$G = \frac{T_c}{T_d} = \frac{P_d(f)}{P_c(f)} \qquad (16)$$

Just as equation 7 was approximated, equation 14, can be expressed by

$$P_c(f) = \frac{(N + V)(N + V - 1)}{2} p^2 \qquad (17)$$

when

$$P < \frac{3}{2(N + V - 2)} \qquad (18)$$

Since this is less stringent than equation 8, equation 8 will be used to define the maximum value of p for which these equations are valid.

Applying equation 10 and 17 to 16, the gain,

$$G_{(N, P)} = \frac{2(N+1)}{(N+V)(N+V-1)\,p} = \frac{K_n}{p} \qquad (19)$$

where values for K_N are found in Table I.

Part IV

Some examples are given below to illustrate the above equations. Example I. Take N = 1 and the largest possible p for our equations to hold.

(11) $$P < \frac{1}{(50)(1)} = .02$$

(8) $$T_d = \frac{1}{(N+1)\,p} = \frac{1}{2(.02)} = 25$$

(19) $G_{(1, .02)} = \frac{.3333}{.02} = 16.67$

(16) $T_c = GT_d = 25\,(16.67) = 416.67$

Thus for small N and large p, the gain can be very good.

Example II. Take $N = 1$ and $T_d = 50$.

(8) $p = \frac{1}{(N+1)\,(T_d)} = \frac{1}{2\,(50)} = .01$

(19) $G_{(1, .01)} = \frac{.3333}{.01} = 33.33$

(16) $T_c = (33.33)\,(50) = 1667$

If the channel is twice as good, the gain will be doubled, T_d is doubled, and T_c is quadrupled.

Example III. Take $N = 6$ and $p = 7.14 \times 10^{-11}$.

(8) $T_d = \frac{1}{7\,(7.14 \times 10^{-11})} = \frac{10^{11}}{49.98} = 2 \times 10^9$

Remembering that T_d is in operations, let us also assume that each operation takes 17 usec.

$$T_d = (2 \times 10^9)\,(17 \times 10^{-6}) = 34 \times 10^3 \text{ sec.} = 9.45 \text{ hr.}$$

This is approximately the error rate on the Type 702. Referring to Table I, $K_n = .1273$.

(19) $G_{(6,\ 7.14 \times 10^{-11})} = \frac{.1273}{(7.14 \times 10^{-11})} = 1.785 \times 10^9$

(16) $T_c = GT_d = (1.785 \times 10^9)\,(9.45 \text{ hr.}) = 1.92 \times 10^4$ centuries

This is a little misleading, for one of the original assumptions was that p and q do not change during the trials. For a week or a month this may be true, but for longer periods we must analyze the effect of channels degrading. This will be done in later parts. Now, note that for $N = 6$, that $V = 5$ which means that the system size increased by 57%. Another approach is to consider 5N channels at a time.

(8) $T_d = \frac{1}{31\,(7.14 \times 10^{-11})} = \frac{10^{11}}{221.34} = 4.52 \times 10^8$

(19) $G_{(30,\ 7.14 \times 10^{-11})} = \frac{.0465}{7.14 \times 10^{-11}} = 6.53 \times 10^8$

The gain is still "out-of-this world" and the system size has increased only 23%.

Example IV. If $N = 64$, equation 11 stated that $p < .0003125$ so let us continue with $p = 7.14 \times 10^{-11}$.

(8) $$T_d = \frac{1}{65(7.14 \times 10^{-11})} = \frac{10^{11}}{456} = 2.19 \times 10^8$$

Letting each operation be 2 usec.

$$T_d = (2.19 \times 10^8)(2 \times 10^{-6}) = 4.38 \times 10^2 \text{ sec.} = 7.3 \text{ min.}$$

This shows how if the <u>same</u> probability of error existed in a bigger and faster machine that it would not be tolerable without error correction. For,

(19) $$G(64,\ 7.14 \times 10^{-11}) = \frac{.0254}{7.14 \times 10^{-11}} = 3.56 \times 10^8$$

(16) $$T_c = GT_d = (3.56 \times 10^8)(7.3 \text{ min}) = 4.94 \times 10^4 \text{ years.}$$

Example V. Take $N = 1$ and $p = .01$ as in example II.

$$(q + p)^{N+1} = (.99+.01)^2 = (.99)^2 + 2(.99)(.01) + (.01)^2$$
$$= .9801 + .0198 + .0001$$

$P_d(s) = .9801$
$P_d(f) = .0199$
$T_d = 50.251256+$

$$(q+p)^{N+V} = (.99+.01)^4 = (.99)^4 + 4(.01)(.99)^3 + \ldots.$$
$$= .960,596,01 + .038,811,96 + \ldots.$$

$P_c(s) = .999,407,97$
$P_c(f) = .000,592,03$
$T_c = 1689.1036-$

$$G(1,.01) = \frac{1689.1036}{50.251256} = 33.613162-$$

These values can be compared to those in example II. Note how the terms in $P_c(s)$ add and thus must for smaller values of p be carefully calculated if this method is used.

Part V

The difficulty noted in example III can be solved in the following manner. Consider a system made up of R+1 channels, except that there will be one channel which has a probability of error, p_B, which is greater than the p of the others. Now,

$$f(0) = (q)^R (q_B) \tag{20}$$

and $$f(1) = R(p)(q)^{R-1}(q_B) + 1(p_B)(q)^R \tag{21}$$

Note, that as $p_B \rightarrow p$ then

$$(q)^R (q_B) \rightarrow (q)^{R+1} \tag{22}$$

and $$R(P)(q)^{R-1}(q_B) + (pB)(q)^R \rightarrow (R+1)(p)(q)^R \tag{23}$$

which are the relations which were stated before.

Part VI

If the system just has a satisfactory failure rate without error correction and error correction is added to allow for channel degradation, the question can be asked how much can one channel be degraded before the system reaches the unsatisfactory failure rate. This implies that p is small enough to give a gain when error correction is added. This condition is satisfied by $p < L_n$. Thus usually,

$$T_d < T_c \tag{24}$$

Now, to relate p_B to p, define

$$p_B = mp \text{ for} \tag{25}$$
$$1 < m \leq \frac{1}{p} \tag{26}$$

Note, that for m = 1 the channel is not degraded and when m = 1/p, $p_B = 1$ and error correction is applied on every cycle of operation. This is then a "solid" failure. If $p_B = 1$, then $q_B = 0$ and equation 20 is zero. Equation 21, then reduces to

$$f(1) = (q)^R \tag{27}$$

Adding a "B" to the former notion to indicate this system, we have when $p_B = 1$,

$$P_{cb}(s) = (q)^R \tag{28}$$
$$P_{cb}(f) = 1-(q)^R \tag{29}$$

or approximately, $$p_{cb}(f) = (N+V-1)p \tag{30}$$

and $$T_{cb} = \frac{1}{(N+V-1)p} \tag{31}$$

From before

$$T_d = \frac{1}{(N + \;\;)p} \tag{10}$$

Therefore, when $p_b = 1$

$$T_d > T_{cb} \tag{32}$$

Comparing this with equation 24, implies that there is some value for m such that the following equation will hold.

$$T_d = T_{cb} \tag{33}$$

and

$$P_d\,(f) = P_{cb}\,(f) \tag{34}$$

Expanding each side of equation 34, gives

$$1-(q)^{n+1} = 1-(q)^{N+V-1}(q_B) - (N+V-1)(p)(q)^{N+V-2}(q_B) - (p_B)(q)^{N+V-1} \tag{35}$$

Applying equations 9 and 25 and simplifying.

$$(N+1)\,p = \frac{(N+V-1)\,(N+V-2)}{2}\,p^2 + (N+V-1)\,m\,p^2 \tag{36}$$

Solving for m

$$m = \frac{N+1}{(N+V-1)p} - \frac{(N+V-2)}{2} \tag{37}$$

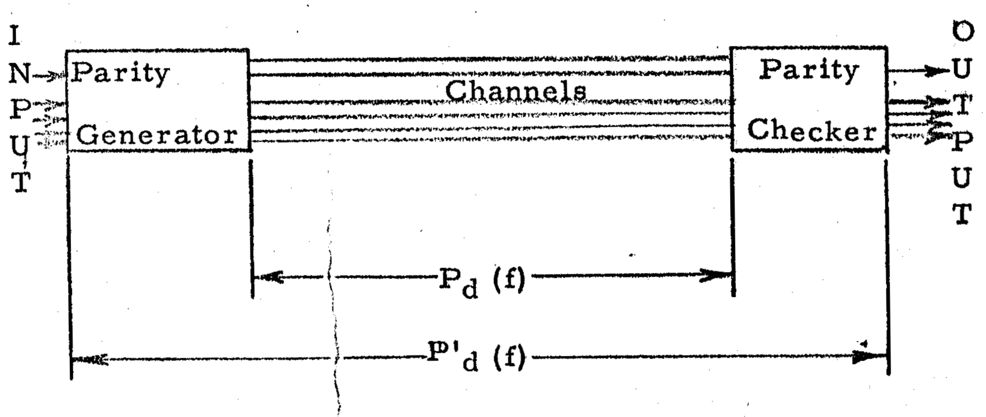
Substituting in equation 25,

$$P_b = mp = \frac{(N+1)}{(N+V-1)} - \frac{(N+V-2)}{2}\,p \tag{38}$$

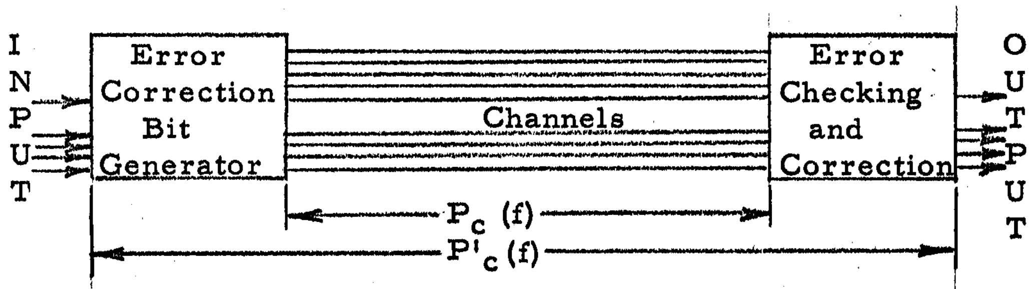
This shows that the tolerated degradation of a channel can be large and the limit of p_b for this degradation is independent of p.

Part VII

In the above discussion the fact that additional circuitry is necessary for either single bit error detection or error correction was not taken into account. The systems for which the operation time is desired is shown in the following figure:



System With Error Detection



System With Error Correction

The probability of error in the channels has been derived. Now it is desired to relate these to the probability of error in the full system. The probability of an error in the system other than in the channels themselves will be denoted by P_{da} (f). An error in the system outside the channels is not mutually exclusive with an error within the channels, that is, both can occur at the same time. Thus, the probability that an error has occured in the system is

$$P'_d\,(f) = P_d(f) + P_{da}\,(f) - (P_d\,(f))\,(P_{da}\,(f)) \qquad (39)$$

Because these probabilities are very small, the third term can be dropped. Equation 39 says that as additions are made to a system with similar components

$$P_t\,(f) = P_o\,(f) + P_a\,(f) \doteq (1 + \alpha)\,(P_o\,(f)) \qquad (40)$$

where α is the added amount of circuitry expressed as a fraction of the original amount. If the added circuitry were made out of a different type of circuitry, then by letting

$$P_a\,(f) = (\beta)\,(P_o\,(f)) \qquad (41)$$

hold, β could be said to be an equivalent amount of added circuitry. Then equation 40 would be

$$P_t\,(f) \doteq (1 + \beta)\,(P_o\,(f)) \qquad (42)$$

Letting δ be the equivalent amount of additional circuitry in the system with error detection, equation 39 can now be written as

$$P'_d\,(f) = (1 + \delta)\,(P_d\,(f)) \qquad (43)$$

Similar to equation 1, we now have

$$T'_d = \frac{1}{P'_d(f)} = \frac{1}{(1+\delta)(P_d(f))} = \frac{T_d}{1+\delta} \tag{44}$$

The above steps for the system with error correction are

$$P'_c(f) = P_c(f) + P_{ca}(f) \tag{45}$$

Letting $P_{ca}(f) = (\gamma)(P_d(f))$ (46)

$$P'_c(f) = P_c(f) + (\gamma)(P_d(f)) \tag{47}$$

Applying equation 16,

$$P'_c(f) = \frac{P_d(f)}{G} + \gamma P_d(f) = \frac{1+\gamma G}{G}(P_d(f)) \tag{48}$$

$$T'_c = \frac{1}{P'_c(f)} = \frac{G}{(1+\gamma G)(P_d(f))} = \frac{GT_d}{(1+\gamma G)} \tag{49}$$

Defining

$$G' = \frac{T'_c}{T'_d} \tag{50}$$

Applying equation 42 and 47

$$G' = \frac{\dfrac{GT_d}{1+\gamma G}}{\dfrac{T_d}{1+\gamma}} = \frac{G(1+\gamma)}{1+\gamma G} = \frac{1+\delta}{\gamma + 1/G} \tag{51}$$

Remembering the large values of G calculated in the previous section equation 49 is reduced to

$$G' = \frac{1+\delta}{\gamma} \tag{52}$$

Thus the equivalent added circuitry outside the channel is the factor that determines the gain if the p of the channel is small.

Part VIII

The results of Part VI have been modified to include the analysis of the preceding section and are as follows:

$$m = \frac{(1 + \delta)(N + 1) - \gamma(N + 1)}{(N + V - 1)\,p} - \frac{(N + V - 2)}{2} \qquad (53)$$

$$P_b = \frac{(1 + \delta - \gamma)(N + 1)}{(N + V - 1)} - \frac{(N + V - 2)p}{2} \qquad (54)$$

Since $\gamma > \delta$ therefore m is less than before as would be expected.

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TABLE I

N	V	N + V	K_n	L_n
1	3	4	.3333	.020,000
2	4	6	.2000	.010,000
3	4	7	.1905	.006,667
4	4	8	.1785	.005,000
5	5	10	.1333	.004,000
6	5	11	.1273	.003,000
7--10	5	12--15	-----	--------
11	5	16	.1000	.001,820
12	6	18	.0850	.001,667
13--25	6	19--31	-----	--------
26	6	32	.0524	.000,769
27	7	34	.0482	.000,741
28--29	7	35--36	-----	--------
30	7	37	.0465	.000,645
31--35	7	38--42	-----	--------
36	7	43	.0410	.000,555
37-56	7	44--63	-----	--------
57	7	6	.0288	.000,351
58	8	66	.0283	.000,345
59--63	8	67--71	-----	--------
64	8	72	.0254	.000,312
65--119	8	73--127	-----	--------
120	8	128	.0149	.000,167

N = Number of Information Channels

V = Number of Additional Channels Necessary to Provide for Error Correction

K_n = Constant for Equation 19

L_n = Maximum Value for p that allow equations to hold.