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PROJECT STRETCH

File

FILE MEMO #36
SUBJECT: Multiple Precision Division
BY: W. Wolensky
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The problem of multiple precision division resolves itself into the problem of finding a way to perform division using simple division, multiplication, addition, or combinations thereof. The only basic machine requirement, which applies only when a simple division process is involved, is that the execution of the divide instruction permit the programmer to prevent a rounding of the quotient and to preserve the remainder for further possible use.

The main consideration in attacking multiple precision problems is that of maintaining the accuracy, precision or number of significant digits. The methods of attack discussed will provide accuracy at least to the same precision as the size of the longest word involved. It is left to the individual programmer to determine the degree of precision required and to provide in his program the facilities for maintaining the same. The methods discussed are nothing more than descriptions of two ways in which multiple precision division can be done accurately.

By definition, a simple division is $\frac{A}{B}$ where both A and B are a maximum of one word length in significance. This type of division is the one built into computers via hardware, and requires that the words have some specific relationship to each other. Word organization, modification or manipulation may be necessary to conform to the rules of a machine divide instruction. Utilizing this type of instruction for multiple precision applications precludes the facility of retaining the remainder for possible further use. (Since the remainder is to be made available for further division, the quotient resulting from the prime division should not be rounded.)

Idiot division is defined as $\frac{A}{B}$ or $\frac{A_1 + A_2 \text{---}}{B}$ where B is a maximum of one word length and A is greater than one word length and may extend to two, three or more word lengths. This is the simplest form of multiple precision division as implied, and is readily solved by dividing B into A_1, adding the remainder to A_2 then dividing the sum of A_2 and the remainder by B, etc. No particular data handling problems are involved other than the normal word adjustment required by the characteristics of the computer involved. (Any adjustment to A_1

must be propagated to A_2, A_3 so as not to change the characteristics of the word A of which A_1, A_2 --- are all parts.)

Double precision division can be illustrated by the formula $\frac{A_1 + A_2}{B_1 + B_2}$. Double precision divisions cannot be solved in a normal division procedure in a fixed word length machine. One approach to the solution is to convert the relationship to $(A_1 + A_2)\left(\frac{1}{B_1 + B_2}\right)$. Then, using the Binomial Expansion theorem on $\frac{1}{B_1 + B_2}$, it is possible to find a solution which involves multiplication, simple division, addition and subtraction. Expansion of $\frac{1}{B_1 + B_2} = \frac{1}{B_1}\left(1 - \frac{B_2}{B_1} + \left(\frac{B_2}{B_1}\right)^2 - \left(\frac{B_2}{B_1}\right)^3 + \left(\frac{B_2}{B_1}\right)^4 ---\right)$ and the complete problem solution is realized by multiplying the calculated expansion of $\frac{1}{B_1 + B_2}$ by the quantity $(A_1 + A_2)$. To guarantee the accuracy of the quotient to two-word lengths (double precision), it is necessary to program all terms of the expansion to be carried out to three-word precision. The expansion term need only be carried to the squared term, in the equation of the form shown. The final two precision word quotient is obtained by rounding the final total accumulation after multiple precision multiplication of the expansion term and the original numerator $A_1 + A_2$. The amount of error incurred in this type of procedure is lesser in value than the term following the least significant term used in the calculating.

Triple precision division according to the definition $\frac{A_1 + A_2}{B_1 + B_2 + B_3}$, can be performed by programming the solution as an extention of the double precision method involving the Binomial Expansion theorem. The derived equation is: $(A_1 + A_2 ---)(Z - B_3 Z^2 + B_3^2 Z^3 - B_3^3 Z^4 ----)$ where $Z =$

$$\frac{1}{B_1}\left(1 - \frac{B_2}{B_1} + \left(\frac{B_2}{B_1}\right)^2 - \left(\frac{B_2}{B_1}\right)^3 ----\right)$$

Obviously the procedure to solve triple precision division has greatly increased in complexity compared to the double precision solution. The accuracy of the triple precision problem necessitates working with words to the fourth precision and carrying the expansions to the cubed term.

Multiple precision devision to any degree may be realized through denominator conversions by the expansion theorem. Beyond triple precision the programming complexity approaches a point of diminishing return, and other methods are probably more advantageous.

The complexity of the solution routine can be reduced slightly, during the multiple precision multiplication process by eliminating the portions of the expanded products whose significance is beyond the point where it can effect the desired accuracy of the quotient being obtained. (Refer to Stretch Memo #34 - Multiple Precision Multiplication)

The solution to multiple precision division problems by use of Newton's formula is similar to the Binomial Expansion theorem method in that the prime objective is to reduce the divisor to a multiple precision word, which can be multiplied with the dividend to produce the quotient desired.

Newton's formula $Xi = \frac{1}{N} \cong Xn\,(2 - NXn)$

where: Xi is the converted multiple precision divisor
Xn is the latest approximation
N is the divisor

The mechanics of solving for $\frac{1}{N}$ are to make an initial approximation Xn, multiple precision multiply the approximation by N and subtract the result from the quantity 2. The result of this process is multiplied by the approximation Xn to obtain the subsequent approximation. It is apparent that the criteria of operation is that of reducing NXn to the quantity 1, in which case $NXn = \frac{1}{N}$. The fact that if NXn is not 1, its value is multiplied by Xn to produce a new Xn which is closer to equalling $\frac{1}{N}$. The process therefore is an iterative one, and the degree of precision for $\frac{1}{N}$ can be determined to any length desired.

Care must be exercised in choosing the initial approximation value of Xn, to ensure convergence of the formula. Mr. D. A. Quarles, Jr. lists these three as possible methods for obtaining a suitable choice for the first guess Xn.

1) Obtain Xo from a single precision reciprocation by division.

2) Decide Xo by a test of the size of the high-order word of the divisor N.

3) Let Xo be a fixed number if N is scaled to a sufficiently restricted range. (e.g. with a significant digit in the left-most position of the high-order word).

Regardless of the method used to determine the first approximation, the only governing factor is that the value of the first approximation Xo should be such that $XoN < 2$.

The number of iterations required to yield the desired accuracy in determining might be predetermined, or a test for convergence could be made. The convergence test could be the comparison of the value of NXn with 1, to any degree of precision required.

The solution is completed when the final value of Xn is multiple precision multiplied with the dividend of the original multiple precision division problem.

Both of the basic methods of multiple precision division described have advantages, and both will provide the degree of accuracy required. It is left to the programmer to choose the technique that best suits his needs. The study of this problem has shown that multiple precision division can be performed on fixed word length machines by employing programming techniques, and by having the computer provide the facility of single word division with retention of the remainder.

Bibliography:

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References:

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