(6)

COMPANY CONDIFENTIAL

PROJECT BETA
FILE MEMO #15
SUBJECT: Complementary Multiplication
BY: W. A. Hunt

January 18, 1956

In the Stretch File Memo #2 by Mr. S. W. Dunwell, the subject of complementary multiply is discussed. This is an attempt to obtain some mathematical results for two different ways of treating the problem. The results obtained here for the "first case" (see the third paragraph of this page) would be identical with those obtained by Mr. S. W. Dunwell in Project Stretch File Memo #2, page 6, if it is noted that he uses a 48 bit word and that in his table, page 6, that the Cycles for the "Multiple 1's Followed by Single Zeros" should be 4 instead of 3. The average number of ones are now 16 which is 1/3 of 48. This agrees with the 2/3 zeros or 1/3 ones obtained here.

The following is an example of complementary multiply. Suppose that the machine encounters the sequence X01110X in the multiplier. It substitutes X100(-1)0X for the previous sequence and continues. The saving of time which is obtained by skipping zeros (see Stretch File Memo #2 and a memorandum to B. Housman, subject: A Method of Decreasing the Multiply Time in the AN/FSQ-7, by W. A. Hunt, 12/9/55), can be further increased by this manipulation.

The first case to be examined is where all groups of ones equal to or greater than two, are changed in the manner described in the previous paragraph. The second case will show the increase in zeros due to changing only groups of ones equal to or greater than three. In these cases, the end result will be indicated as an increase in the percentage of zeros for an infinitely long multiplier.

Table 1 lists the probability of certain types of groups of ones or zeros occurring in the infinitely long binary word where ones and zeros have equal probability of occurring in any bit position.

Group			Probability of Occurrence
X101X	or	X010X	1/8
X1001X		X0110X	1/16
X10001X		X01110X	1/32

If the complementary multiplication is not done the maximum saving in time due to skipping zeros may be calculated by summing the series obtained by multiplying the probability of occurrence by the number of zeros per group. The sum of this series: (See Appendix I.)

$$(1/8 + 2/16 + 3/32 + \ldots) = 1/2$$

This verifies the original supposition of half zeros and half ones. When complementary multiplication is done, there are two effects which enter into the calculations. These are:

1. The increase in the size of the one groups due to a one being

added from the right because of a previous complementary multiplication, and,

2. The decrease in the size of the zero groups due to shifting a one into them because of a previous complementary multiplication.

Number one is a special case of number two where the zero group is the single zero case. Therefore, the series sum of one should be equal to the probability of decrease in the single zero group.

The increase in the size of the one group, i.e., the fraction of the times that a one will be shifted in from the right due to a previous complementary multiply is equal to the probability that a one occurrs following the right zero, followed by a one, or a zero which in turn is followed by a one or a zero followed by a one ... This is indicated by the following:

	(X10)	11X	
or	(X10)	1011X	The parenthesis indicates the original one
or	(X10)	101011X	group.
or	(X10)	10101011X	

etc.

This can be written in series form as follows: (See Appendix II)

$$1/4 + 1/16 + 1/64 + \ldots = 1/3$$

This indicates that the probability of increase of any one group is 1/3 times the probability of that group occurring.

The probability that a one will be shifted into the zero group from the right by the complementary multiply is two times that obtained in the previous calculation. This is true because the right most bit in the definition of the zero group is always a one by definition. This is indicated by the following:

Any zero group

(X01) 1X...
(X01) 1011X...
(X01) 101011X...
etc.

This can be written in series form as: $(1/2 + 1/8 + 1/32 + \ldots) = 2/3$

If the probabilities of increase in size of each of the one groups times the probability of occurrence of each of the one groups are summed, the following series and results are obtained.

$$1/3(1/8 + 1/16 + 1/32 + \ldots) = 1/3 \times 1/4 = 1/12$$

This is equal to the probability of decrease in the single zero group (X101X) times the probability of occurrence or $2/3 \times 1/8 = 1/12$

The statements of the last few paragraphs are given in the following tables.

Group	Probability of Occurrence	Probable number of zeros occupying the former one-positions during complementary multiply
X010X	1/8	(0 + 1/3)
X0110X	1/16	(1 + 1/3)
X01110X	1/32	(2 + 1/3)

Group	Probability of Occurence	Probable number of zeros occupying the zero positions when complementary multiply is used.
X101X	1/8	(1 - 2/3) = 1/3
X1001X	1/16	(2 - 2/3) = 1 + 1/3
X10001X	1/32	(3 - 2/3) = 2 + 1/3

Summing all of the probable number of zeros per group times the probability of occurrence of the group, the probable number of zeros, when complementary multiply is used in this fashion, is 2/3 or $66\frac{2}{3}$ %.

$$S = 2\left[1/3 \times 1/8 + (1 + 1/3)\, 1/16 + (2 + 1/3)\, 1/32 + \ldots\right]$$

$$S = 2(1/3 \times 1/4 + 1/4)$$

$$S = 2/3$$

If the calculations are carried through for the second case, i.e., complementary multiply is used for groups of three ones or more, the probable number of zeros obtained is 64.3%.

These results indicate that the use of complementary multiply for two or more ones, and skipping all zero groups may decrease the multiply time by a maximum of $66\frac{2}{3}$ %. This is a $16\frac{2}{3}$ % increase over that obtained by skipping zeros and not using complementary multiply.

W A Hunt

W. A. Hunt

WAH:SG

APPENDIX I

$S_1 = (1/8 + 1/16 + 1/32 + \ldots)$

$S_1 - 1/8 = (1/16 + 1/32 + 1/64 + \ldots)$

$S_1 - 1/8 = 1/2\,(1/8 + 1/16 + 1/32 \ldots)$

$S_1 - 1/8 = 1/2\, S_1$

$S_1 = 1/4$

$S_2 = (1/8 + 2/16 + 3/32 + 4/64 + \ldots)$

$S_2 - 1/8 = (1/16 + 1/32 + 1/64 + \ldots) + (1/16 + 2/32 + 3/64 + \ldots)$

$S_2 - 1/8 = 1/2\, S_1 + 1/2\, S_2$

$S_2 = 1/2$

APPENDIX II

$S_3 = (1/4 + 1/16 + 1/64 + \ldots)$

$S_3 - 1/4 = 1/4\, S_3$

$S_3 = 1/3$

$S_4 = (1/2 + 1/8 + 1/32 + \ldots)$

$S_4 = 2S_3 = 2/3$

/SG