CHAPTER I

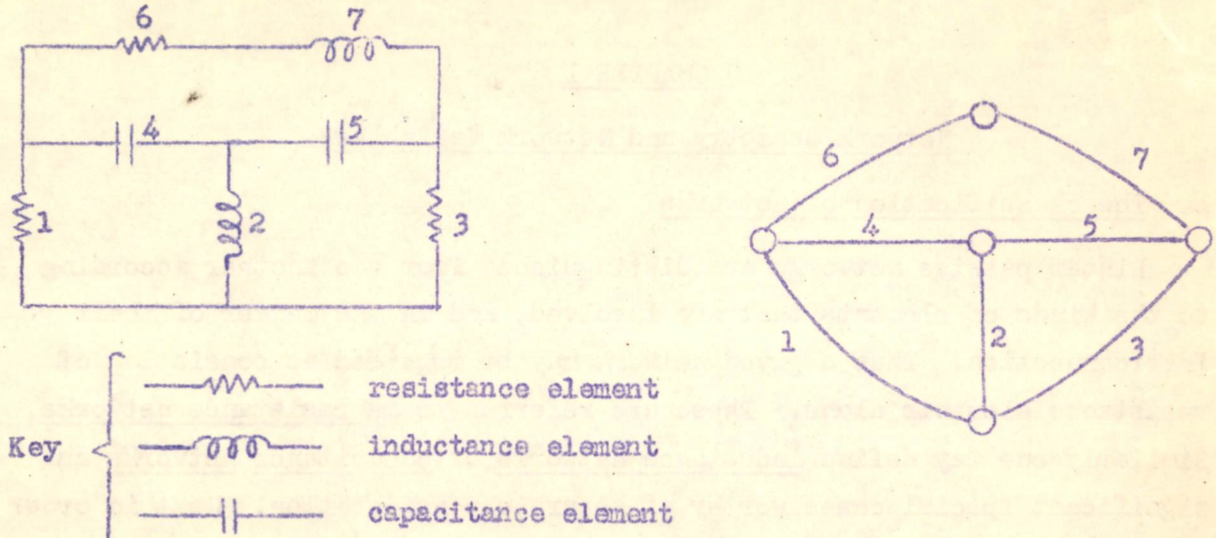
Network Geometry and Network Variables

1. The classification of networks.

Linear passive networks are distinguished from one another according to the kinds of elements that are involved, and in the manner of their interconnection. Thus a given network may be regarded as consisting of resistance elements alone. These are referred to as resistance networks. Similarly one may define inductance networks or capacitance networks as significant special cases worthy of separate consideration. Next in order of complexity come the so-called two-element types, more precisely the LC-networks (those containing inductance and capacitance elements but, by assumption, no resistances), the RC-networks in which inductive effects are absent, and RL-networks in which capacitive effects are absent. The RLC-network then represents the general case in the category of linear passive networks.

2. The graph of a network.

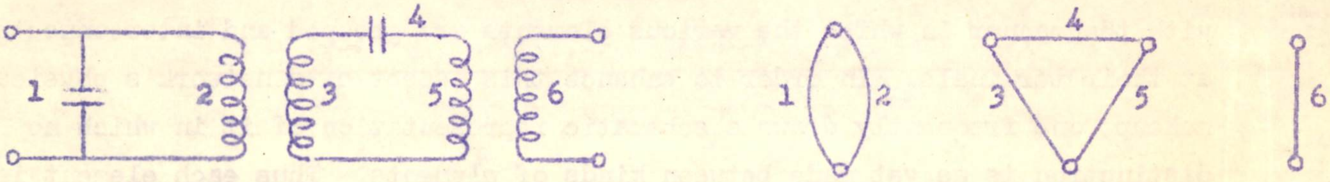
Quite apart from the kinds of elements involved in a given network is the all-important question of network geometry that concerns itself solely with the manner in which the various elements are grouped and interconnected at their terminals. In order to enhance this aspect of a network's physical makeup, one frequently draws a schematic representation of it in which no distinction is as yet made between kinds of elements. Thus each element is represented merely by a line with small circles at the ends denoting terminals. Such a graphical portrayal showing the geometrical interconnection of elements only, is called a graph of the given network. Figure 1 shows an example of a network as it is usually drawn so as to distinguish the various kinds of elements (part (a)), and how this same network appears when only its geometrical aspects are retained (the graph of part (b)). The numbers associated with the various branches are added for their identification only. The terminals of the branches (which are common to two or more branches where these are confluent) are referred to as nodes.



(a) (b)

Fig. 1

There are situations in which various parts of a network are only inductively connected as in part (a) of Fig. 2 where two pairs of mutually coupled inductances are involved. Here the corresponding graph (shown in part (b) of Fig. 2) consists of three separate parts; and it is seen also that a node may be simply the terminus of a single branch as well as the point of confluence of several branches.



Given network (a) Network graph (b)

Fig. 2

With the graph of a network there are thus associated three things or concepts; namely, branches, nodes, and separate parts. The graph is the skeleton of a network; it retains only its geometrical features. It is useful when discussing how one should best go about characterizing the network behavior in terms of voltages and currents, and in deciding whether a selected set of these variables are not only independent but also adequate for the unique characterization of the state of a network at any moment.

In this regard it is apparent that an economy can be effected in situations like the one in Fig. 2 through permitting one node in each of the separate parts to become coincident, thus uniting these parts, as shown in the graph of Fig. 3.

Except for the fact that the superimposed nodes are constrained to have the same electrical potential, no restrictions are imposed upon any of the branch voltages or currents through this modification which reduces the total number of nodes and the number of separate parts by equal integer values. In subsequent discussions it is thus possible without loss in generality to consider only graphs having one separate part.

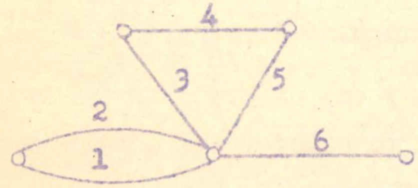
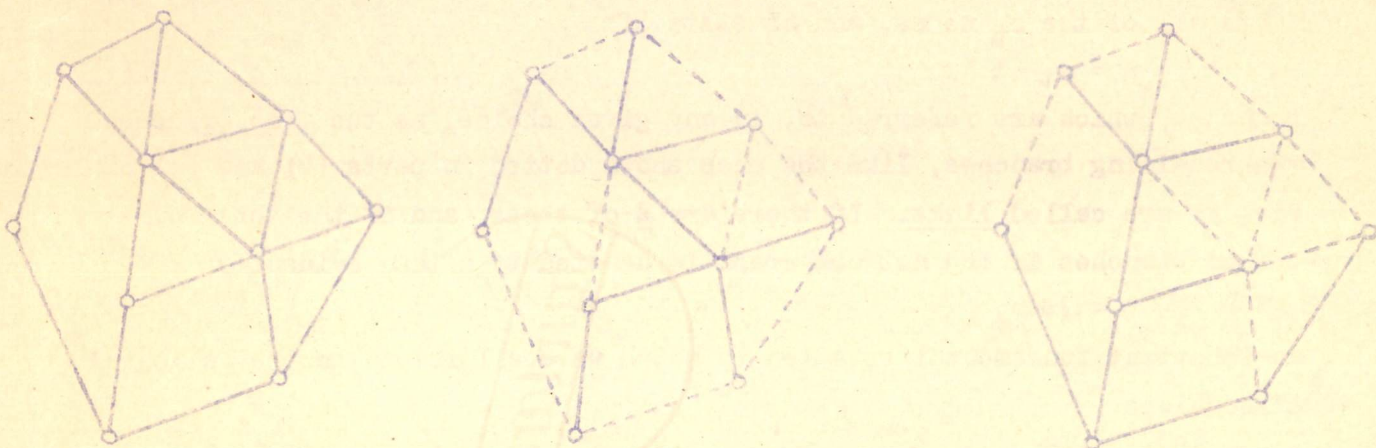


Fig. 3

3. The concept of a "tree."

The graph of a network places in evidence a number of closed paths upon which currents can circulate. This property of a graph (that it contain closed paths) is obviously necessary to the existence of currents in the associated network. It is a property that can be destroyed through the removal of judiciously chosen branches.



(a) (b) (c)

Fig. 4

In Fig. 4, the graph of a given network is shown in part (a), and again in parts (b) and (c) with some of the branches represented by dotted lines. If the branches shown dotted were removed, there would remain in each of the cases shown in (b) and in (c), a graph having all of the nodes of the original graph (a) but no closed paths. This remnant of the original graph is called a "tree" for the reason that its structure (like that of any tree) possesses the significant property of having no closed paths.

More specifically, a tree is defined as any set of branches in the original graph that is just sufficient in number to connect all of the nodes. It is not difficult to see that this number is always $n_t - 1$ where n_t denotes the total number of nodes. For if we start with only the nodes drawn and no branches, it is clear that the first added branch connects two nodes, but thereafter one additional branch is needed for each node contacted. If no more than the minimum number of $n_t - 1$ branches are used to connect all of the nodes, then it is likewise clear that the resulting structure contains no closed paths, for the creation of a closed path involves the linking of two nodes that are already contacted, and hence involves the use of more branches than are actually needed merely to connect all of the nodes.

For a given network graph it is possible to draw numerous trees, since the process just described is not a unique one. Each tree, however, connects all of the n_t nodes, and consists of

$$n = n_t - 1 \tag{1}$$

branches, which are referred to, in any given choice, as the <u>tree branches</u>. The remaining branches, like the ones shown dotted in parts (b) and (c) of Fig. 4, are called <u>links</u>. If there are ℓ of these, and if the total number of branches in the network graph is denoted by b, then evidently

$$b = \ell + n, \tag{2}$$

an important fundamental relation to which we shall return in the following discussions.

4. Network variables.

The response or behavior of a network is completely known if the currents and the voltages in all of its branches are known. The branch currents, however, are related to the branch voltages through fundamental equations that characterize the voltampere behavior of the separate elements. For instance, in a resistance branch the voltage drop (by Ohm's law) equals the current in that branch times the pertinent branch resistance. In a capacitance branch the voltage equals the reciprocal capacitance value times the time integral of the branch current; and in an inductance branch the voltage is given by the time derivative of the current with the inductance as a proportionality factor. Although the latter relations become somewhat more elaborate when several inductances in the network are mutually coupled (as will later be discussed in detail), their determination in no way involves the geometrical interconnection of the elements. One can always, in a straightforward manner, relate the branch-voltages directly and reversibly to the branch currents.

We may, therefore, regard either the branch currents alone or the branch-voltages alone as adequately characterizing the network behavior. If the total number of branches is denoted by b, then from either point of view we have b quantities that play the role of unknowns or variables in the problem of finding the network response. We shall now show that either set of b quantities is not an independent one, but that fewer variables suffice to characterize the network equilibrium, whether on a current or a voltage basis.

If in a given network a tree is selected, then the totality of b branches is separated into two groups: the tree branches and the links. Correspondingly, the branch currents are separated into tree-branch currents, and link currents. Since a removal or opening of the links destroys all closed paths and hence by force renders all branch currents zero, it becomes clear that the act of setting only the link currents equal to zero forces all currents in the network to be zero.* The link currents alone hold the power of life and death, so to speak, over the entire network. Their values fix all the

*In these considerations it is not necessary that we concern ourselves with the manner in which the network is energized although some sort of excitation is implied since all currents and voltages would otherwise be zero regardless of whether the links are removed or not. If the reader insists upon being specific about the nature of the excitation he may picture in his mind a small boy tossing coulombs into the capacitances at random intervals.

current values; that is, it must be possible to express all of the tree-branch currents uniquely in terms of the link currents.

The inference to be drawn from this argument is that, of the $\underline{b}$ branch currents in a network, only $\underline{\ell}$ are independent; $\underline{\ell}$ is the smallest number of currents in terms of which all others can be expressed uniquely. This situation may be seen to follow from the fact that all currents become zero when the link currents are zero. Thus it is clear that the number of independent currents is surely not larger than ℓ, for if one of the tree-branch currents were claimed also to be independent, then its value would have to remain non-zero when all the link currents are set equal to zero, and this condition is manifestly impossible physically. It is equally clear on the other hand that the number of independent currents is surely not less than ℓ, for then it would have to be possible to render all currents in the network zero with one or more links still in place, and this result is not possible because closed paths exist so long as some of the links remain.

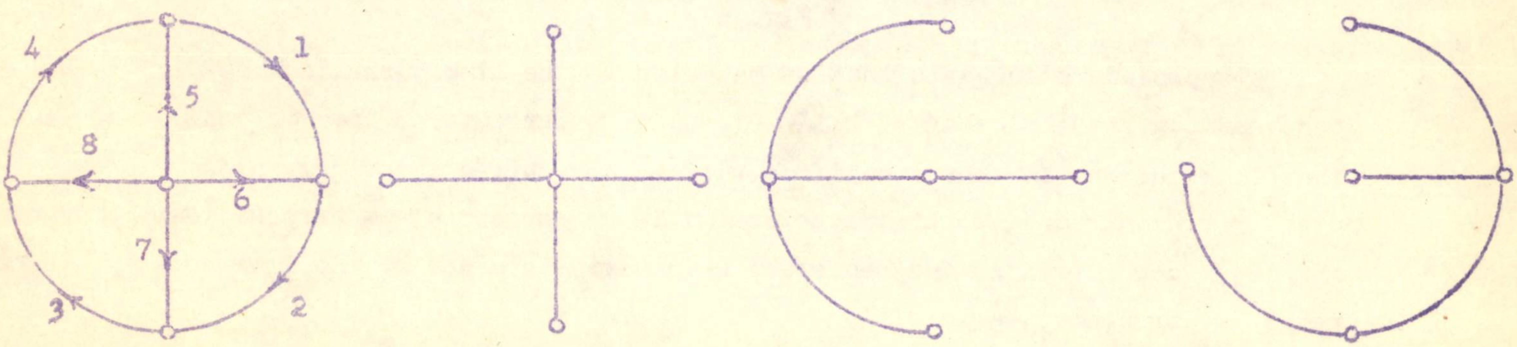
Thus, in terms of currents, it must be possible to express uniquely the state of a network in terms of $\underline{\ell}$ variables alone. As will be shown later, these variables may be any appropriate set of link currents (according to the specific choice made for a tree), but more generally they may be chosen in a large variety of ways so that numerous specific requirements can be accommodated.

Analogously one may regard the branch voltages as separated into two groups: the tree-branch voltages, and the link voltages. Since the tree branches connect all of the nodes, it is clear that if the tree-branch voltages are forced to be zero (through short-circuiting the tree-branches, for example) then all the node potentials become coincident and hence all branch voltages are forced to be zero. Thus, the act of setting only the tree-branch voltages equal to zero, forces all voltages in the network to be zero. The tree-branch voltages alone hold the power of life and death, so to speak, over the entire network. It must be possible, therefore, to express all of the link-voltages uniquely in terms of the tree-branch voltages.

Exactly n of the branch-voltages in a network are independent, namely, those pertaining to the branches of a selected tree. Surely no larger number than this can be independent because one or more of the link voltages would then have to be independent, and this assumption is contradicted by the fact that all voltages become zero through short-circuiting the tree branches alone. On the other hand, no smaller number than n voltages can form the controlling set, for it is physically not possible to force all of the node potentials to coincide so long as some tree-branch voltages remain nonzero.

Thus one recognizes that the state of a network can uniquely be characterized either by means of $\ell = b - n_t + 1$ currents or by $n = n_t - 1 = b - \ell$ voltages. The currents may, for example, be any set of link currents, and the voltages may be any set of tree-branch voltages. Since, in general, $n \neq \ell$, the characterization of a network in terms of current variables involves a different number of unknowns than does its characterization in terms of voltage variables. There is nothing inconsistent about this conclusion since we are at present considering the question of independence among voltages or currents from a geometrical point of view only. Dynamically, the number of independent variables associated with a given physical system determines uniquely its so-called degrees of freedom; their number depends neither upon any algebraic method of derivation, nor upon the manner in which the variables are defined. It is, however, not appropriate to raise these questions at this time, since we are at the moment considering only those features of our problem that are controlled by the geometrical aspects of the given network.

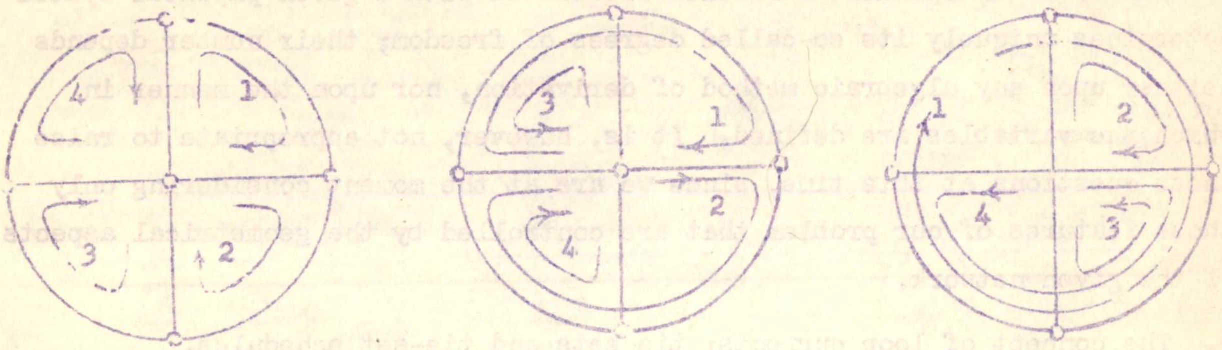
5. The concept of loop currents; tie sets and tie-set schedules.



(a) (b) (c) (d)

Fig. 5

It is possible to give to the link currents an interesting geometrical interpretation that is useful when these are selected as a set of variables. This interpretation is best presented in terms of a specific example. In Fig. 5 (a) is shown a simple network graph and in parts (b), (c), and (d) of the same figure are several possible choices for a tree. For the tree of part (b), the branches numbered 1, 2, 3, 4 are the links. If one of these is inserted into the tree, the resulting structure has just one closed path or loop, which is different for each link. Thus, for this choice of tree, a distinct set of closed paths is associated with the respective links. In Fig. 6 (a) these are indicated by loop arrows, numbered to correspond to the similarly numbered links with which they are associated and directed (as indicated by these loop arrows) so as to be confluent with the respective link currents. Thus loop No. 1 is formed by placing link No. 1 alone into the tree of Fig. 5 (b); loop No. 2 is formed by placing link No. 2 alone into this tree; etc.



(a) (b) (c)

Fig. 6

This procedure suggests that we may give to the link currents a new interpretation, namely, that of being circulatory currents or loop currents. Each link current is thus identified with a loop current; the remaining tree-branch currents are clearly expressible as appropriate superpositions of these loop currents, and hence are uniquely determined by the link currents, as predicted earlier.

If the branch currents in the network graph of Fig. 5 (a) are denoted by $j_1, j_2, \cdots j_8$, numbered to correspond to the branch numbering, and if the loop currents of the graph of Fig. 6 (a) are denoted by i_1, i_2, i_3, i_4, then we can make the identifications

$$j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \quad j_4 = i_4. \tag{3}$$

Through comparison of Figs. 5 (a) and 6 (a) one can then readily express the remaining tree-branch currents as appropriate superpositions of the loop currents, thus

$$\begin{aligned} j_5 &= i_1 - i_4, \\ j_6 &= i_2 - i_1, \\ j_7 &= i_3 - i_2, \\ j_8 &= i_4 - i_3, \end{aligned} \tag{4}$$

or, being mindful of the relations 3, have

$$\begin{aligned} j_5 &= j_1 - j_4, \\ j_6 &= j_2 - j_1, \\ j_7 &= j_3 - j_2, \\ j_8 &= j_4 - j_3. \end{aligned} \tag{5}$$

These last four equations express the tree-branch currents, uniquely and unambiguously, in terms of the link currents. Thus, of the eight branch currents in the graph of Fig. 5 (a), only four are geometrically independent. These four are appropriate to the set of links associated with any selected tree. For the tree of Fig. 5 (b), the link currents are j_1, j_2, j_3, j_4. For the tree of Fig. 5 (c) they are j_1, j_2, j_5, j_7. Here we may write, in place of Eqs. 3,

$$j_1 = i_1, \quad j_2 = i_2, \quad j_5 = i_3, \quad j_7 = i_4. \tag{6}$$

These loop currents circulate on the contours indicated in Fig. 6 (b), which again are found through inserting, one at a time, the branches 1, 2, 5, 7 into the tree of Fig. 5 (c). The tree-branch currents in this case are expressed in terms of the loop currents by the relations

$$
\begin{aligned}
j_3 &= i_2 + i_4, \\
j_4 &= i_1 - i_3, \\
j_6 &= i_2 - i_1, \\
j_8 &= i_1 - i_2 - i_3 - i_4,
\end{aligned}
\qquad (7)
$$

which are found by inspection of Figs. 5 (a) and 6 (b) through noting that the currents in the tree branches result from the superposition of pertinent loop currents.

Through substitution of Eqs. 6 into 7, one again obtains the tree-branch currents expressed in terms of the link currents

$$
\begin{aligned}
j_3 &= j_7 - j_2, \\
j_4 &= j_1 - j_5, \\
j_6 &= j_2 - j_1, \\
j_8 &= j_1 - j_2 - j_5 - j_7,
\end{aligned}
\qquad (8)
$$

thus making evident once more the fact that only four of the eight branch currents are geometrically independent.

The reader is cautioned against concluding that <u>any</u> four of the eight branch currents may be regarded as an independent set. The branches pertaining to a set of independent currents must be the links associated with a tree, for it is this circumstance that assures the independence of the currents. Thus the branch currents j_5, j_6, j_7, j_8, for example, could not be a set of independent currents because the remaining branches 1, 2, 3, 4 do not form a tree. The concept of a tree is recognized as useful because it yields a simple and unambiguous method of deciding whether any selected set of branch-currents is an independent one. Or one can say that the tree concept provides a straightforward method of determining a possible set of independent current variables for any given network geometry.

Part (d) of Fig. 5 shows still another possible choice for a tree appropriate to the graph of part (a), and in Fig. 6 (c) is shown the corresponding set of loops. In this case one has

$$
j_4 + i_1, \quad j_5 = i_2, \quad j_7 = i_3, \quad j_8 = i_4, \qquad (9)
$$

and through superposition there follows that

$$
\begin{aligned}
j_1 &= i_1 + i_2 = j_4 + j_5 \\
j_2 &= i_1 - i_3 - i_4 = j_4 - j_7 - j_8 \\
j_3 &= i_1 - i_4 = j_4 - j_8 \\
j_6 &= -i_2 - i_3 - i_4 = -j_5 - j_7 - j_8.
\end{aligned}
\tag{10}
$$

When dealing with networks having large numbers of branches and correspondingly elaborate geometries, one must have a less cumbersome and more systematic procedure for obtaining the algebraic relationships between the branch currents and the loop-current variables. Thus it is readily appreciated that the process of drawing and numbering the reference arrows for the loops, and subsequently obtaining by inspection the appropriate expressions for the branch currents as algebraic sums of pertinent loop currents, can become both tedious and confusing in situations involving complex geometries.

A systematic way of indicating the loops associated with the selection of a particular tree is had through use of a schedule such as the following, which pertains to the graph of Fig. 5 (a) with the tree of part (c) and hence for the loops shown in Fig. 6 (b)

Loop No. \ Branch No.	1	2	3	4	5	6	7	8
1	1	0	0	1	0	-1	0	1
2	0	1	1	0	0	1	0	-1
3	0	0	0	-1	1	0	0	-1
4	0	0	1	0	0	0	1	-1

(11)

To interpret this schedule we note that the first row, pertaining to loop No. 1, indicates that a circuit around this loop is equivalent to traversing in the positive reference direction, branches 1, 4 and 8, and in the negative reference direction, branch 6. None of the remaining branches participate in forming the contour of loop No. 1, and so their corresponding spaces in the first row of the schedule are filled in with zeros. The second row is similarly constructed, noting that the pertinent loop contour is formed through traversing branches 2, 3, and 6 positively, and branch 8 negatively. Thus the successive rows in this schedule indicate the confluent sets of branches that participate in forming the various correspondingly numbered loops, due attention being given (and indicated through appropriate algebraic signs) to the confluence or counterfluence of respective branch and loop reference arrows.

A most interesting and useful property of this schedule is now revealed through attention to its columns, for we note that their elements are the coefficients in an orderly written set of equations expressing the branch currents in terms of the loop currents. Interpretation of the columns of schedule 11 in this way yields the equations

$$
\begin{aligned}
j_1 &= i_1 \\
j_2 &= i_2 \\
j_3 &= i_2 + i_4 \\
j_4 &= i_1 - i_3 \\
j_5 &= i_3 \\
j_6 &= -i_1 + i_2 \\
j_7 &= i_4 \\
j_8 &= i_1 - i_2 - i_3 - i_4,
\end{aligned}
\qquad (12)
$$

which are seen to agree with Eqs. 6 and 7.

The reason why this schedule has the property just mentioned may best be seen through supposing that it is originally constructed, by columns, according to the relationships expressed in Eqs. 12. One subsequently can appreciate why the resulting rows of the schedule indicate the pertinent closed paths, through noting that the nonzero elements of a row are associated with branches traversed by the same loop current, and these collectively must form the closed path in question.

The actual construction of the schedule may thus be done in either of two ways, viz.: by rows, according to a set of independent closed paths (for example, those associated with a selected tree), or by columns, according to a set of equations expressing branch currents in terms of loop currents. If constructed by columns, the rows of the schedule automatically indicate the closed paths upon which the associated loop currents circulate; and if constructed by rows from a given set of closed paths, the columns of the resulting schedule automatically yield the pertinent relations for the branch currents in terms of the loop currents. This type of schedule (which for reasons given later is called a tie-set schedule) is thus revealed to be a compact and effective means for indicating both the geometrical structure of the closed paths and the resulting algebraic relations between branch currents and loop currents.

Regarding this relationship, one may initially be concerned about its uniqueness, since there are fewer loop currents than branch currents. Thus, if asked to solve the Eqs. 12 for the loop currents in terms of branch currents, one might be puzzled by the fact that there are more equations than unknowns. However, the number of independent equations among this set just equals the number of unknown loop currents (for reasons given in the preceding discussion), and the equations collectively form a consistent set. Therefore the desired solution is effected through separating from the Eqs. 12 an independent subset, and solving these. Knowing that the equations were originally obtained through choice of the tree of Fig. 5 (c), thus designating branch currents j_1, j_2, j_5, j_7 as a possible independent set, indicates that the corresponding equations among those given by 12 may be regarded as an independent subset. These yield the identifications $i_1 = j_1$, $i_2 = j_2$, $i_3 = j_5$, $i_4 = j_7$ as indicated in Eqs. 6 for this choice of tree.

It is, however, not essential that the independent subset chosen from the Eqs. 12 be this particular one. Thus, if we consider the tree of Fig. 5 (d) as a possible choice, it becomes clear that branch currents j_4, j_5, j_7, j_8 are an independent set. The corresponding equations separated from 12, namely,

$$\begin{aligned} j_4 &= i_1 - i_3 \\ j_5 &= i_3 \\ j_7 &= i_4 \\ j_8 &= i_1 - i_2 - i_3 - i_4 \end{aligned} \qquad (13)$$

may alternatively be regarded as an appropriate independent subset. Their solution reads

$$\begin{aligned} i_1 &= j_4 + j_5 \\ i_2 &= j_4 - j_7 - j_8 \\ i_3 &= j_5 \\ i_4 &= j_7 . \end{aligned} \qquad (14)$$

Noting from Eqs. 8 that $j_4 + j_5 = j_1$, and that $j_4 - j_7 - j_8 = j_2$, it is clear that 14 agrees with the former result.

Four of the eight Eqs. 12 are independent. A simple rule for picking four independent ones is to choose those corresponding to the link currents associated with a possible tree. Any four independent ones may be solved for the four loop currents. Substitution of these solutions into the remaining equations then yields the previously discussed relations between tree-branch currents and link currents.

There should be no difficulty in understanding this situation since the previous discussion has made it amply clear that the link currents or loop currents are an independent set and all other branch currents are uniquely related to these. The Eqs. 12 are consistent with this viewpoint and contain all of the implicit and explicit relations pertinent thereto. Hence their solution cannot fail to be unique, no matter what specific approach one may take to gain this end.

Although a schedule like 11 may be constructed either by columns or by rows, the usual viewpoint will be that it is constructed by rows from an observation of those sets of confluent branches forming the pertinent closed paths. The latter are placed in evidence, one by one, through imagining that all the links are opened except one, thus forcing all but one of the link or loop currents to be zero. The existence of a single loop current energizes a set of branches forming the closed path on which this loop current circulates. This set of branches, called a tie set, is indicated by the elements in the pertinent row of the tie set schedule.

If the geometry of the network graph permits its mappability upon a plane or spherical surface without crossed branches, then we may regard any tie set as forming a boundary that divides the total network into two portions*. Hence if the branches in such a set are imagined to shrink longitudinally until they reduce to a single point, the network becomes "tied off" so to speak (as a fish net would by means of a draw string), and the two portions bounded by the tie set become effectively separated except for a common node. It is this interpretation of the tie set that suggests its name.

Although there are several important variations in this procedure for establishing an appropriate set of current variables, we shall leave these for subsequent discussion, and turn our attention now to the alternate procedure (dual to the one just described) of formulating a set of network variables on a voltage basis.

6. The concept of node-pair voltages; cut sets and cut-set schedules.

On the voltage side of the network picture, an entirely analogous situation prevails. Here we begin by regarding the tree-branch voltages as a possible set of independent variables in terms of which the state of a network may uniquely be expressed. Since the tree branches connect all of the nodes, it is possible to trace a path from any node to any other node in the network by traversing tree branches alone; and therefore it is possible to express the difference in potential between any pair of nodes in terms of the tree-branch voltages alone. Moreover, the path connecting any two nodes via tree branches is unique since the tree has no closed loops and hence offers no alternate paths between node-pairs. Therefore, the potential

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*For a graph not mappable on a sphere (for example one that requires a a doughnut-shaped surface), some but not all tie sets have this property. This point is discussed further in Art. 9.

difference between any two nodes, referred to as the pertinent node-pair voltage, is uniquely expressible in terms of the tree-branch voltages. The link voltages, which are a particular set of node-pair voltages, are thus recognized to be uniquely expressible in terms of the tree-branch voltages.

Let us illustrate these principles with the network graph of Fig. 5 (a), and choose initially the tree given in part (b) of this same figure. If the branch voltages are denoted by $v_1, v_2, \cdots v_8$, numbered to correspond to the given branch numbering, then the quantities v_5, v_6, v_7, v_8 are the tree-branch voltages and hence may be regarded as an independent set. They may simultaneously be regarded as node-pair voltages, and since they are to serve as the chosen set of variables, we distinguish them through an appropriate notation and write

$$e_1 = v_5,\ e_2 = v_6,\ e_3 = v_7,\ e_4 = v_8. \tag{15}$$

This part of the procedure parallels the use of a separate notation for the loop currents $i_1, i_2, \cdots$ when choosing variables on a current basis. There the link currents are identified with loop currents; in Eq. 15 the tree-branch voltages are identified with node-pair voltages.

The remaining branch voltages, namely the link voltages, are now readily expressible in terms of the four tree-branch or node-pair voltages 15. Thus, by inspection of Fig. 5 (a) we have

$$\begin{aligned} v_1 &= -v_5 + v_6 = -e_1 + e_2 \\ v_2 &= -v_6 + v_7 = -e_2 + e_3 \\ v_3 &= -v_7 + v_8 = -e_3 + e_4 \\ v_4 &= -v_8 + v_5 = -e_4 + e_1. \end{aligned} \tag{16}$$

The procedure in writing these equations is to regard each link voltage as a potential difference between the nodes terminating the pertinent link, and to pass from one of these nodes to the other via tree branches only, adding algebraically the several tree-branch voltages encountered.

If the tree of Fig. 5 (c) is chosen, the branch voltages v_3, v_4, v_6, v_8 become the appropriate independent set, and we make the identifications

$$e_1 = v_3, \ e_2 = v_4, \ e_3 = v_6, \ e_4 = v_8. \tag{17}$$

The expressions for the link voltages in terms of these read

$$\begin{aligned} v_1 &= -v_4 + v_6 - v_8 = -e_2 + e_3 - e_4 \\ v_2 &= -v_3 - v_6 + v_8 = -e_1 - e_3 + e_4 \\ v_5 &= v_4 + v_8 = e_2 + e_4 \\ v_7 &= -v_3 + v_8 = -e_1 + e_4. \end{aligned} \tag{18}$$

The results expressed in Eqs. 16 and 18 bear out the truth of a statement made in Art. 4 to the effect that any set of tree-branch voltages may be regarded as an independent group of variables in terms of which the remaining branch voltages (link voltages) are uniquely expressible. In the network graph of Fig. 5, any tree has four branches. Hence, of the eight branch voltages, only four are geometrically independent. These may be the ones pertinent to any selected tree; and the rest are readily expressed in terms of them.

In dealing with more complex network geometries it becomes useful to establish a systematic procedure for the selection of node-pair voltage variables and the unique expression of the branch voltages in terms of them. The accomplishment of this end follows a pattern that is entirely analogous (yet dual) to that described in the previous article for the current basis. That is to say, we seek to construct a schedule appropriate to the voltage basis in the same way that the tie-set schedule is pertinent to the current basis. To this end we must first establish the geometrical interpretation for a set of branches which, for the voltage basis, plays a role analogous to that defined for the current basis by a tie set (or confluent set of branches forming a closed loop). The latter is placed in evidence through opening all of the links but one, so that all loop currents are zero except one. The analogous procedure on a voltage basis is to force all but one of the node-pair (i.e., tree-branch) voltages to be zero, which is accomplished through short-circuiting all but one of the tree branches. This act will in general simultaneously short-circuit some of the links, but there will in any nontrivial case be left some links in addition to the one nonshorted tree branch that are likewise not short circuited and will appear to form connecting links between the pair of nodes terminating the pertinent tree branch. This set of branches, which is called a <u>cut set</u>, is the desired analogue of a tie set, as the following detailed elaboration will clarify.

Consider again the network of Fig. 5 (a) and the tree of part (b) of this figure, together with the pertinent stipulation of node-pair voltages as expressed by Eqs. 15. The cut-set schedule appropriate to this situation reads

Node-pair No. \ Branch No.	1	2	3	4	5	6	7	8
1	-1	0	0	1	1	0	0	0
2	1	-1	0	0	0	1	0	0
3	0	1	-1	0	0	0	1	0
4	0	0	1	-1	0	0	0	1

(19)

It is customary to regard the node-pair voltages e_1, e_2, e_3, e_4 as rises, while the branch voltages are drops. For this reason the reference arrow for an "e" is opposite to that for the "v" that it is numerically equal to. With this fact in mind let us consider in detail the construction of schedule 19 by its rows, referring for this purpose to Fig. 5. Since by Eq. 15, $e_1 = v_5$, we observe that the terminals of branch 5 constitute node pair No. 1, and the tip end of the reference arrow for e_1 is at the tail end of the reference arrow for branch 5. Analogous remarks apply to the other three node pairs which are the terminals of branches 6, 7, and 8.

To determine the cut set associated with node pair No. 1, we regard e_1 as the only nonzero node-pair voltage; that is, branches 6, 7, and 8 are short-circuited. Under these conditions it should be clear that the links 2 and 3 are likewise short-circuited, but that links 1 and 4 together with tree branch 5 remain nonshorted. These three branches, therefore, constitute the cut set pertinent to node pair No. 1, and the corresponding elements in the first row of schedule 19 thus are the only nonzero ones.

Now as to the algebraic signs of these nonzero elements, we note that a positive e_1 would cause currents that are confluent with the reference arrows in branches 4 and 5, and counterfluent with the reference arrow in branch 1. The sign given to the nonzero element is chosen positive for confluence and negative for counterfluence.

Similarly, node pair No. 2 is identified with the terminals of branch 6. To place in evidence the associated cut set, we imagine the other tree branches to be short-circuited, whence the nonshorted branches are 1, 2, and 6. The sign convention just described yields plus signs for branches 1 and 6, and a minus sign for branch 2. Construction of the remaining rows in this cut-set schedule follows the same pattern.

If we now regard the columns in this schedule as containing the coefficients in an orderly written system of equations expressing the branch voltages in terms of the node-pair voltages, we have

$$
\begin{aligned}
v_1 &= -e_1 + e_2 \\
v_2 &= -e_2 + e_3 \\
v_3 &= -e_3 + e_4 \\
v_4 &= e_1 - e_4 \\
v_5 &= e_1 \\
v_6 &= e_2 \\
v_7 &= e_3 \\
v_8 &= e_4
\end{aligned}
\qquad (20)
$$

These agree with the results expressed by Eqs. 15 and 16. Hence we see that the cut-set schedule 19 could alternatively have been constructed by columns according to Eqs. 15 and 16, or the equivalent Eqs. 20.

The cut set is that group of branches that becomes energized when its pertinent node-pair voltage is the only nonzero one among all of the node-pair voltages, just as the tie set is that group of branches that becomes energized when its pertinent loop current is the only nonzero one among all of the loop currents. It is clear, therefore, that the cut sets may alternatively be found from the Eqs. 20 through picking out the branches corresponding to those nonzero v's that result from considering the e's to be nonzero one at a time; and it is thus appreciated that the construction of a cut-set schedule may be done either by rows according to the method first described or by columns according to a set of equations like those given by 20. If constructed by rows, through picking out the cut sets, the columns of the schedule automatically yield the associated relations for the branch voltage drops in terms of the node-pair voltages; and if constructed by columns, the rows of the schedule automatically yield the associated cut sets.

An important geometrical characteristic of a cut set is recognized from the following consideration. If all but one of the branches in a tree become short circuits, then only the two nodes at the ends of the nonshorted branch survive. Or we may say that the totality of nodes in the network separate into two groups; all the nodes of one group become superimposed at one end of the nonshorted branch, and all the nodes of the second group coincide at the other end of this branch. If we think of grasping these two groups of nodes separately, one in each hand, and regard the branches as though they were elastic bands, then the cut set in question is that set of branches that is stretched as we pull our hands apart. If we were to cut these stretched branches, the original network graph would be cut into two subgraphs, one of which we would be holding in each hand. It is this interpretation that suggests the name "cut set."

Cut Set

A cut set quite generally is thus seen to be any group of branches so selected that the act of cutting them separates the network into two parts. We can visualize the selection of a cut set as being done by picking up in one hand some of the nodes in the network and pulling these away from the rest (which may be thought of as fastened somehow to the plane of the paper); the stretched branches form a cut set.

With this interpretation in mind let us again consider the formation of the cut set schedule 19 based upon the graph of Fig. 5 (a) with the tree of part (b) of that figure and hence for the node-pair voltages defined by Eq. 15. In Fig. 7 this graph is redrawn with the nodes lettered so we can refer to them specifically. Branches 5, 6, 7, 8 constitute the tree. Node-pair voltage e_1 equals v_5, and the tip of its reference arrow is at node o. To find the cut set corresponding to this node-pair voltage, the remaining tree branches are regarded as short circuits, whence the nodes b, c, d are seen to coincide with the node o. In our right hand we, therefore, imagine picking up nodes o, b, c, d, and with our left hand, holding node a. The tip end of the reference arrow for e_1 is in our right hand; hence the positive reference direction for branches in the associated cut set is from our right hand to our left hand. The stretched branches clearly are those numbered 1, 4, 5. The reference arrows on 4 and 5 are confluent with the positive reference direction for this node pair while that on branch 1 is counterfluent. Construction of the first row of schedule 19 is thus clear.

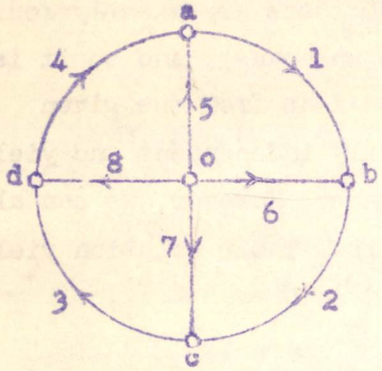


Fig. 7.

To construct the second row we observe that $e_2 = v_6$, and so the tip end of e_2 is again at node o. This time branches 5, 7, 8 are regarded as short circuits; so the nodes picked up in our right hand become o, a, c, d, and the positive reference direction for the cut-set branches is again divergent from our right hand. The cut set consists of branches 1, 2, 6, with 1 and 6 positive and 2 negative.

For e_3 the picked-up nodes are o, a, b, d, and for e_4 they are o, a, b, c. It is useful always to consider picking up that group of nodes that coincides at the tip end of the pertinent node-pair voltage; then the positive reference direction for the branches in the associated cut set is consistently divergent from the picked-up nodes.

Returning now to the Eqs. 20, a few additional remarks may be in order regarding their inversion. That is to say, if we were asked to solve these equations for the e's in terms of the v's, the question of uniqueness may arise since there are more equations than unknowns. However, the situation here is the same as has already been discussed for the current basis in connection with Eqs. 12. Namely, among the Eqs. 20 there are exactly four independent ones (as many as there are independent unknowns), and so it is merely a matter of separating four independent equations from the given group and solving these. The last four are obviously independent and yield the definitions chosen for the e's in the first place. However, we can alternatively choose say the first three and the fifth. Their solution yields

$$
\begin{aligned}
e_1 &= v_5 \\
e_2 &= v_1 + v_5 \\
e_3 &= v_1 + v_2 + v_5 \\
e_4 &= v_1 + v_2 + v_3 + v_5,
\end{aligned}
\qquad (21)
$$

but Eqs. 16 show that

$$
\begin{aligned}
v_1 + v_5 &= v_6 \\
v_1 + v_2 + v_5 &= v_7 \\
v_1 + v_2 + v_3 + v_5 &= v_8.
\end{aligned}
\qquad (22)
$$

Hence the solutions 21 again agree with the definitions 15.

Of the Eqs. 20, four are independent. Not <u>any</u> four are independent, but there are no more than four independent ones in this group, and there are several different sets of four independent ones that can be found among them. A simple rule for picking four independent ones is to choose those corresponding to the branch voltages of a possible tree. The solution to these yields the expressions for the e's in terms of the v's; and substitution of these solutions for the e's into the remaining equations yields the previously discussed relations between link voltages and tree-branch voltages. The cut-set schedule which contains the information regarding the geometrical character of the cut sets, as well as the algebraic relationships between the implied node-pair voltages and the branch-voltages, is thus seen to be a compact and effective mode of expressing these things. It does for the formulation of variables on the voltage basis what the tie-set schedule does for the establishment of a system of variables on the current basis. Continued use will be made of both types of schedules in the following discussions.

7. Alternative methods of choosing current variables.

The procedure for selecting an appropriate set of independent current variables in a given network problem can be approached in a different manner which may sometimes be preferred. Thus, the method given in Art. 5, which identifies the link currents with a set of loop current variables, leaves the tie sets or closed paths upon which these currents circulate, to be determined from the choice of a tree, whereas one may prefer to specify a set of closed paths for the loop currents at the outset.

Consider in this connection the graph of Fig. 8. In addition to providing the branches with numbers and reference arrows, a set of loops have also been chosen and designated with the circulatory arrows numbered 1, 2, 3, 4. These loops, incidentally, are referred to as meshes because they have the appearance of the meshes in a fish net. It is a common practice in network analysis to choose, as a set of current variables, the currents that are assumed to circulate on the contours of these meshes. Having made such a choice, we must know how to relate in an unambiguous and reversible manner, the branch currents to the chosen mesh currents.

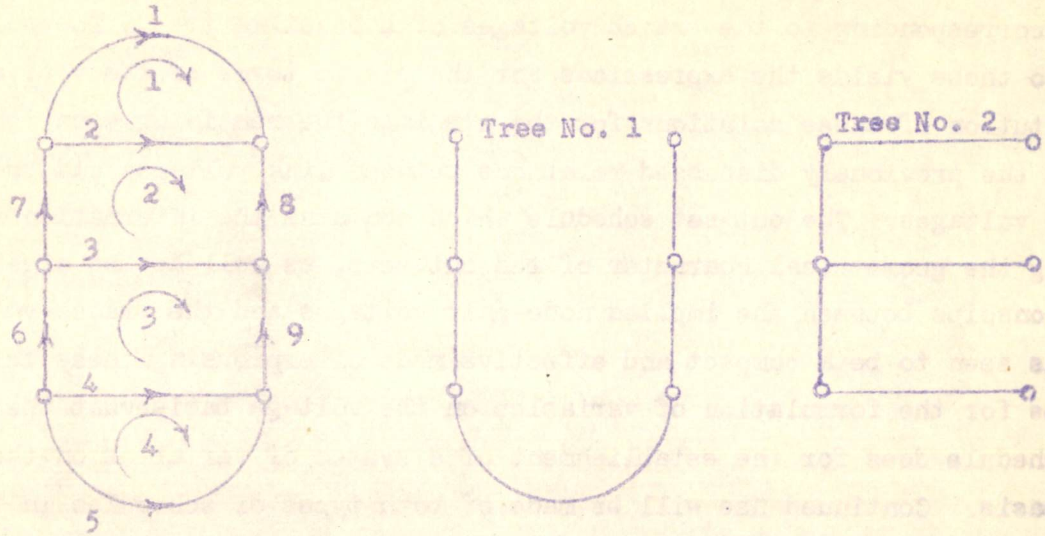


Fig. 8

This end is accomplished through setting down the tie-set schedule corresponding to the choice made for the closed paths defining the tie sets. With reference to the graph of Fig. 8 one has by inspection

Mesh No. \ Branch No.	1	2	3	4	5	6	7	8	9
1	1	-1	0	0	0	0	0	0	0
2	0	1	-1	0	0	0	1	-1	0
3	0	0	1	-1	0	1	0	0	-1
4	0	0	0	1	-1	0	0	0	0

(23)

and the columns yield

$$
\begin{aligned}
j_1 &= i_1 \\
j_2 &= -i_1 + i_2 \\
j_3 &= -i_2 + i_3 \\
j_4 &= -i_3 + i_4 \\
j_5 &= -i_4 \\
j_6 &= i_3 \\
j_7 &= i_2 \\
j_8 &= -i_2 \\
j_9 &= -i_3 .
\end{aligned}
\qquad (24)
$$

The mesh currents in terms of the branch currents are found through solving any four independent equations in this group. If we consider tree No. 1 in Fig. 8, the links are given by branches 1, 2, 3, 4, thus indicating that the first four of the Eqs. 24 are independent. These yield

$$
\begin{aligned}
i_1 &= j_1 \\
i_2 &= j_1 + j_2 \\
i_3 &= j_1 + j_2 + j_3 \\
i_4 &= j_1 + j_2 + j_3 + j_4 .
\end{aligned}
\qquad (25)
$$

Substitution into the remaining Eqs. 24, gives

$$
\begin{aligned}
j_5 &= -j_1 - j_2 - j_3 - j_4 \\
j_6 &= j_1 + j_2 + j_3 \\
j_7 &= j_1 + j_2 \\
j_8 &= -j_1 - j_2 \\
j_9 &= -j_1 - j_2 - j_3 .
\end{aligned}
\qquad (26)
$$

These express the tree-branch currents in terms of the link currents.

If instead, we choose tree No. 2 in Fig. 8, the branches 1, 5, 8, 9 become links. The corresponding equations in the group 24, namely

$$
\begin{aligned}
i_1 &= j_1 \\
i_4 &= -j_5 \\
i_2 &= -j_8 \\
i_3 &= -j_9
\end{aligned}
\qquad (27)
$$

are independent and give the expressions for the mesh currents in terms of the link currents. With these, the remaining Eqs. 24 yield again the tree-branch currents in terms of the link currents, thus

$$
\begin{aligned}
j_2 &= -j_1 - j_8 \\
j_3 &= j_8 - j_9 \\
j_4 &= -j_5 + j_9 \\
j_6 &= -j_9 \\
j_7 &= -j_8 .
\end{aligned}
\qquad (28)
$$

It is readily seen that the results expressed by Eqs. 25 and 26 are consistent with those given by Eqs. 27 and 28. That is to say, the choice of a tree has nothing to do with the algebraic relations between the loop currents and the branch currents; it merely serves as a convenient way of establishing an independent subset among the Eqs. 24. In the present very simple example, one can just as easily pick an independent subset without the aid of the tree concept; however, in more complex problems the latter can prove very useful.

In approaching the establishment of a set of current variables through making at the outset a choice of closed paths, a difficulty arises in that the independence of these paths is in general not assured. A necessary (though not sufficient) condition is that all branches must participate in forming these paths, for if one or more of the branches were not traversed by loop currents, then the currents in these branches in addition to the loop currents would appear to be independent. Actually, the loop currents chosen in this manner could not be independent since altogether there can be only ℓ independent currents.

A sufficient (though not necessary) procedure to insure the independence of the closed paths (tie sets) is to select them successively in such a way that each additional path involves at least one branch that is not part of any of the previously selected paths. This statement follows from the fact that the paths or tie sets form an independent set if the ℓ rows in the associated tie-set schedule are independent; that is, if it is not possible to express any row in this schedule as a linear combination of the other rows. If, as we write down the successive rows in this schedule, each new row involves a branch that has not appeared in any of the previous rows, that row can surely not be formed from a linear combination of those already chosen, and hence must be independent of them.

A glance at the schedule 23 shows that this principle is met. Thus, construction of the first row involves only branches 1 and 2. The second row introduces the additional branches 3, 7, 8; the third row adds branches 4, 6 and 9, and the last row involves the previously unused branch No. 5. It is not difficult to convince oneself that if one designates only meshes as closed paths (which is, of course, possible only in a graph that is mappable on a plane or sphere), then the rows in the associated tie-set schedule can always be written in such a sequence that the principle just described will be met. This simple choice in a plane mappable graph, therefore, always assures the independence of the closed paths and hence does the same for the implied mesh-current variables.

However, it is quite possible for the ℓ rows in a tie-set schedule to be independent while not fulfilling the property just pointed out. Thus as already stated above, this property of the rows is a sufficient though not necessary condition to insure their independence. When closed paths are chosen in a more general manner, as they sometimes may be, it is not always evident at the outset whether the choice made is acceptable. To illustrate this point, let us reconsider the network graph of Fig. 8 with the choice of closed paths shown in Fig. 9.

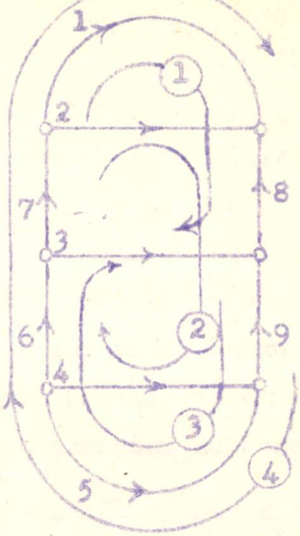


Fig. 9

The tie-set schedule reads

Loop No. \ Branch No.	1	2	3	4	5	6	7	8	9
1	1	0	-1	0	0	0	1	-1	0
2	0	1	0	-1	0	1	1	-1	-1
3	0	0	1	0	-1	1	0	0	-1
4	1	0	0	0	-1	1	1	-1	-1

(29)

and hence the expressions for the branch currents in terms of the loop currents are

$$j_1 = i_1 + i_4 \qquad j_5 = -i_3 - i_4$$
$$j_2 = i_2 \qquad j_6 = i_2 + i_3 + i_4$$
$$j_3 = -i_1 + i_3 \qquad j_7 = i_1 + i_2 + i_4 \qquad (30)$$
$$j_4 = -i_2 \qquad j_8 = -i_1 - i_2 - i_4$$
$$j_9 = -i_2 - i_3 - i_4,$$

as may also be verified by inspection of Fig. 9.

To investigate the independence of the chosen loops, we observe that the choice of tree No. 1 in Fig. 8 indicates that the branch currents j_1, j_2, j_3, j_4 form an independent set. Hence the first four of the Eqs. 30 should be independent. They obviously are not, since the second and fourth equations are identical except for a change in algebraic sign. Hence the loops indicated in Fig. 9 are not an independent set, notwithstanding the fact that they are ℓ in number and collectively traverse all of the branches.

If we modify the choice of loops in Fig. 9 merely by shifting the part of loop 4 that traverses branch 1 onto branch 2, the resultant set is found to be an independent one.* It should thus be clear that the choice of an independent set of loops (or tie sets) is in general not a matter that is evident by inspection, although one has a straightforward procedure for checking a given selection. Namely, the chosen set of loops are independent if the ℓ rows of the associated tie-set schedule are independent; and they are, if it is possible to find in this schedule a subset of ℓ independent columns (i.e., ℓ independent equations among a set like 30). The simplest procedure for making this check among the columns is to pick those columns corresponding to the links of <u>any</u> chosen tree. These must be independent if the ℓ rows of the schedule are to be independent. They are if the pertinent equations (like the first four of 30 in the test discussed in the previous paragraph) have unique solutions. Usually one can readily see by inspection whether or not such solutions exist. An elegant algebraic method is to see if the determinant of these equations is nonzero. Thus the nonvanishing of the determinant formed from the subset of columns corresponding to the links of a chosen tree suffices to prove the independence of an arbitrarily selected set of closed paths.

---

*It might be noted, incidentally, that the rows of the correspondingly modified schedule 29 do not fulfill the property that each successive one involves at least one new branch. Nevertheless these rows are independent.

In the case of graphs having many branches this method may prove tedious, and so it is useful to be aware of alternative procedures for arriving at more general current variable definitions, should this be desirable. Thus one may make use of the fact that the most general tie-set schedule is obtainable through successive elementary transformations of the rows of any given one, and that such transformations leave the independence of the rows invariant. We may, for example, start with a schedule like 23 that is based upon a choice of meshes so that its rows are surely independent. Suppose we construct a new first row through adding to the elements of the present one, the respective elements of the second row. The new schedule is then

Loop No. \ Branch No.	1	2	3	4	5	6	7	8	9
1	1	0	-1	0	0	0	1	-1	0
2	0	1	-1	0	0	0	1	-1	0
3	0	0	1	-1	0	1	0	0	-1
4	0	0	0	1	-1	0	0	0	0

. (31)

Loops 2, 3, 4 are still the meshes 2, 3, 4 of Fig. 8. However, loop 1 is now the combined contour of meshes 1 and 2, as a comparison of the first row of the new schedule with the graph of Fig. 8 reveals. If we modify this new schedule further througn constructing a new second row with elements equal to the sum of the respective ones of the present row 2, 3, and 4, there results another schedule that implies a loop No. 2 with the combined contours of meshes 2, 3, and 4. It should thus be clear that more general loops or tie sets are readily formed through combining linearly a set of existing simple ones. So long as only one new row is constructed from the combination of rows in a given schedule, and if the pertinent old row is a constituent part of this combination, the procedure cannot destroy the independence of a given set of rows.

Each new schedule has the property that its columns correctly yield the expressions for the branch currents in terms of the implied new loop currents. That is to say, since transformation of the schedule through making linear row combinations implies a revision in the choice of loops, it likewise implies a revision in the algebraic definitions of the loop currents. Nevertheless the relations expressing the branch currents in terms of these new loop currents is still given by the coefficients in the columns of the schedule. For example, we would get for schedule 31 the relations

$$\begin{aligned}
j_1 &= i_1' \\
j_2 &= i_2' \\
j_3 &= -i_1' - i_2' + i_3' \\
j_4 &= -i_3' + i_4' \\
j_5 &= -i_4' \\
j_6 &= i_3' \\
j_7 &= i_1' + i_2' \\
j_8 &= -i_1' - i_2' \\
j_9 &= -i_3'
\end{aligned} \tag{32}$$

where primes are used on the i's to distinguish them from those in Eqs. 24 which are pertinent to schedule 23.

Comparison of Eqs. 24 and 32 reveals the transformation in the loop currents implied by the transformation of schedule 23 to the form 31, namely

$$\begin{aligned}
i_1 &= i_1' \\
i_2 &= i_1' + i_2' \\
i_3 &= i_3' \\
i_4 &= i_4' .
\end{aligned} \tag{33}$$

This result is at first sight somewhat unexpected. Thus the transformation from schedule 23 to schedule 31 implies leaving the contours for the loop currents i_2, i_3, i_4 the same as in the graph of Fig. 8, but changes the contour for loop current i_1. Offhand we would expect the algebraic definition for i_1 to change and those for i_2, i_3, and i_4 to remain the same. Instead we see from Eqs. 33 that i_1, i_3, and i_4 are unchanged while i_2 changes. Nevertheless the Eqs. 32 are correct as we can readily verify through sketching in Fig. 8 the altered contour for loop 1 and expressing the branch currents as linear superpositions of the loop currents, noting this altered path for i_1. The results expressed by Eqs. 33, therefore, are surely correct also.

The mental confusion temporarily created by this result disappears if we concentrate our attention upon schedule 23 and the Eqs. 24, and ask ourselves: What change in the relations 24 will bring about the addition of row 2 to row 1 in the schedule 23 and leave rows 2, 3, and 4 unchanged? The answer is that we must replace the symbol i_2 by $i_1 + i_2$, for then every element in row 2 will also appear in row 1 in addition to the elements that are already in row 1, and nothing else will change. The lesson to be learned from this example is that we should not expect a simple and obvious connection between the contours chosen for loop currents and the algebraic definitions for these currents, nor should we expect to be able to correlate by inspection, changes in the chosen contours (tie sets) and corresponding transformations in the loop currents until experience with these matters has given us an adequate insight into the rather subtle relationships inplied by such transformations.

The reason for our being misled in the first place is that we are too prone to regard the choice of contours for loop currents as <u>equivalent</u> to their definition in terms of the branch currents, whereas in reality the fixing of these contours merely <u>implies</u> the algebraic relationships between the loop currents and branch currents (through fixing the tie-set schedule); it does <u>not</u> place them in evidence.

The most general form which a linear transformation of the tie-set schedule may take is indicated through writing in place of 33

$$
\begin{aligned}
i_1 &= \alpha_{11} i_1' + \alpha_{12} i_2' + \cdots + \alpha_{1\ell} i_\ell' \\
i_2 &= \alpha_{21} i_1' + \alpha_{22} i_2' + \cdots + \alpha_{2\ell} i_\ell' \\
&\cdots\cdots\cdots\cdots\cdots \\
i_\ell &= \alpha_{\ell 1} i_1' + \alpha_{\ell 2} i_2' + \cdots + \alpha_{\ell\ell} i_\ell',
\end{aligned}
\qquad (34)
$$

in which the α's are any real numbers. If $i_1 \cdots i_\ell$ are an independent set of current variables, then $i_1' \cdots i_\ell'$ will be independent if the Eqs. 34 are independent; that is, if they possess unique solutions (which they will if their determinant is nonzero). In general the currents $i_1' \cdots i_\ell'$ will no longer have the signficance of circulatory currents or loop currents, although for convenience they may still be referred to by that name. They will turn out to be some linear combinations of the branch currents.

If such a very general set of definitions for the loop currents is desired, one can approach the construction of an appropriate tie-set schedule directly from this point of view, which we will illustrate for the network graph of Fig. 8. Thus let us suppose that one wishes to introduce current variables which are the following linear combinations of the branch currents

$$\begin{aligned} i_1 &= -j_1 + j_2 - j_3 + j_4 - 3j_9 \\ i_2 &= j_2 + 2j_3 + j_6 - j_8 \\ i_3 &= j_1 + j_3 + j_5 + j_7 + j_9 \\ i_4 &= j_2 + 2j_4 + j_6 + j_8 . \end{aligned} \qquad (35)$$

The first step is to rewrite these expressions in terms of ℓ (in this case four) branch currents. To do this we may follow the usual scheme of picking a tree and finding the relations for the tree-branch currents in terms of the link currents. For tree No. 1 of Fig. 8, these are given by Eqs. 26. Their use transforms Eqs. 35 into

$$\begin{aligned} i_1 &= 2j_1 + 4j_2 + 2j_3 + 1j_4 \\ i_2 &= 2j_1 + 3j_2 + 3j_3 + 0j_4 \\ i_3 &= 0j_1 - 1j_2 - 1j_3 - 1j_4 \\ i_4 &= 0j_1 + 1j_2 + 1j_3 + 2j_4 , \end{aligned} \qquad (36)$$

having the solutions

$$\begin{aligned} j_1 &= 0\, i_1 + \tfrac{1}{2}\, i_2 + 3\, i_3 + \tfrac{3}{2}\, i_4 \\ j_2 &= \tfrac{1}{2}\, i_1 - \tfrac{1}{2}\, i_2 - \tfrac{3}{2}\, i_3 - 1\, i_4 \\ j_3 &= -\tfrac{1}{2}\, i_1 + \tfrac{1}{2}\, i_2 - \tfrac{1}{2}\, i_3 + 0\, i_4 \\ j_4 &= 0\, i_1 + 0\, i_2 + 1\, i_3 + 1\, i_4 , \end{aligned} \qquad (37)$$

Using Eqs. 26 again we have the additional relations

$$
\begin{aligned}
j_5 &= 0\, i_1 - \tfrac{1}{2}\, i_2 - 2\, i_3 - \tfrac{3}{2}\, i_4 \\
j_6 &= 0\, i_1 + \tfrac{1}{2}\, i_2 + 1\, i_3 + \tfrac{1}{2}\, i_4 \\
j_7 &= \tfrac{1}{2}\, i_1 + 0\, i_2 + \tfrac{3}{2}\, i_3 + \tfrac{1}{2}\, i_4 \\
j_8 &= -\tfrac{1}{2}\, i_1 + 0\, i_2 - \tfrac{3}{2}\, i_3 - \tfrac{1}{2}\, i_4 \\
j_9 &= 0\, i_1 - \tfrac{1}{2}\, i_2 - 1\, i_3 - \tfrac{1}{2}\, i_4
\end{aligned}
\qquad (38)
$$

The results in Eqs. 37 and 38 yield the following tie-set schedule, which more compactly contains this same information

Loop No. \ Branch No.	1	2	3	4	5	6	7	8	9
1	0	1/2	-1/2	0	0	0	1/2	-1/2	0
2	1/2	-1/2	1/2	0	-1/2	1/2	0	0	-1/2
3	3	-3/2	-1/2	1	-2	1	3/2	-3/2	-1
4	3/2	-1	0	1	-3/2	1/2	1/2	-1/2	-1/2

(39)

This is the schedule that is implied by the definitions 35 for the loop-current variables, which no longer possess the geometrical interpretation of being circulatory currents.

As we shall see in the following chapter, the tie-set schedule plays an important role in the formulation of the equilibrium equations appropriate to the chosen definitions for the current variables. The present discussions, therefore, provide the basis for accommodating such a choice regardless of its generality or mode of inception. Thus we have shown that the process of selecting an appropriate set of current variables can take one of essentially three different forms. (1) The approach through choice of a tree and identification of the link currents with loop currents. In this process the algebraic definitions for the loop current variables are as simple as they can be, but one exercises no direct control over the nature of the closed paths or loops. (2) The approach through a forthright choice of loops or tie sets. Here the procedure exercises direct control over the paths upon which the current variables are assumed to circulate (a simple choice being the meshes of a mappable network), but no facile control is had regarding the associated algebraic definitions of the loop currents. (3) The approach through making an initial and arbitrarily general choice for the algebraic definitions of the current variables (like those given by Eqs. 35). In this case the variables no longer possess the simple geometrical significance of circulatory currents. This approach will probably seldom be used, and is given largely for the sake of its theoretical interest.

8. <u>Alternative methods of choosing voltage variables.</u>

When voltages are chosen as variables, we similarly have three possible variations which the form of the approach may take. The first, which is discussed in Art.6, proceeds through choice of a tree and the identification of tree-branch voltages with node-pair voltage variables. In this process (like procedure 1 mentioned above for the choice of current variables), the algebraic definitions for the node-pair voltages are as simple as they can be, but little or no direcy control can be exercised over the geometrical distribution of node pairs. A second form of procedure which permits a forthright choice of node pairs at the outset, and a third in which the process is initiated through an arbitrarily general choice for the algebraic definitiated of the voltage variables, are now presented in detail.

To illustrate how a designation of node-pair voltage variables may be approached through the initial selection of an appropriate set of node-pairs, let us consider the network of Fig. 8. In Fig. 10 are indicated the nodes of this network, lettered a, b, ... f for ease of reference, and a system of lines with arrow heads intended to indicate a choice of node pairs and reference directions for the voltage variables e_1, e_2, ... e_5. These arrows are not to be confused with branches of the network, yet if we momentarily think of them as such, we notice that the structure in Fig. 10 has the characteristics of a tree, for it connects all of the nodes, and involves the smallest number of branches needed to accomplish this end. Hence this choice for the variables $e_1 \cdots e_5$ is an appropriate one since the variables surely form an independent set, and their number equals the number of branches in any tree associated with a network having these nodes. In making a forthright choice of node pairs it is sufficient to see to it that the system of reference arrows accompanying this choice (whether actually drawn or merely implied) forms a structure that has a tree-like character.

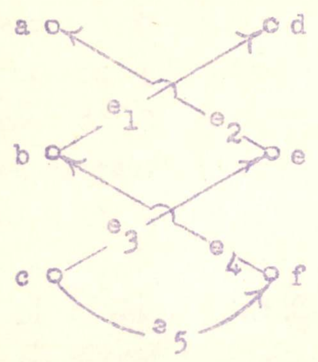


Fig. 10

Using the principles set forth in Art. 6, one can construct the following cut-set schedule appropriate to the choice of node pairs indicated in Fig. 10 for the network graph of Fig. 8

Node pair No. \ Branch No.	1	2	3	4	5	6	7	8	9	Nodes picked up
1	-1	-1	0	0	0	0	0	-1	0	d
2	1	1	0	0	0	0	-1	0	0	a
3	1	1	-1	0	0	0	-1	1	-1	a,e
4	-1	-1	1	0	0	-1	1	-1	0	b,d
5	-1	-1	1	-1	-1	-1	1	-1	1	b,d,f

(40)

In this schedule we have included a last column indicating the nodes picked up in the formation of the respective cut sets. Thus, with reference to Fig. 10, if e_1 is regarded as the only nonzero node-pair voltage, nodes a, b, c, e, f coincide at the tail end of e_1 while node d alone occupies the tip end. Hence cut set No. 1 is found through picking up node d alone and (all other nodes remaining fixed) noting the set of branches in the network of Fig. 8 that are stretched by this procedure. The associated signs are positive or negative according to whether the respective branch reference arrow is divergent from or convergent upon the picked-up node or nodes.

The cut set pertinent to e_2 is found through regarding all other node-pair voltages as being zero so that nodes b, c, d, e, f coincide at the tail end of e_2 while node a marks the tip end. Picking up this node stretches the branches 1, 2, and 7 in the graph of Fig. 8, and the reference arrows on branches 1 and 2 are divergent while that on branch 7 is convergent. Similarly, with e_3 alone nonzero, nodes a, e coincide at the tip end of e_3 while nodes b, c, d, f coincide at the tail end. The picked-up nodes are a and e, and the corresponding cut set is then seen by inspection of Fig. 8. The reader can thus readily check the remaining rows in the cut-set schedule 40.

According to the columns of this schedule we can now immediately write

$$
\begin{aligned}
v_1 &= -e_1 + e_2 + 2 \;\; - e_4 - e_5 \\
v_2 &= -e_1 + e_2 + e_3 - e_4 - e_5 \\
v_3 &= -e_3 + e_4 + e_5 \\
v_4 &= -e_5 \\
v_5 &= -e_5 \\
v_6 &= -e_4 - e_5 \\
v_7 &= -e_2 - e_3 + e_4 + e_5 \\
v_8 &= -e_1 + e_3 - e_4 - e_5 \\
v_9 &= -e_3 + e_5 .
\end{aligned}
\qquad (41)
$$

The correctness of these may readily be checked with reference to Figs. 8 and 10, remembering again that the v's are drops and the e's are rises. For example, v_1 is the voltage drop from node <u>a</u> to node <u>d</u>. If we pass from <u>a</u> to <u>d</u> via the system of node-pair voltage arrows in Fig. 10, we observe that we first traverse the arrows for e_2 and e_3 counterfluently, and then the arrows for e_5, e_4, and e_1 confluently. Since confluence indicates a rise in voltage, the terms for e_1, e_4, and e_5 are negative. There should be no difficulty in thus verifying the remaining equations in the set 41.

One could have written the Eqs. 41 from inspection of Figs. 8 and 10 to start with and thus constructed the schedule 40 by columns, whence the rows would automatically yield the cut sets. This part of the procedure is thus seen to be the same as with the alternate approach given in Art. 6. So is the matter regarding the solution of the Eqs. 41 for the node-pair voltages in terms of the branch voltages. One selects any five independent equations from this group and solves them. Again the selection of a tree in the associated network graph (such a tree No. 1 or No. 2 in Fig. 8) is a quick and sure way to spot an independent subset among the Eqs. 41, and the remaining ones will then yield the appropriate expressions for the link voltages in terms of tree-branch voltages as discussed previously.

In this method of approach to the problem of defining an appropriate set of independent voltage variables, a rather common procedure is to choose the potential of one arbitrarily selected node as a reference, and designate as variables the potentials $e_1 \cdots e_n$ of the remaining nodes with respect to this reference. Thus, one node serves as a datum or reference, and the node pairs defining the variables $e_1 \cdots e_n$ all have this datum node in common. The quantities $e_1 \cdots e_n$ in this arrangement are spoken of as node-potentials and are referred to as a "node-to-datum" set of voltage variables.

The rather simplified choice of node pairs implied in this specialized procedure is in a sense the parallel of choosing meshes for loops in the specification of current variables. This theme is elaborated upon in Art. 9 where the dual character of the loop and node procedures is stressed and the implications of this duality are partially evaluated.

The equivalent of Fig. 10 for a choice of node-pair voltages of this sort is shown in Fig. 11, pertinent to the network graph of Fig. 8. Again for the moment regarding the arrows in this diagram as branches, we see that it has tree-like character and hence that such a node-to-datum set of voltages is always independent.

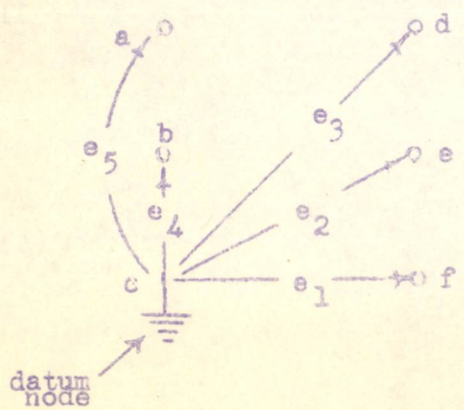


Fig. 11

The cut sets appropriate to this group of node-pair voltages are particularly easy to find since we observe that setting all but one of the node-pair voltages equal to zero causes all of the nodes to coincide at the datum except the one at the tip end of the nonzero voltage. Hence the branches divergent from this single node form the pertinent cut set. With reference to Fig. 8, the following cut-set schedule is thus readily obtained

Node No. \ Branch No.	1	2	3	4	5	6	7	8	9	Picked-up nodes
1	0	0	0	-1	-1	0	0	0	1	f
2	0	0	-1	0	0	0	0	1	-1	e
3	-1	-1	0	0	0	0	0	-1	0	d
4	0	0	1	0	0	-1	1	0	0	b
5	1	1	0	0	0	0	-1	0	0	a

(42)

Since the node-pair voltages are the potentials of the separate nodes with respect to a common datum, each branch voltage drop is given by the difference of two node potentials, namely those associated with the nodes terminating the pertinent branch. If the latter touches the datum node, then its voltage drop is given by a single node potential (with proper algebraic sign). These observations are borne out by the fact that all of the columns in the schedule 42 have either two or one nonzero elements; those with a single nonzero element pertain to branches touching the datum node. The branch voltages in terms of the node potentials are thus formed either by inspection of Figs. 8 and 11 or from the columns of schedule 42 to be

$$
\begin{aligned}
v_1 &= -e_3 + e_5 & \qquad v_5 &= -e_1 \\
v_2 &= -e_3 + e_5 & v_6 &= -e_4 \\
v_3 &= -e_2 + e_4 & v_7 &= e_4 - e_5 \\
v_4 &= -e_1 & v_8 &= e_2 - e_3 \\
& & v_9 &= e_1 - e_2 .
\end{aligned}
\qquad (43)
$$

The node potentials in terms of the branch voltages are found from these by the usual process of selecting from these equations a subset of five independent ones. According to tree No. 1 of Fig. 8, the last five are such a subset. Their solution yields

$$\begin{aligned} e_1 &= -v_5 \\ e_2 &= -v_5 - v_9 \\ e_3 &= -v_5 - v_8 - v_9 \\ e_4 &= -v_6 \\ e_5 &= -v_6 - v_7, \end{aligned} \tag{44}$$

and the remaining equations in the set 43 then give the following expressions for the link voltages in terms of the tree-branch voltages

$$\begin{aligned} v_1 &= v_5 - v_6 - v_7 + v_8 + v_9 \\ v_2 &= v_5 - v_6 - v_7 + v_8 + v_9 \\ v_3 &= v_5 - v_6 + v_9 \\ v_4 &= v_5. \end{aligned} \tag{45}$$

It is interesting to observe how more general node-pair voltage definitions are derivable from the simple node-to-datum set through carrying out linear transformations on the rows of the cut-set schedule 42. Thus, suppose we form from this one a new schedule through adding the elements of the second row of 42 to the respective ones of the first row, giving

Node-pair No. \ Branch No.	1	2	3	4	5	6	7	8	9	Picked-up nodes
1	0	0	-1	-1	-1	0	0	1	0	e,f
2	0	0	-1	0	0	0	0	1	-1	e
3	-1	-1	0	0	0	0	0	-1	0	d
4	0	0	1	0	0	-1	1	0	0	b
5	1	1	0	0	0	0	-1	0	0	a

(46)

The columns yield

$$
\begin{aligned}
v_1 &= -e_3' + e_5' \\
v_2 &= -e_3' + e_5' \\
v_3 &= -e_1' - e_2' + e_4' \\
v_4 &= -e_1' \\
v_5 &= -e_1' \\
v_6 &= -e_4' \\
v_7 &= e_4' - e_5' \\
v_8 &= e_1' + e_2' - e_3' \\
v_9 &= -e_2'.
\end{aligned} \tag{47}
$$

Solution of the last five now reads

$$
\begin{aligned}
e_1' &= -v_5 \\
e_2' &= -v_9 \\
e_3' &= -v_5 - v_8 - v_9 \\
e_4' &= -v_6 \\
e_5' &= -v_6 - v_7,
\end{aligned} \tag{48}
$$

and comparison with Eqs. 44 reveals that

$$
\begin{aligned}
e_1 &= e_1' \\
e_2 &= e_1' + e_2' \\
e_3 &= e_3' \\
e_4 &= e_4' \\
e_5 &= e_5'.
\end{aligned} \tag{49}
$$

This result suggests that the node-pair voltage diagram has changed from the form shown in Fig. 11 to that shown in Fig. 12 since the potential of node e with respect to the datum (which in Fig. 11 is e_2) now is equal to the sum of e_1' and e_2'. We note further that when e_1' is the only nonzero voltage, nodes e and f coincide at the tip end of e_1'; so the

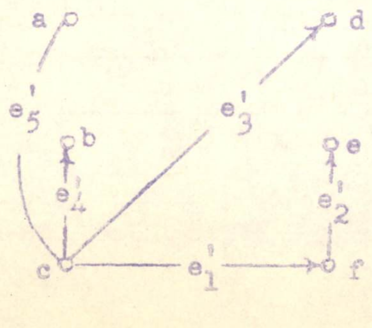


Fig. 12

associated cut set is found through picking up these two nodes, as is also indicated in the schedule 46. The picked-up nodes corresponding to the remaining node-pair voltages evidently remain the same as before, and hence the rest of the cut sets are unchanged.

Other simple transformations in the schedule 46 may similarly be interpreted. For example, if row 3 is added to row 4, the picked-up nodes for cut set No. 4 become d and b, which in Fig. 12 implies that the tail end of e_3' shifts from the datum to node b, and we will find that now $e_3 = e_3'' + e_4''$ where the double prime refers to the latest revision of the set of node-pair voltages (the rest of the e's remain as in Eqs. 49 with double primes on the right-hand quantities).

One soon discovers, upon carrying out additional row combinations in the schedules 42 or 46 that it is by no means always possible to associate a node-pair voltage diagram like the ones in Figs. 10, 11, or 12 with the resulting node-pair voltages for the reason that some of these are likely no longer to be simply potential differences between node pairs but instead are more general linear combinations of the branch voltages.

The same is true if one constructs a cut-set schedule (as is also a possible procedure) through making arbitrary choices for the picked-up nodes. To illustrate such a method we may consider again the graph of Fig. 8 which is redrawn in Fig. 13 with the nodes lettered as in Figs. 10, 11, and 12.

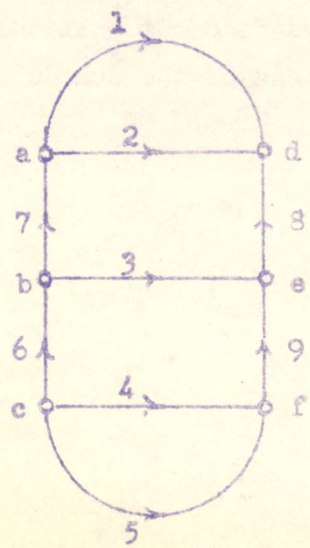


Fig. 13

The following cut set schedule is constructed by simply making an arbitrary choice for the picked-up nodes relating to the pertinent cut sets

Node-pair No. \ Branch No.	1	2	3	4	5	6	7	8	9	Picked-up nodes
1	0	0	0	0	0	0	-1	-1	0	a,d
2	0	0	0	0	0	1	0	0	1	c,f
3	1	1	1	1	1	0	0	0	0	a,b,c
4	-1	-1	-1	0	0	0	0	0	-1	d,e
5	0	0	-1	-1	-1	0	0	1	0	e,f

(50)

The term "node pair" here retains only a nominal significance since we are not at all assured that the implied voltage variables are potential differences between node pairs. We are at this stage in the procedure not even assured that the implied voltage variables form an independent set. What we have done is to choose cut sets in an altogether random manner, which is analogous to choosing closed loops in a random manner when approaching the problem on a current basis. We can readily check whether the implied voltage variables are an independent set and find out what they are by writing down the relations specified by the columns of the schedule 50, thus

$$
\begin{aligned}
v_1 &= e_3 - e_4 \\
v_2 &= e_3 - e_4 \\
v_3 &= e_3 - e_4 - e_5 \\
v_4 &= e_3 - e_5 \\
v_5 &= e_3 - e_5 \\
v_6 &= e_2 \\
v_7 &= -e_1 \\
v_8 &= -e_1 + e_5 \\
v_9 &= e_2 - e_4.
\end{aligned}
\qquad (51)
$$

That such a set of equations exist is manifest since the schedule 50 is obtainable through a linear transformation of the rows in schedule 42, and the property of the columns to yield the branch voltages in terms of the implied node-pair voltages is not lost through such a transformation. Hence, if the Eqs. 51 possess unique solutions for the voltage variables $e_1 \cdots e_5$, these exist, and the cut-set schedule 50 is appropriate to them. The Eqs. 51 do possess unique solutions if there exists among them an independent subset of five equations appropriate to any tree of the graph of Fig. 13 or 8. Picking tree No. 1 in Fig. 8 designates v_5, v_6, v_7, v_8, v_9 as independent tree-branch voltages and hence stipulates that the last five equations in the set 51 should be independent. It is readily seen that they are, for they yield the solutions

$$
\begin{aligned}
e_1 &= -v_7 \\
e_2 &= v_6 \\
e_3 &= v_5 - v_7 + v_8 \\
e_4 &= v_6 - v_9 \\
e_5 &= -v_7 + v_8 .
\end{aligned}
\qquad (52)
$$

We may conclude that the cut sets in the schedule 50 are independent, and Eqs. 52 tell us what the implied voltage variables are in terms of the branch voltages. The first two are simple potential differences between nodes, but the remaining three are not. There is no reason why the selected voltage variables have to be potential differences between nodes. So long as they form an independent set, and we know the algebraic relations between them and the branch voltages, they are appropriate.

Lastly let us consider for the same network of Fig. 8 the following set of independent linear combinations of the branch voltages as a starting point

$$
\begin{aligned}
e_1 &= -v_1 + v_3 + v_4 + 2v_6 + 2v_7 + 5v_8 + 5v_9 \\
e_2 &= -v_2 + v_3 + v_4 - v_5 + v_6 + v_7 + 4v_8 + 4v_9 \\
e_3 &= -v_2 + 2v_3 - v_5 + v_6 + 3v_8 + 2v_9 \\
e_4 &= -v_1 + 2v_3 - v_5 + v_6 - v_7 + 2v_8 + v_9 \\
e_5 &= v_3 - v_5 + v_6 .
\end{aligned}
\qquad (53)
$$

Through use of Eqs. 45 one can eliminate all but five of the branch voltages and get the definitions 53 into the form

$$\begin{aligned} e_1 &= v_5 + 2v_6 + 3v_7 + 4v_8 + 5v_9 \\ e_2 &= v_6 + 2v_7 + 3v_8 + 4v_9 \\ e_3 &= v_7 + 2v_8 + 3v_9 \\ e_4 &= v_8 + 2v_9 \\ e_5 &= v_9. \end{aligned} \tag{54}$$

The solutions to these equations together with Eqs. 45 yield the complete set of relations for the branch voltages in terms of $e_1 \cdots e_5$, thus

$$\begin{aligned} v_1 &= e_1 - 3e_2 + 2e_3 - e_5 \\ v_2 &= e_1 - 3e_2 + 2e_3 - e_5 \\ v_3 &= e_1 - 3e_2 + 3e_3 - e_4 + e_5 \\ v_4 &= e_1 - 2e_2 + e_3 \\ v_5 &= e_1 - 2e_2 + e_3 \\ v_6 &= e_2 - 2e_3 + e_4 \\ v_7 &= e_3 - 2e_4 + e_5 \\ v_8 &= e_4 - 2e_5 \\ v_9 &= e_5. \end{aligned} \tag{55}$$

These results may be summarized in the cut set schedule

Node pair No. \ Branch No.	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	0	0	0	0
2	-3	-3	-3	-2	-2	1	0	0	0
3	2	2	3	1	1	-2	1	0	0
4	0	0	-1	0	0	1	-2	1	0
5	-1	-1	1	0	0	0	1	-2	1

(56)

Thus we see that any stated algebraic definitions for the voltage variables can be accommodated, although their usual simple geometrical interpretation is no longer applicable. The independence of the definitions 53, incidentally, is verified through noting that Eqs, 54 possess solutions.

9. Duality.

In the foregoing discussions regarding the appropriate selection of current or voltage variables, the reader will undoubtedly have noticed a number of similarities and analogies between the procedures pertinent to these two methods of approach. We wish now to call specific attention to this aspect of our problem so that we may gain the circumspection that later will enable us to make effective use of its implications. In a word, this usefulness stems from the fact that two situations which, on a current and voltage basis respectively, are entirely analogous, have identical behavior patterns except for an interchange of the roles played by voltage and current, while physically and geometrically they are distinctly different. Not only can one recognize an obvious economy in computational effort resulting from this fact since the analysis of only one of two networks so related yields the behavior of both, but one can sense as well that an understanding of these ideas may lead to other important and practically useful applications, as indeed the later discussions of our subject substantiate.

A careful review of the previous articles in this chapter shows that essentially the same sequence of ideas and procedures characterize both the loop and node methods, but with an interchange in pairs of the principal quantities and concepts involved. Since the latter are thus revealed to play a dual role, they are referred to as dual quantities and concepts. First among such dual quantities are current and voltage; and first among the dual concepts involved are meshes and nodes or loops and node-pairs. Since a zero current implies an open circuit and a zero voltage a short circuit, these two physical constraints are seen to be duals. The identification of loop currents with link currents and of node-pair voltages with tree-branch voltages shows that the links and the tree branches likewise are dual quantities. The following table gives a more complete list of such pairs.

Dual quantities or concepts

current	voltage
branch current	branch voltage
mesh or loop	node or node pair
number of loops (l)	number of node pairs (n)
loop current	node-pair voltage
mesh current	node potential
link	tree branch
link current	tree-branch voltage
tree-branch current	link voltage
tie set	cut set
short circuit	open circuit
parallel paths	series paths

It should be emphasized that duality is strictly a mutual relationship. There is no reason why any pair of quantities in the above table cannot be interchanged, although each column as written associates those quantities and concepts that are pertinent to one of the two procedures commonly referred to as the loop and node methods of analysis.

Two network graphs are said to be duals if the characterization of one on the loop basis leads to results identical in form to those obtained for the characterization of the other on the node basis. Both graphs will have the same number of branches, but the number of tree branches in one equals the number of links in the other; or the number of independent node pairs in one equals the number of independent loops in the other. More specifically, the equations relating the branch currents and loop currents for one network are identical in form to the equations relating the branch voltages and the node-pair voltages for the other, so that these sets of equations become interchanged if the letters i and j are replaced respectively by e and v, and vice versa. For appropriately chosen elements in the branches of the associated dual networks, the electrical behavior of one of these is obtained from that of the other simply through an interchange in the identities of voltage and current.

Apart from the usefulness that will be had from later applications of these ideas, a detailed consideration of the underlying principles is advantageous at this time because of their correlative value with respect to the foregoing discussions of this chapter.

Geometrically, two graphs are dual if the relationship between branches and node pairs in one is identical with the relationship between branches and loops in the other. The detailed aspects involved in such a mutual relationship are best seen from actual examples. To this end, consider the pair of graphs in Fig. 14. Suppose the one in part (a) is given and we are

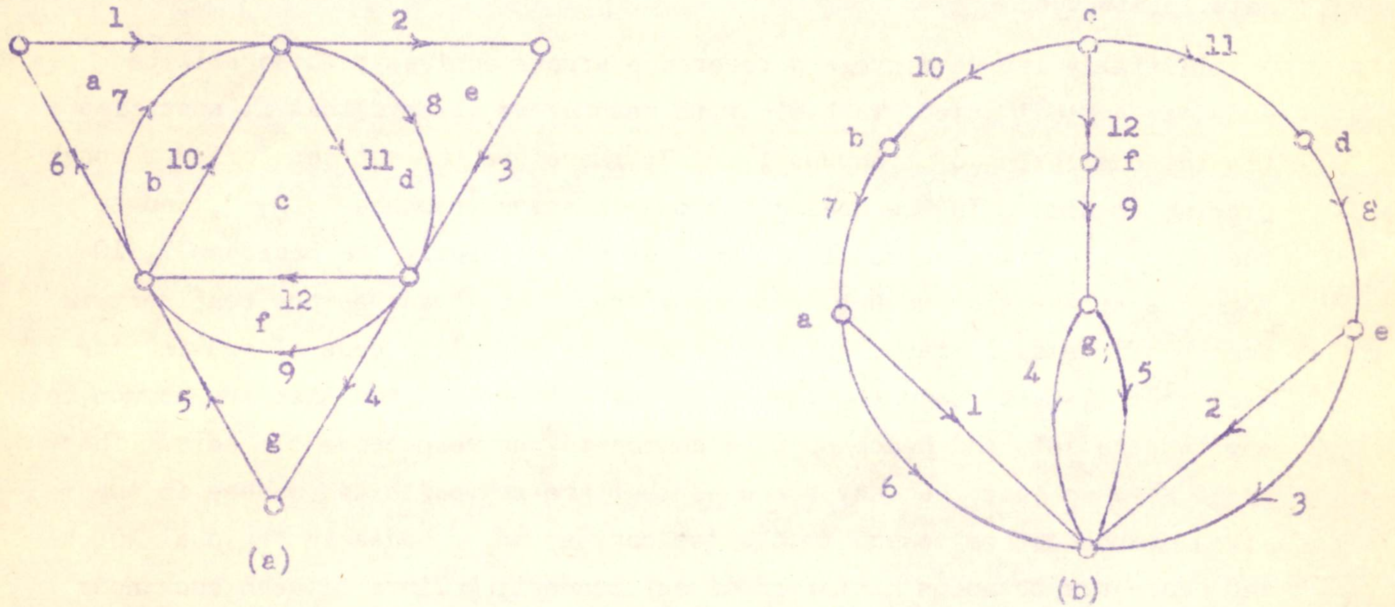


Fig. 14

to construct its dual as shown in part (b). At the outset we observe that the graph of part (a) has seven meshes and five independent node pairs (a total of six nodes). Hence the dual graph must have seven independent node pairs (a total of eight nodes) and five meshes. The total number of branches must be the same in both graphs.

In proceeding with the construction of the dual of (a), one may begin by setting down eight small circles as nodes--one for each mesh in the graph of part (a), and an extra one that can play the part of a datum node if we wish to regard it as such, although any or none of the eight nodes needs to be considered in this light. We next assign each of these seven nodes to one of the seven meshes in the given graph, as is indicated in Fig. 14 through the letters a, b, ... g. The procedure so far implies that we are considering as tie sets those confluent branches in the graph (a) that form the contours of meshes, and as cut sets, those branches in the dual graph that are stretched in the process of picking up single nodes. At least, this implication is true of the nodes a, ... g that are assigned to specific meshes; the cut set pertaining to the remaining unassigned node will correspond to a tie set in the graph (a) that will reveal itself as we now proceed to carry out the process of making all tie sets in the given graph identical to all the cut sets in its dual.

Initially let us disregard reference arrows entirely; these will be added as a final step. To begin with mesh a, we observe that it specifies a tie set consisting of branches 1, 6, 7; therefore the cut set formed through picking up node a in the dual graph must involve branches 1, 6, 7, and so these are the branches confluent in node a. Similarly the branches 7, 10 form the tie set for mesh b, and therefore, these branches are confluent in node b of the dual graph; and so forth. The actual process of drawing the dual graph is best begun by inserting only those branches that are common to any two tie sets and hence must be common to the respective cut sets. That is to say, we note that any branches that are common to two meshes in the given graph must be common to the two corresponding nodes in the dual graph and hence are branches that form direct connecting links between such node pairs. For example, branch 7 is common to meshes a and b and hence branch 7 in the dual graph connects nodes a and b; similarly branch 10 links nodes b and c; branch 11 links nodes c and d, and so forth.

In this way we readily insert branches 7, 10, 11, 8, 12, 9, and then note that the remaining branches 1, 2, 3, 4, 5, 6 in the original graph form a tie set that must be identical with the cut set of the dual graph that is associated with the remaining unassigned node. Hence these branches, which have one terminus in an assigned node, are the ones that must be confluent in the remaining node. The latter is thus seen to be assignable to the loop formed by the periphery of the given graph. In a sense we may regard this periphery as a "reference loop" corresponding to the originally unassigned node playing the role of a "reference node," although the following discussion will show that this view is a rather specialized one and need not be considered unless it seems desirable to do so.

Now as to reference arrows on the branches of the dual graph we note, for example, that the traversal of mesh a in a clockwise direction is confluent with the reference arrow of branches 1 and 6, and counterfluent with the reference arrow of branch 7. Hence on the dual graph we attach reference arrows to branches 1 and 6 that are divergent from node a, and provide branch 7 with an arrow that is convergent upon this node. That is to say, we correlate clockwise traversal of the meshes with divergence from the respective nodes, and then assign branch arrows in the dual graph that agree or disagree with this direction according to whether the corresponding branch arrows in the given graph agree or disagree with the clockwise direction for each corresponding mesh. We could, of course, choose a consistent counterclockwise traversal of the meshes, or in the dual graph choose convergence as a corresponding direction. Such a switch will merely reverse all reference arrows in the dual graph (which we can do anyway), but we must in any case be consistent and stick to the same chosen convention throughout the process of assigning branch reference arrows. This is done in the construction of the graph of Fig. 7 (b), as the reader may readily verify by inspection.

Being mindful of the fact that duality is in all respects a mutual relationship, we now expect to find that the graph (a) of Fig. 7 is related to the graph (b) in the same detailed manner that (b), through the process of construction just described, is related to (a). Thus we expect the meshes of (b) to correspond to nodes in (a) as do the meshes of (a) to the nodes in (b). However, we find upon inspection that such is not consistently the case. For example, the mesh in graph (b) having its contour formed by the consecutively traversed branches 1, 7, 10, 12, 9, 4 corresponds in graph (a) not to a single node but instead is seen to be the dual of the group of three nodes situated at the vertexes of the triangle formed by the branches 2, 3, 11, since the act of simultaneously picking up these nodes reveals the same group of branches 1, 7, 10, 12, 9, 4 in graph (a) to be a cut set.

This apparent inconsistency is easily resolved through consideration of a slight variation in the construction of the dual of graph (a) as shown in Fig. 15. Here all meshes correspond to the nodes of graph (a) in Fig. 14 in the same way that the meshes of graph (a) correspond to nodes in the graph of Fig. 15, as the reader should carefully verify. The additional principle observed in the construction of the graph of Fig. 15 is that the sequence of branches about any node is chosen to be identical to that of the similarly numbered branches around the respective mesh, assuming a consistent clockwise (or counterclockwise) direction of circuitation around meshes and around nodes. For example, the branches taken in clockwise order around mesh a of the graph of Fig. 14 (a) are numbers 1, 7, 6; around node a in the graph of Fig. 15 this sequence of branches corresponds to counterclockwise rotation. Correspondingly the clockwise sequence of branches around mesh c in Fig. 14 (a) is 10, 11, 12, and this is the counterclockwise sequence of the corresponding branches around node c in Fig. 15. This correspondence in the sequence of branches is seen to hold for all meshes and their corresponding nodes not only between meshes in Fig. 14 (a) and nodes in Fig. 15 but also between the meshes in Fig. 15 and and their corresponding nodes in Fig. 14 (a). The duality between these two graphs is indeed complete in every respect.*

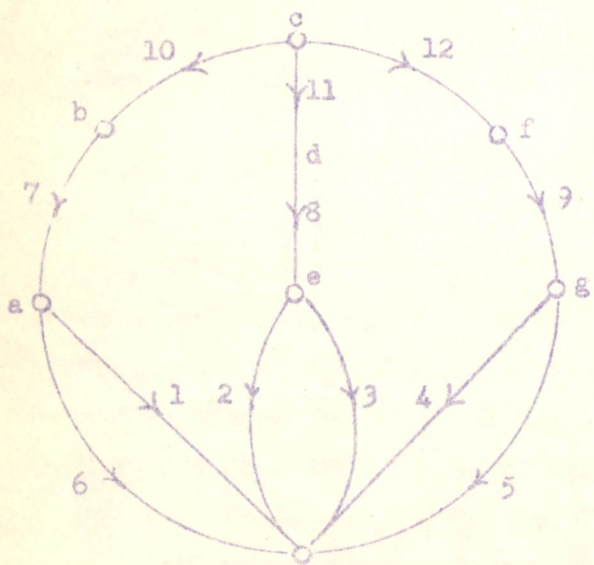


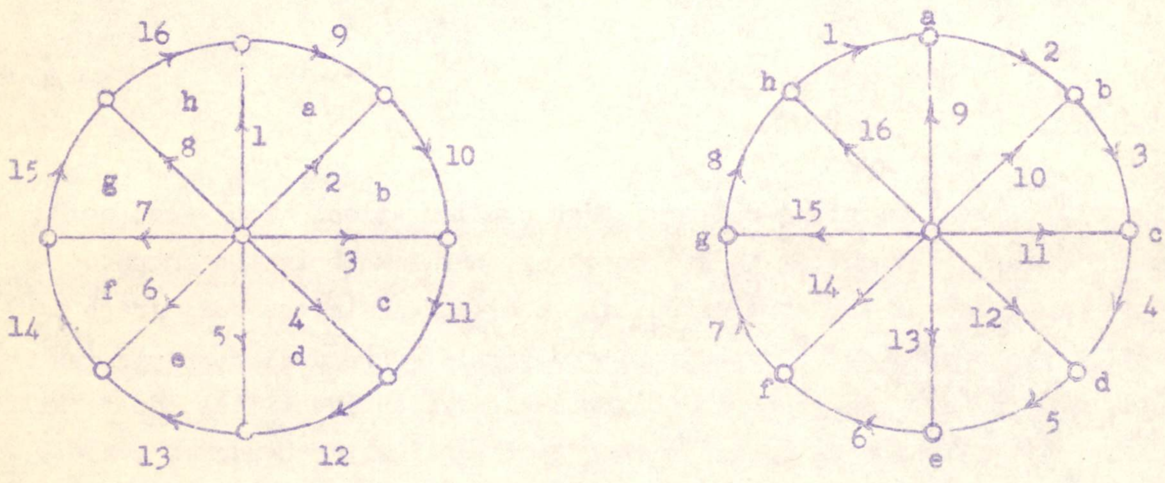
Fig. 15

So far as the relationships between branch currents and loop currents or between branch voltages and node-pair voltages are concerned, however, these must be the same for the graph of Fig. 14 (b) as they are for the graph of

*The correlation of clockwise rotation in one graph with counterclockwise rotation in its dual is an arbitrary choice. One can as well choose clockwise rotation in both, the significant point being that a consistent pattern is adhered to.

Fig. 15, since both involve fundamentally the same geometrical relationship between nodes and branches, as a comparison readily reveals. For this reason it is not essential in the construction of a dual graph to preserve branch number sequences around meshes and nodes as just described unless one wishes for some other reason to make meshes in the dual graph again correspond to single nodes in the original graph. From the standpoint of their electrical behavior, the networks whose graphs are given by Figs. 14 (b) and 15 are entirely identical. These graphs are, therefore, referred to as being <u>topologically* equivalent</u>, and either one may be regarded as the dual of Fig. 14 (a), or the latter as the dual of either of the networks of Figs. 14 (b) and 15.

An additional interesting example of dual graphs is shown in Fig. 16. The meshes a, b, c, ... in the graph of part (a) correspond to similarly lettered nodes in the graph of part (b); and conversely, the meshes in graph (b)



(a) Fig. 16 (b)

correspond to nodes in part (a). It will also be observed that the sequences of branches around meshes and around corresponding nodes agree; and it is interesting to note in this special case that although both graphs have the form of a wheel, the spokes in one are the rim segments of the other. It is further

*The mathematical subject dealing with the properties of linear graphs is known as <u>topology</u>.

useful to recognize that these graphs may be redrawn as shown in Fig. 17

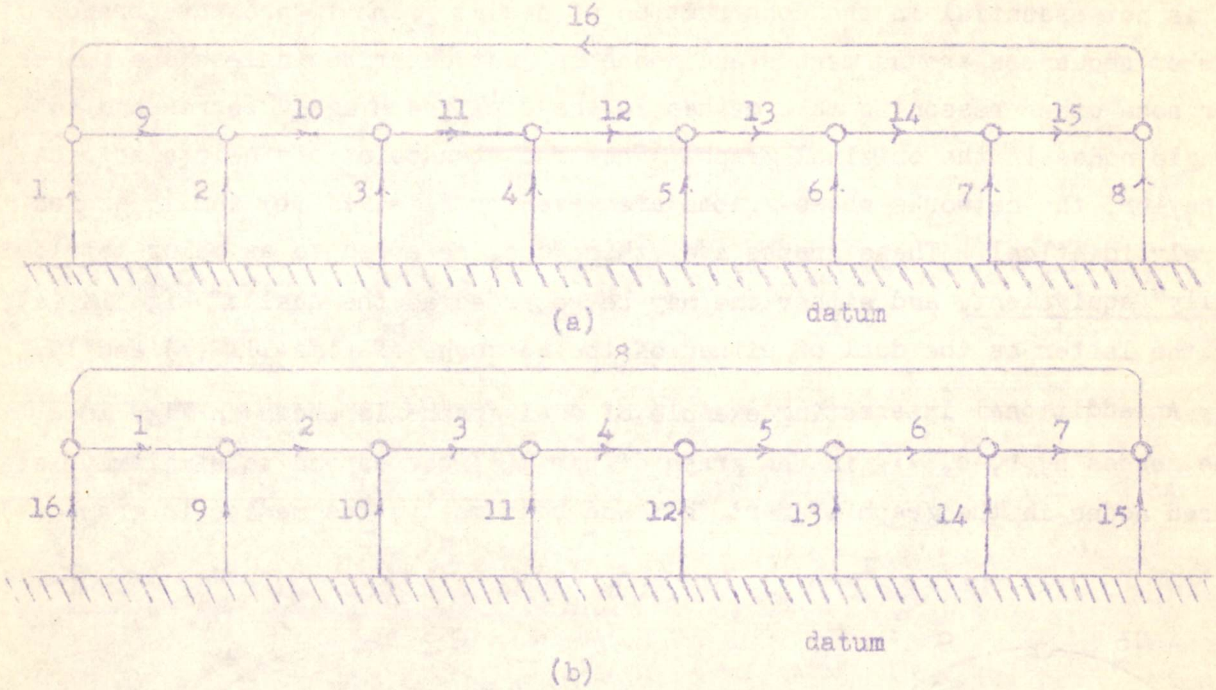


Fig. 17

where they take the form of so-called <u>ladder</u> configurations with "feedback" between their input and output ends. Removing the link 16 in the graph of Fig. 16 (a) corresponds to short-circuiting the link 16 in the dual graph of part (b), since open and short-circuit constraints are dual concepts (as previously mentioned). In graph 17 (b) this alteration identifies the first node on the left with the datum, thus in effect paralleling branches 1 and 9 at the left and branches 8 and 15 on the right. Such ladder configurations are much used in practice and it is, therefore, well to know that the dual of a ladder is again a ladder with the essential difference that its series branches correspond to shunt branches in the given ladder, and vice versa.

It is helpful, in the process of constructing a dual graph, to visualize the given one as mapped upon the surface of a sphere instead of on a plane. If this is done, then the periphery appears as an ordinary mesh when viewed from the opposite side of the sphere. For example, if the graph of Fig. 16 (a) is imagined to consist of an elastic net and is stretched over the surface of a sphere until the periphery contracts upon the opposite hemisphere, and if one now views the sphere from the opposite side so as to look directly at this hemisphere, then the periphery no longer appears to be fundamentally different in character from an ordinary mesh, for it now appears as a simple opening in the net, like all the other meshes. Thus the branches 9, 10, 11, 12, 13, 14, 15, 16 forming the contour of this mesh appear more logically to correspond to the similarly numbered group of branches in the dual graph 16 (b) emanating from the central node which, like all the other nodes, now corresponds to a simple mesh in the given graph.

When, in the choice of network variables, one identifies loop currents with link currents and node-pair voltages with tree-branch voltages, it will be recalled that each tie set consists of one link and a number of tree branches, while each cut set consists of one tree branch and a number of links. Since the tie sets of a given graph correspond to cut sets in the dual graph, one recognizes that the tree branches in one of these graphs are links in the other. That is to say, corresponding trees in dual graphs involve complementary sets of branches. In Fig. 16, for example, if one chooses the branches 1, 2, 3, 4, 5, 6, 7, 8 in graph (a) as forming a tree, then the corresponding tree in graph (b) is formed by the branches 9, 10, 11, 12, 13, 14, 15, 16. Or, if in graph (a) we choose branches 1, 2, 3, 4, 12, 13, 14, 15 as forming a tree, then in graph (b) the corresponding tree is formed by branches 5, 6, 7, 8, 9, 10, 11, 16.

It should now be clear, according to the discussion in the preceding articles, that if in a given graph we pick a tree and choose the complementary set of branches as forming a tree in the dual graph, then the resulting equations between branch currents and loop currents in one of these graphs becomes identical (except for a replacement of the letters j and i respectively by v and e) with those relating branch voltages and node-pair voltages in the dual graph. In the graphs of Fig. 16, for example, we may choose branches 1 to 8 inclusive as the tree of graph (a) and branches 9 to 16 inclusive as the tree of graph (b). Then in graph (a), the branch currents j_9, j_{10}, ... j_{16}

are respectively identified with loop currents $i_1, i_2, \cdots i_8$, while in graph (b) the branch voltages $v_9, v_{10}, \cdots v_{16}$ are respectively identified with node-pair voltages $e_1, e_2, \cdots e_8$. For the tree-branch currents in graph (a) we then have, for example, $j_2 = -i_1 + i_2 = -j_9 + j_{10}$; $j_3 = -i_2 + i_3 = -j_{10} + j_{11}$, etc.; while for the link voltages in graph (b) we have correspondingly $v_2 = -v_9 + v_{10} = -e_1 + e_2$; $v_3 = -v_{10} + v_{11} = -e_2 + e_3$, etc. The reader may complete these equations as an exercise, and repeat the process for several other trees as well as for the graphs of Fig. 14.

It should likewise be clear that similar results for a pair of dual graphs and their current and voltage variables are obtained if for one graph one chooses meshes as loops and in the other the corresponding nodes as a node-to-datum set of node pairs. In this case it may be desirable to regard the unassigned node as a datum and the corresponding peripheral mesh as playing the role of a datum mesh. Since more general choices of loops or of node pairs may be expressed as linear combinations of these simple ones, it is seen that the parallelism between the current and voltage relations of dual networks holds in all cases regardless of the approach taken in formulating defining relations for network variables.

It is important, however, to note a restriction with regard to the existence of a dual graph. This restriction may most easily be understood through recognizing that all possible choices of tie sets in a given network must correspond to cut sets in its dual, and vice versa. In this connection, visualize the given graph as some net covering the surface of a sphere, and a tie set as any confluent group of branches forming a closed path. As mentioned at the close of Art. 5, let us think of inserting a draw string along this path and then tying off, as we might if the sphere were an inflated balloon. We would thus virtually create two balloons, fastened one to the other only at a single point where the contracted tie set has become a common node for the two subgraphs formed by the nets covering these balloons. Whether we thus regard the tie set as contracted or left in its original form upon the sphere, its primary characteristic so far as the present argument is concerned lies in the fact that it forms a boundary along which the given network is divided into two parts, and correspondingly the totality of meshes is divided into two groups.

In the dual graph these correspond to two groups of nodes. If we think of grasping one of these node groups in each of our two hands and pulling them apart, the stretched branches place in evidence the cut set corresponding to the tie set of the original graph. The act of cutting this set of branches is dual to the tying off process described above, since by this means the dual graph is separated into two parts which are respectively dual to the two subgraphs created by contracting the tie set.

Duality between the original graph and its dual demands that to every creatable cut set in one of these there must correspond in the other a tie set with the property just described. It should be clear that this requirement cannot be met if either network is not mappable upon a sphere but requires the surface of some multiply connected space like that occupied by a doughnut or a pretzel. For example, if the mapping of a graph requires the surface of a doughnut, then it is clear that a closed path passing through the hole is not a tie set because the doughnut is not separated into two parts through the contraction of this path. The surface of a simply connected region like that of a sphere, is the only surface on which <u>all</u> closed paths are tie sets. There is obviously no corresponding restriction on the existence of cut sets, since we can visualize grasping complementary groups of nodes in our two hands and, through cutting the stretched branches, separating the graph into two parts regardless of whether the geometry permits its being mapped upon a sphere or not.

Thus, mappability upon a sphere is revealed as a necessary condition that a tie set in the original graph shall correspond to every possible cut set in its dual, and hence the latter is contructible only if the graph of the given network is so mappable.

10. Concluding remarks.

As expressed in the opening paragraphs of the previous article, the object in discussing the subject of duality is twofold. First, duality is a means for recognizing the analytical equivalence of pairs of physically dissimilar networks; so far as mappable networks are concerned, it essentially reduces by a factor of two the totality of distinct network configurations that can occur. Second, and no less useful, is the result that the principle of duality gives us two geometrically different ways of interpreting a given situation; if one of these proves difficult to comprehend, the other frequently turns out to be far simpler. This characteristic of the two geometrical interpretations of dual situations to reinforce the mental process of comprehending the significance of either one, we wish now to present through a few typical examples.

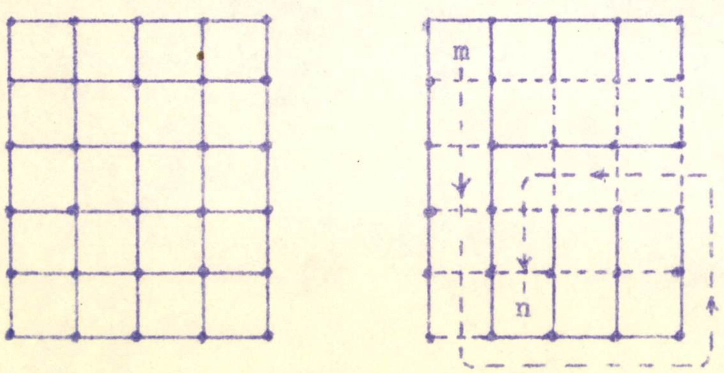
Suppose, for a given mappable graph, we consider a node-to-datum set of voltage variables. That is to say, we pick a datum node, and choose as variables the potentials of the remaining nodes with respect to this datum. If we now wish to obtain algebraic expressions for these node voltages in terms of a like number of independent branch voltages, the simplest procedure is to select a tree and recognize that each node potential is then uniquely given by an algebraic sum of tree-branch voltages, since the path from any node to the datum via tree branches is a unique one. The geometrical picture involved and the pertinent algebraic procedure are simple and easily comprehensible.

Contrast with this the completely dual situation. For a given mappable graph, we consider the mesh currents as a set of appropriate variables, and ask for the algebraic expressions for these in terms of a like number of independent branch currents. Since the latter may be regarded as the currents in a set of links associated with a chosen tree, the initial step in the procedure is clearly the same as in the previous situation. At this point, however, the lucidity of the picture is suddenly lacking, for we do not appear to have a procedure for expressing each mesh current as an algebraic sum of link currents that has a geometric clarity and straightforwardness comparable to the process of expressing node potentials in terms of tree-branch voltages, and yet we feel certain that there must exist a picture of equivalent clarity since, to every mappable situation there exists a dual which possesses all of the same features and with the same degree of lucidity. Our failure to find the mesh situation as lucid as the one involving node potentials must be due to our inability to construct in our minds the completely dual geometry. Once we achieve the latter, our initial objective will easily be gained and our understanding of network geometry will correspondingly be enhanced.

It turns out that our failure to recognize the dual geometry stems from an initial misconception of what is meant by a mesh. Since we use the term mesh to connote a particular kind of loop, namely the simplest closed path that one can trace, we establish in our minds the view that the term mesh refers to the contour (the associated tie set) instead of the thing that it should refer to, namely the space surrounded by that contour! A mesh is an opening--not the boundary of that opening. This opening is the dual of a node--the point of confluence of branches.

A tree consists of nodes connected by tree branches. The dual of a tree branch is a link. Therefore the dual of a tree should be something that consists of spaces (meshes) connected by links. If we add to the mental picture created by these thoughts the fact that traversing a branch longitudinally and crossing it at right angles are geometrically dual operations (since a branch voltage is found through a longitudinal summation process while a branch current is given by a summation over the cross section), we arrive without further difficulty at the geometrical entity that must be recognized as the dual of a tree. It is the space surrounding the tree.

This space is subdivided into sections by the links. Each of these sections is a mesh; and one passes from mesh to mesh by crossing the links, just as in the tree one passes from node to node by following along the tree branches. Fig. 18 shows in part (a) a graph in the form of a rectangular grid, and in part (b) a possible tree with the links included as dotted lines. The space surrounding the tree, and dual to it, is best described by the word maze as used to denote a familiar kind of picture puzzle where one is asked to trace a continuous path from one point in this



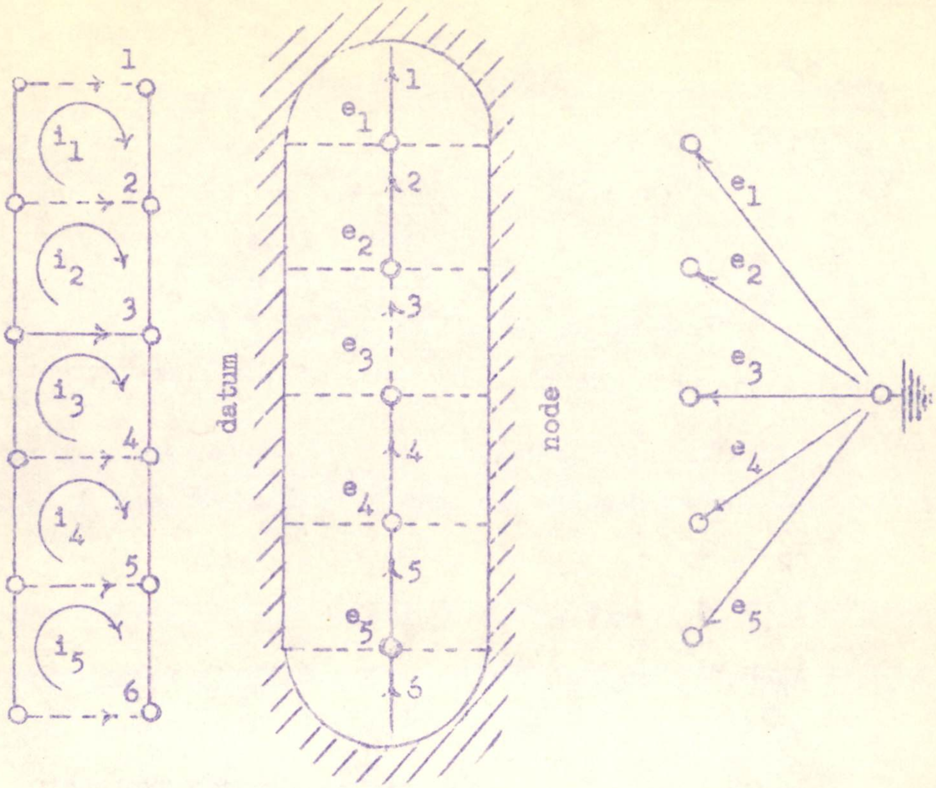
(a) (b)

Fig. 18

space to another without crossing any of the barriers formed by the tree-like structure.

Such a path connecting meshes m and n is shown dotted in part (b) of the figure. It is clear that the path leading from one mesh to any other, is unique, just as is the path from one node to another along the tree branches. In passing along a path such as the one leading from mesh m to mesh n, one crosses a particular set of links. These links characterize this path just as a set of confluent tree branches characterize the path from one node to another in a given tree.

Having recognized those dual processes, we now realize that we have not been entirely accurate in the foregoing discussions where we refer to a loop current as being dual to a node-pair voltage. The latter is the difference between two node potentials, and its dual is, therefore, the difference between two mesh currents, like the currents in meshes m and n in Fig. 18 (b). The difference $(i_m - i_n)$ is algebraically given by the summation of those link currents (with due attention to sign) characterizing the path from m to n, just as a node-pair voltage (potential difference between two nodes) equals the algebraic sum of tree-branch voltages along the path connecting this node pair. The difference $(i_m - i_n)$, which might be called a mesh-pair current, is the real dual of a node-pair voltage. With the addition of the maze concept to our interpretation of network geometry, we have acquired a geometrical picture for the clarification of the algebraic connection between mesh-current differences and link currents that is as lucid as the familiar one used to connect node-potential differences with tree-branch voltages.



(a) (b) (c)

Fig. 19

These matters are further clarified through more specific examples. In Fig. 19 is shown a simple network graph (part (a)), its dual (part (b)), and a schematic indicating a choice of node-to-datum voltages characterizing the dual graph (part (c)). In the graph of part (a) the tree branches are the solid lines and the links (branches 1, 2, 4, 5, 6) are shown dotted. In the dual graph of part (b), these same branches (1, 2, 4, 5, 6) form the tree, and the rest are links. The datum node surrounds the whole dual graph. Mesh currents i_1, i_2, $\cdots$ i_5 are chosen to characterize the graph (a), while correspondingly the node potentials e_1, e_2, $\cdots$ e_5 characterize the dual graph (b).

Starting with the dual graph, it is evident that the expressions for the e's in terms of the tree branches read

$$
\begin{aligned}
e_1 &= v_1 \\
e_2 &= v_2 + v_1 \\
e_3 &= -v_4 - v_5 - v_6 \\
e_4 &= -v_5 - v_6 \\
e_5 &= -v_6
\end{aligned}
\qquad (57)
$$

Analogously, the mesh currents in terms of the link currents in graph (a) must be given by

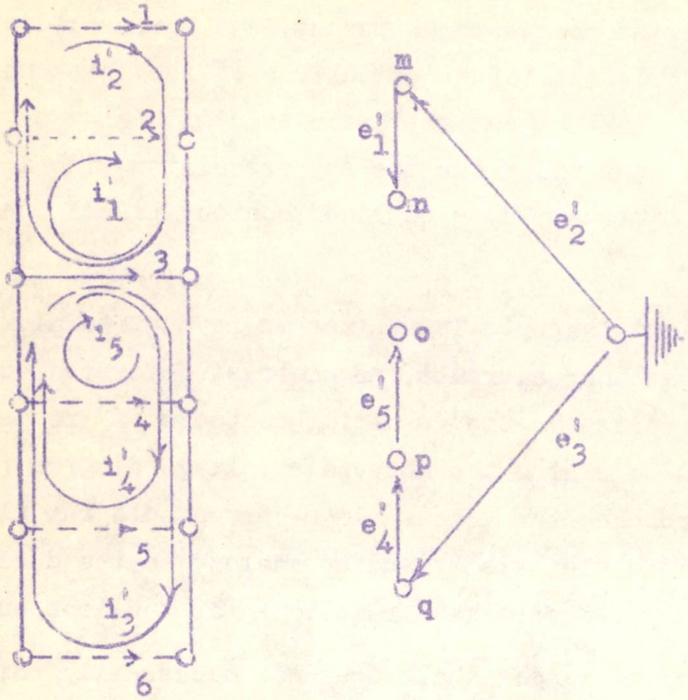
$$
\begin{aligned}
i_1 &= j_1 \\
i_2 &= j_2 + j_1 \\
i_3 &= -j_4 - j_5 - j_6 \\
i_4 &= -j_5 - j_6 \\
i_5 &= -j_6.
\end{aligned}
\qquad (58)
$$

One can verify these last results either through expressing the link currents as superpositions of the loop currents in the following manner

$$
\begin{aligned}
j_1 &= i_1 \\
j_2 &= i_2 - i_1 \\
j_4 &= i_4 - i_3 \\
j_5 &= i_5 - i_4 \\
j_6 &= -i_5
\end{aligned}
\qquad (59)
$$

and solving for the i's, or through noting that each mesh current (like a node potential) is the difference between the current circulating on the contour of that mesh and the datum mesh current, which is visualized as circulating on the periphery of the entire graph. In this sense the datum mesh is the entire space outside the graph, just as the datum node in the dual graph surrounds it. Following the pattern set in Fig. 18 (b) for expressing mesh-current differences in terms of link currents, one readily establishes the Eqs. 58 as representing the situation depicted in graph (a) of Fig. 19, and simultaneously recognizes how the algebraic signs in these equations are related to the reference arrows involved.

Consider now the same networks, but with an altered choice for the voltage and current variables. In Fig. 20 (a) are shown the paths for the new loop currents. The dual graph is not repeated in this figure, but part (b) shows the diagram for the choice of node-pair voltages in the dual graph that corresponds to the new loop currents in graph (a). All variables corresponding to this revised choice are distinguished by primes. So far as the voltage picture is concerned, one has little difficulty in recognizing that one now has



(a) (b)

Fig. 20

$$
\begin{aligned}
e_1' &= e_2 - e_1 = v_2 \\
e_2' &= e_1 = v_1 \\
e_3' &= e_5 = -v_6 \\
e_4' &= e_4 - e_5 = -v_5 \\
e_5' &= e_3 - e_4 = -v_4,
\end{aligned}
\qquad (60)
$$

and so, by analogy, the corresponding relations for the loop currents in terms of the link currents of the graph in part (a) of Fig. 20 must be

$$
\begin{aligned}
i_1' &= i_2 - i_1 = j_2 \\
i_2' &= i_1 = j_1 \\
i_3' &= i_5 = -j_6 \\
i_4' &= i_4 - i_5 = -j_5 \\
i_5 &= i_3 - i_4 = -j_4.
\end{aligned}
\qquad (61)
$$

These can readily be verified through the usual procedure of writing expressions for the link currents in terms of the loop currents and solving. It is more interesting, however, to establish them entirely by analogy to the dual voltage situation, for we learn in this way more about the manner in which the loop currents are related to the link currents. Thus a loop current like i_3', for example, surrounds three meshes, and correspondingly the node-pair voltage e_3' contributes to the potentials of the three nodes o, p, q (Fig. 20 (b). In forming the cut set associated with e_3', we would pick up nodes o, p, q; while in forming the tie set associated with i_3' we may say that we "pick up" the meshes whose combined contour places that tie set in evidence.

Having established the fact that picking up meshes is dual to picking up nodes, and recognizing that loop currents, as contrasted to mesh currents, are currents that circulate on the resulting contours of groups of meshes, we are in a position to sketch the node-pair voltage diagram (like part (b) of Fig. 20) corresponding to a chosen loop-current diagram (like part (a) of Fig. 20), provided one exists, and by analogy to the dual voltage equations, obtain directly the pertinent relations for the loop currents.

Since for cut sets picked at random there does not necessarily correspond a set of "node-pair voltages" that are simple potential differences between pairs of nodes, it is analogously true that for loops (i.e. tie sets) picked at random there does not necessarily correspond a set of "mesh-pair currents" that are simple differences between currents in pairs of meshes. In the example of Fig. 20, pertinent to the Eqs. 60 and 61, the conditions are chosen so that one does obtain e's that are potential differences between nodes and i's that are mesh-current differences, but when loops are picked at random, it is in general no longer possible to give any simple geometrical interpretation to the implied current relationships just as on the voltage side of the picture a straightforward interpretation fails when cut sets are chosen at random.

Wherever simple relationships do exist, the principle of duality is distinctly helpful in clarifying them. For example, in comparing parts (a) of Figs. 19 and 20, one might be tempted to conclude offhand that $i_1' = i_2$, or $i_5' = i_3$ because the contours on which these pairs of currents circulate are the same. As pointed out in Art. 7, it is fallacious to imply that there be any direct relation between the contours chosen for loop currents and their algebraic expressions in terms of link currents. Eqs. 61 show that the above offhand conclusions are false. Use of the duality principle, as in the preceding discussion, shows why they are false.

CHAPTER II

The Equilibrium Equations

1. Kirchhoff's laws

Having chosen an appropriate set of geometrically independent variables, either on the voltage or current basis, one is interested next in expressing the equilibrium of the network in terms of these. The means available for doing this, is given by the so-called Kirchhoff laws, of which there are two. One of these laws expresses a fundamental equilibrium condition in terms of voltages; the other expresses an analogous condition in terms of currents. When currents are chosen as variables, equilibrium of the network is expressed by means of the voltage law; when voltages are chosen as variables, equilibrium is expressed by the current law. This seeming inconsistency will adequately be elaborated upon in the following paragraphs, but first let us become acquainted with the Kirchhoff laws themselves.

We shall begin with a discussion of the voltage law, and in preparation for this discussion let us recall what is meant by voltage. "Voltage" is a shorter way of saying "electrical potential difference". Electric potential is work or energy; that is to say, it is a scalar quantity like temperature, or quantity of water, or altitude above sea level, etc. The fact that it is a scalar is the important thing so far as Kirchhoff's voltage law is concerned. Moreover, it is important to recognize that it is a single-valued scalar. Thus we can speak of the electric potential of any point in a network with respect to the potential of some arbitrary point chosen as reference, just as we can speak of the altitude of any point in a mountainous terrain with respect to sea level chosen as an arbitrary reference. By the single-valued character of either of these functions we imply that the value found for the function at some point relative to that at another, is independent of the route chosen in traversing from one of these points to the other in the course of actually carrying through a measurement or computation.

Suppose, for example, that we are to measure the altitude of the tip of Mt. Washington in New Hampshire with respect to some bench mark at its base by the customary methods used in surveying. We do this through successively measuring the differences in altitude between appropriately chosen intermediate points extending over some circuitous route having its ends respectively at the bench mark and at the mountain top. Assuming errors in measurement to be negligible, the net difference in altitude is expected to be independent of the route chosen. A similar conclusion applies in the case of any other single-valued scalar function, and the electric potential is such a one.

Now suppose we were to go on a surveying trip, start from some arbitrary bench mark, traverse all over the mountainous terrain, and finally return to the same bench mark. In this case we would expect to find a zero net difference in altitude, as is manifestly clear from the fact that the end points of our circuit are identical. Formal expression of this obvious fact in the analogous electrical case is the essence of Kirchhoff's voltage law. Let us elaborate slightly.

Refer to the network geometry shown in Fig. 1, and suppose we proceed around the periphery, touching in succession upon nodes a, b, c, d, e, f, g, h, and returning to node a. The potential of node a minus the potential of node b is the "drop" in potential from a to b, or the voltage drop v_2 in branch 2. Similarly v_3 is the voltage drop in branch 3 and equals the potential of node b minus that of node c. Proceeding in this way around the periphery, we see that the following equation is true

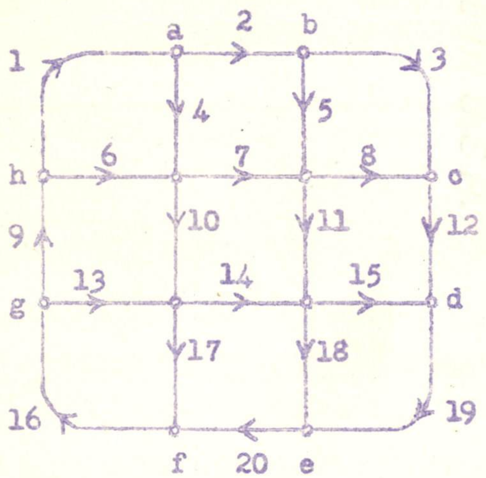


Fig. 1

$$v_2 + v_3 + v_{12} + v_{19} + v_{20} + v_{16} + v_9 + v_1 = 0 . \qquad (1)$$

This is Kirchhoff's voltage law expressed for the closed loop formed by the periphery. Numerically the values of some of these voltage drops must, of course, be negative; otherwise the sum of all of them could not be zero. We speak of Eq. 1 as representing an algebraic sum. If some voltage drop like v_2 is numerically negative, then evidently the potential of node a is less than that of node b (both referred to the same reference, of course).

Interpretation of the physical meaning of Eq. 1 is aided through use of the altitude analogue; that is, through regarding the nodes in the graph of Fig. 1 as bench marks in a mountainous terrain, and the voltage drops like v_2, v_3, etc. as drops in altitude between the pertinent bench marks in the reference arrow directions. Thus a rise in altitude in a given direction between two bench marks can alternately be regarded as a numerically negative drop.

One can trace through many additional closed paths in the network graph of Fig. 1. For example, starting again at node a and proceeding in succession to nodes b, c, d, e, and f, one may return to node a via the confluent set of branches 17, 10, and 4. In this case the voltage-law equation reads

$$v_2 + v_3 + v_{12} + v_{19} + v_{20} - v_{17} - v_{10} - v_4 = 0. \qquad (2)$$

Observe that a voltage-drop term is algebraically negative when traversal of the pertinent branch is contrary to its reference arrow. Thus the reference arrow on any branch is just what its name implies, namely an arbitrarily selected direction which we agree to call positive for the voltage drop in question. If the voltage drop actually has this direction, its value is numerically positive; if it has the opposite direction, its value is negative. In traversing a closed circuit within the terrain, the algebraic summation of altitude drops (potential drops between pertinent node pairs) must take account of reference arrow directions, as must also the process of deciding whether a given drop has a numerically positive or negative value.

Thus if bench mark e, for example, is higher than bench mark d, then v_{19} is numerically negative; and since its algebraic sign in Eq. 2 is plus, we see that this term involves an arithmetic subtraction. In branch 10, on the other hand, the actual drop in altitude may be contrary to the arrow direction so that v_{10} has a negative value. The corresponding term in Eq. 2 becomes numerically positive, as is appropriate since we actually experience a drop in altitude when we encounter branch 10 in traversing the circuit to which Eq. 2 applies.

The Kirchhoff voltage law thus expresses the simply understandable fact that the algebraic sum of voltage drops in any confluent set of branches forming a closed circuit or loop must equal zero. Symbolically this fact may be expressed by writing

$$\sum \pm v = 0, \quad (3)$$

where the Greek capital sigma is interpreted as a summation sign and the quantities $\pm v$ which are summed, are voltage drops with due regard to the possible agreement or disagreement of their pertinent reference arrows with the (arbitrary) direction of traversal around the loop, thus indicating the choice of the plus or minus sign respectively.

It is interesting to observe an important property of equations of this type with reference to a given network geometry such as that shown in Fig. 1. Suppose we write voltage-law equations for the upper left-hand corner mesh and its right-hand neighbor, thus

$$v_1 + v_4 - v_6 = 0, \quad (4)$$

$$v_2 + v_5 - v_7 - v_4 = 0.$$

Addition of these two equations gives

$$v_1 + v_2 + v_5 - v_7 - v_6 = 0, \quad (5)$$

which we recognize as an equation pertinent to the closed loop which is the periphery of the two meshes combined. The reason for this result is that branch 4, which is common to both meshes, injects the terms $+v_4$ and $-v_4$ respectively into the two Eqs. 4, and hence cancels out in their addition.

It is immediately clear that such cancellation of voltage terms will take place in the summation of any group of equations relating to meshes for which these terms correspond to branches common to the group of meshes. Suppose we write separate equations for the meshes immediately below those to which Eqs. 4 refer, thus

$$v_6 + v_{10} - v_{13} + v_9 = 0,$$
$$v_7 + v_{11} - v_{14} - v_{10} = 0\ . \qquad (6)$$

Adding Eqs. 4 and 6 we have

$$v_1 + v_2 + v_5 + v_{11} - v_{14} - v_{13} + v_9 = 0\ . \qquad (7)$$

This equation is pertinent to the periphery of the block of four upper left-hand meshes in the graph of Fig. 1. If all the equations for the separate meshes in this graph are added, one obtains Eq. 1 relating to the periphery of the whole graph. The student should try this as an exercise.

We now turn our attention to an analogous law in terms of branch currents: the so-called Kirchhoff current law. The electric current in a branch is the time-rate at which charge flows through that branch. Unless the algebraic sum of currents for a group of branches confluent in the same node is zero, electric charge will be either created or destroyed at that node. Kirchhoff's current law, which in essence expresses the principle of the conservation of charge, states therefore that an algebraic summation of branch currents confluent in the same node must equal zero. Symbolically this fact is expressed by writing (as in Eq. 3)

$$\sum \pm j = 0\ . \qquad (8)$$

As illustrations of this law suppose we write equations of this sort for nodes a and h and the one immediately to the right of h in Fig. 1. These read

$$-j_1 + j_2 + j_4 = 0,$$
$$j_1 + j_6 - j_9 = 0, \qquad (9)$$
$$-j_4 + j_7 + j_{10} - j_6 = 0\ .$$

Each equation states that the net current diverging from a pertinent node equals zero.

Now suppose we add the three equations 9. This gives

$$j_2 + j_7 + j_{10} - j_9 = 0. \tag{10}$$

Branch currents j_1, j_4, and j_6 cancel out in the process of addition. Reference to the graph of Fig. 1 reveals that these branches are common to the group of three nodes in question, while the branches to which the remaining currents in Eq. 10 refer, terminate only in one of these nodes.

An interesting interpretation may be given the resulting Eq. 10. If we regard the portion of the graph of Fig. 1 formed by branches 1, 4, and 6 alone (referred to as a subgraph of the entire network) as enclosed in a box, then Eq. 10 expresses the fact that the algebraic sum of currents divergent from this box equals zero. In other words, the current law applies to the box containing a subgraph the same as it does to a single node. That is to say, it is not possible for electric charge to pile up or diminish within a box containing a lumped network any more than it is possible for charge to pile up or diminish at a single node. This fact follows directly from the current law applied to a group of nodes, as shown above, and yet students usually have difficulty recognizing the truth of this result. They somehow feel that in a box there is more room for charge to pile up and so it may perhaps do this, whereas at a single node it is clear that the charge would have to jump off into space if more entered than left the node in any time interval. The above analysis shows, however, that what holds for a simple node must hold also for a box full of network.

2. Independence among the Kirchhoff law equations.

Equilibrium equations are a set of relations which uniquely determine the state of a network at any moment. They may be written in terms of any appropriately chosen variables; the uniqueness requirement demands however, that the number of independent equations shall equal the number of independent variables involved. We have seen earlier that the state of a network is expressible either in terms of $\ell = b - n_t + 1$ independent currents (for example, the loop currents), or in terms of $n = n_t - 1 = b - \ell$ independent voltages (for example, the node pair voltages). On a current basis we shall, therefore, require exactly ℓ independent equations, and on a voltage basis exactly n independent equations will be needed.

For these equations we turn our attention to the Kirchhoff laws. It is essential to determine how many independent equations of each type (the voltage-law and the current-law types) may be written for any given network geometry. Consider first the voltage-law equations and assume that these have been written for all of the nine meshes of the network graph in Fig. 1. Incidentally, this graph has 20 branches and a total of 12 nodes ($b = 20$, $n_t = 12$). Hence $\ell = 20 - 12 + 1 = 9$, which just equals the number of meshes. Any tree in this network involves $n = 11$ branches. There are 9 links and hence there are 9 geometrically independent loop currents.

From what has been pointed out in the previous article, it is clear that a voltage-law equation written for any other loop enclosing a group of meshes in Fig. 1 may be formed by adding together the separate equations for the pertinent meshes. Such additional voltage-law equations clearly are not independent. The inference is that one can always write exactly ℓ independent equations of the voltage-law type.

This conclusion is supported by the following reasoning. Suppose, for any network geometry a tree is chosen and the link currents identified with loop currents. For the correspondingly determined loops a set of voltage-law equations are written. These equations are surely independent, for the link voltages appear separately, one in each equation, so that it certainly is not possible to express any equation as a linear combination of the others. Each of these equations could be used to express one link voltage in terms of tree-branch voltages. This fact incidentally, substantiates what was said earlier with regard to the tree-branch voltages being an independent set and the link voltages being expressible uniquely in terms of them (see Art. 6, Ch. I).

Now any other closed loop for which a voltage-law equation could be written must traverse one or more links since the tree branches alone can form no closed paths. If in this equation the previous expressions for the pertinent link voltages are substituted, the resultant equation must reduce to the trivial identity $0 \equiv 0$, since no nontrivial relation can exist among tree-branch voltages alone (the tree-branch voltages are independent and hence are not expressible in terms of each other). It follows, therefore, that the voltage-law equation written for the additional closed loop expresses no independent result. There are indeed exactly ℓ independent voltage-law equations.

Let us turn our attention now to the Kirchhoff current-law equations and see how many of these may be independent. Referring again to the graph of Fig. 1, suppose we begin writing equations for several nodes adjacent to each other. If we examine these equations carefully we observe that each contains at least one term that does not appear in the others. For example, if we consider the equations written for nodes a and h, it is clear that the terms involving j_2 and j_4 do not appear in the equation for node h, and that the j_6 and j_9 terms in the equation for node h do not appear in the one for node a. If we also write an equation for the node immediately to the right of h, this one contains terms with j_7 and j_{10} which are not contained in either of the equations for nodes a or h. Such sets of equations are surely independent, for it is manifestly not possible to express any one as a linear combination of the others so long as each has terms that the others do not contain.

As we proceed to write current-law equations for additional nodes in the graph of Fig. 1, the state of affairs just described continues to hold true until equations have been written for <u>all but one</u> of the nodes. The inference is that exactly $n = n_t - 1$ independent equations of the current-law type can always be written. This conclusion is supported by the following reasoning.

Suppose, for any network geometry, a tree is chosen and the tree-branch voltages are identified with node-pair voltages. For the correspondingly determined node pairs, a set of Kirchhoff current-law equations are written. The set of branches taking part in the equation for any node pair is the pertinent cut set, just as the group of branches involved in the voltage-law equation for any loop is the tie set for that loop. The cut set pertinent to the node pair defined by any tree branch evidently involves that tree branch in addition to those links having one of their ends terminating upon the picked up nodes (see Art. 8, Ch. I).

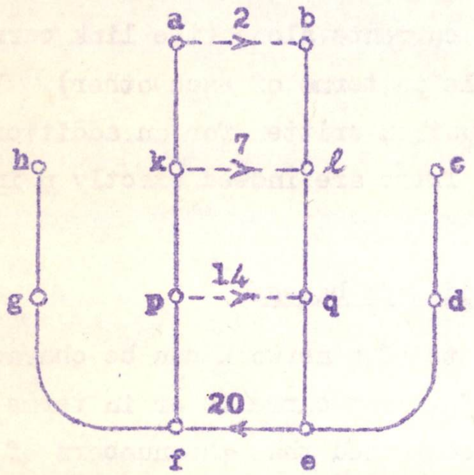


Fig. 2

Figure 2 illustrates the choice of a tree for the network graph of Fig. 1, and, with respect to the node pair f, e joined by branch 20, indicates by dotted lines the links that take part in the pertinent cut set. Since the tree-branch voltage v_{20} is identified with the respective node-pair voltage, the latter has its reference arrow pointing from f to e. That is to say, the picked-up nodes are e, q, ℓ, b, c, d. Hence the pertinent current-law equation reads

$$j_{20} - j_{14} - j_7 - j_2 = 0 \quad (11)$$

Schedules like 40, 42, 46 in Art. 8 of Ch. I are helpful in writing the current-law equations for a chosen set of node pairs, for the elements in the rows of such a schedule are the coefficients appropriate to these equations.

Suppose that current-law equations like 11 are written for all of the node pairs corresponding to the n tree branches. These equations are surely independent, for the tree-branch currents appear separately, one in each equation, so that it certainly is not possible to express any equation as a linear combination of the others. Each of these equations could be used to express one tree-branch current in terms of the link currents. This fact incidentally, substantiates what was said earlier with regard to the link-currents being an independent set and the tree-branch currents being expressible uniquely in terms of them (see Art. 5, Ch. I).

Now any other node pair for which a current-law equation could be written would have to involve one or more tree branches since the tree connects all of the nodes and therefore no node exists that has not at least one tree-branch touching it. If in such an additional current-law equation one substitutes the expressions already obtained for the pertinent tree-branch currents, the resultant equation must reduce to the trivial identity $0 \equiv 0$, since no nontrivial relation can exist among link currents alone (the link currents are independent and hence are not expressible in terms of each other). It follows, therefore, that the current-law equation written for an additional node pair expresses no independent result. There are indeed exactly n independent current-law equations.

3. The equilibrium equations on the loop and node bases.

Having established the fact that the state of a network can be characterized uniquely either in terms of a set of ℓ loop currents or in terms of a set of n node-pair voltages, and having recognized that the numbers of independent Kirchhoff voltage-law and current-law equations are ℓ and n respectively, the conclusion is imminent that the equilibrium condition for a network can be expressed in either of two ways: (a) through a set of ℓ voltage-law equations in which the loop currents are the variables, or (b) through a set of n current-law equations in which the node-pair voltages are the variables. These procedures, which are referred to respectively as the loop and node methods of expressing network equilibrium, are now discussed in further detail.

Consider first the loop method. The voltage-law equations, like Eq. 1 above, involve the branch-voltage drops. If these equations are to be written with the loop currents as variables, we must find some way of expressing the branch voltages in terms of the loop currents. These expressions are obtained in two successive steps.

The branch voltages are related to the branch currents by the volt-ampere equations pertaining to the kinds of elements (inductance, resistance, or capacitance) that the branches represent; and the branch currents in turn are related to the loop currents in the manner shown in Ch. I. Detailed consideration of the relations between branch currents and branch voltages is restricted at present to networks involving resistances only. Appropriate extensions to include the consideration of inductance and capacitance elements will follow in the later chapters.

Let the resistances of branches 1, 2, 3,··· be denoted by r_1, r_2, r_3, etc. Then the relations between all the branch voltages and all the branch currents are expressed by

$$v_k = r_k\, j_k, \quad \text{for } k = 1, 2, \cdots b. \tag{12}$$

The complete procedure for setting up the equilibrium equations on the

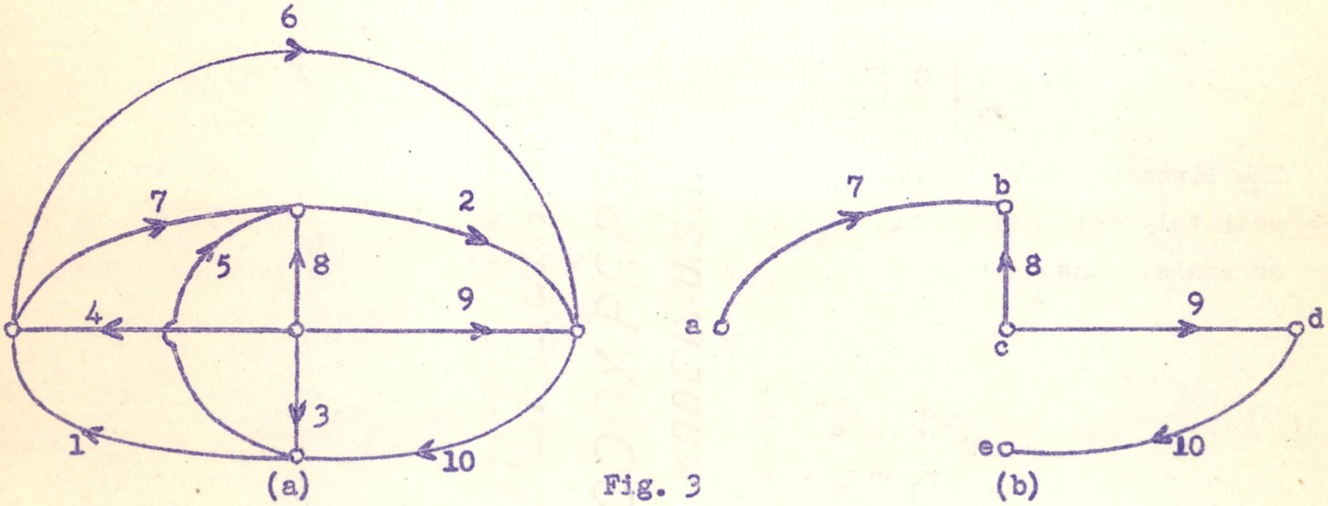


Fig. 3

loop basis will be illustrated for the network graph shown in Fig. 3. Part (a) is the complete graph, and part (b) is a chosen tree. Branches 1, 2,···6 are links, and the link currents j_1, j_2,··· j_6 are identified respectively with the loop currents i_1, i_2,··· i_6.

The following tie set schedule is readily constructed from an inspection of the resulting closed paths pertinent to these six loop currents (as the reader should check through placing the links 1, 2,···6, one at a time, into the tree of Fig. 3b).

Loop No. \ Branch No.	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	1	-1	1	1
2	0	1	0	0	0	0	0	1	-1	0
3	0	0	1	0	0	0	0	0	-1	-1
4	0	0	0	1	0	0	1	-1	0	0
5	0	0	0	0	1	0	0	-1	1	1
6	0	0	0	0	0	1	-1	1	-1	0

(13)

The Kirchhoff voltage-law equations written for these same loops are immediately obtained through use of the coefficients in the rows of this schedule, thus

$$
\begin{aligned}
v_1 + v_7 - v_8 + v_9 + v_{10} &= 0 \\
v_2 + v_8 - v_9 &= 0 \\
v_3 - v_9 - v_{10} &= 0 \\
v_4 + v_7 - v_8 &= 0 \\
v_5 - v_8 + v_9 + v_{10} &= 0 \\
v_6 - v_7 + v_8 - v_9 &= 0,
\end{aligned}
\tag{14}
$$

while the columns of the same schedule furnish the coefficients in the following equations for the branch currents in terms of the loop currents

$$
\begin{aligned}
j_1 &= i_1\\
j_2 &= i_2\\
j_3 &= i_3\\
j_4 &= i_4\\
j_5 &= i_5\\
j_6 &= i_6\\
j_7 &= i_1 + i_4 - i_6\\
j_8 &= -i_1 + i_2 - i_4 - i_5 + i_6\\
j_9 &= i_1 - i_2 - i_3 + i_5 - i_6\\
j_{10} &= i_1 - i_3 + i_5 .
\end{aligned}
\tag{15}
$$

If we assume for the branch resistances the values

$$
\begin{aligned}
&r_1 = 2,\ r_2 = 1,\ r_3 = 5,\ r_4 = 3,\ r_5 = 4,\\
&r_6 = 7,\ r_7 = 6,\ r_8 = 10,\ r_9 = 8,\ r_{10} = 9 \text{ ohms},
\end{aligned}
\tag{16}
$$

then $v_1 = 2j_1$, $v_2 = j_2$, $v_3 = 5j_5$, $v_4 = 3j_4$ volts, and so forth. Use of Eqs. 15 then gives

$$
\begin{aligned}
v_1 &= 2i_1\\
v_2 &= i_2\\
v_3 &= 5i_3\\
v_4 &= 3i_4\\
v_5 &= 4i_5\\
v_6 &= 7i_6\\
v_7 &= 6(i_1 + i_4 - i_6)\\
v_8 &= 10(-i_1 + i_2 - i_4 - i_5 + i_6)\\
v_9 &= 8(i_1 - i_2 - i_3 + i_5 - i_6)\\
v_{10} &= 9(i_1 - i_3 + i_5).
\end{aligned}
\tag{17}
$$

The desired loop equilibrium equations are obtained through substituting these values for the v's into Eqs. 14. After proper arrangement of the results, one finds

$$\begin{aligned}
35i_1 - 18i_2 - 17i_3 + 16i_4 + 27i_5 - 24i_6 &= 0 \\
-18i_1 + 19i_2 + 8i_3 - 10i_4 - 18i_5 + 18i_6 &= 0 \\
-17i_1 + 8i_2 + 22i_3 + 0i_4 - 17i_5 + 8i_6 &= 0 \\
16i_1 - 10i_2 + 0i_3 + 19i_4 + 10i_5 - 16i_6 &= 0 \\
27i_1 - 18i_2 - 17i_3 + 10i_4 + 31i_5 - 18i_6 &= 0 \\
-24i_1 + 18i_2 + 8i_3 - 16i_4 - 18i_5 + 24i_6 &= 0 .
\end{aligned} \tag{18}$$

Considering next the node method of writing equilibrium equations we observe first that the current-law equations, like Eq. 11 above, involve the branch currents. If these equations are to be written with the node-pair voltages as variables, we must express the branch currents in terms of the node-pair voltages. To do this, we note that the branch currents are related to the branch voltages through the Eqs. 12, and the branch voltages in turn are related to the node-pair voltages in the manner shown in Ch. I. The Eqs. 12 are now more appropriately written in the form

$$j_k = g_k \, v_k, \text{ for } k = 1, 2, \ldots b, \tag{19}$$

in which g_1, g_2, g_3, ... are respectively the reciprocals of r_1, r_2, r_3..., and are referred to as the branch conductances (expressed in mhos).

With reference to the network graph of Fig. 3 and the tree shown in part (b) of that figure, let the tree-branch voltages v_7, v_8, v_9, v_{10} be identified respectively with the node-pair voltages e_1, e_2, e_3, e_4. The following cut-set schedule is then readily constructed from an inspection of Fig. 3, noting the picked-up nodes pertinent to these four node pairs

Node Pair No. \ Branch No.	1	2	3	4	5	6	7	8	9	10	Picked up Nodes
1	-1	0	0	-1	0	1	1	0	0	0	a
2	1	-1	0	1	1	-1	0	1	0	0	c d e
3	-1	1	1	0	-1	1	0	0	1	0	a b c
4	-1	0	1	0	-1	0	0	0	0	1	a b c d

(20)

The Kirchhoff current-law equations corresponding to this choice of node pairs are immediately obtained through use of the coefficients in the rows of this schedule (the algebraic sum of currents in the branches of any cut set must equal zero), thus

$$\begin{aligned} -j_1 - j_4 + j_6 + j_7 &= 0 \\ j_1 - j_2 + j_4 + j_5 - j_6 + j_8 &= 0 \\ -j_1 + j_2 + j_3 - j_5 + j_6 + j_9 &= 0 \\ -j_1 + j_3 - j_5 + j_{10} &= 0, \end{aligned} \qquad (21)$$

while the columns of the same schedule furnish the coefficients in the following equations for the branch voltages in terms of the node-pair voltages

$$\begin{aligned} v_1 &= -e_1 + e_2 - e_3 - e_4 \\ v_2 &= -e_2 + e_3 \\ v_3 &= e_3 + e_4 \\ v_4 &= -e_1 + e_2 \\ v_5 &= e_2 - e_3 - e_4 \\ v_6 &= e_1 - e_2 + e_3 \\ v_7 &= e_1 \\ v_8 &= e_2 \\ v_9 &= e_3 \\ v_{10} &= e_4 . \end{aligned} \qquad (22)$$

For branch conductances corresponding to the resistance values 16, one has $j_1 = 0.5v_1$, $j_2 = v_2$, $j_3 = 0.2v_3$, $j_4 = 0.333v_4$, and so forth. Use of Eqs. 22 then gives

$$\begin{aligned} j_1 &= 0.5\,(-e_1 + e_2 - e_3 - e_4) \\ j_2 &= -e_2 + e_3 \\ j_3 &= 0.2\,(e_3 + e_4) \\ j_4 &= 0.333\,(-e_1 + e_2) \\ j_5 &= 0.25\,(e_2 - e_3 - e_4) \\ j_6 &= 0.143\,(e_1 - e_2 + e_3) \\ j_7 &= 0.167e_1 \\ j_8 &= 0.1e_2 \\ j_9 &= 0.125e_3 \\ j_{10} &= 0.111e_4 . \end{aligned} \qquad (23)$$

The desired node equilibrium equations are obtained through substituting these values for the j's into Eqs. 21. After proper arrangement, the results read

$$
\begin{aligned}
1.142e_1 - 0.976e_2 + 0.643e_3 + 0.500e_4 &= 0 \\
-0.976e_1 + 2.326e_2 - 1.893e_3 - 0.750e_4 &= 0 \\
0.643e_1 - 1.893e_2 + 2.218e_3 + 0.950e_4 &= 0 \\
0.500e_1 - 0.750e_2 + 0.950e_3 + 1.061e_4 &= 0 .
\end{aligned}
\qquad (24)
$$

In summary it is well to observe that the procedure for setting up equilibrium equations involves, for either the loop or node method, essentially three sets of relations:

(a) The Kirchhoff equations in terms of pertinent branch quantities.

(b) The relations between branch voltages and branch currents.

(c) The branch quantities in terms of the desired variables.

The coefficients in the rows and in the columns of the appropriate tie set or cut set schedule, supply the means for writing the relations (a) and (c) respectively. The relations (b), in the form either of Eqs. 12 or Eqs. 19, are straightforward in any case.

The desired equilibrium equations are obtained through substituting relations (c) into (b), and the resulting ones into (a). In the loop method, the branch quantities in the voltage-law equations (a) are voltages while the branch quantities in (c) are currents. In the node method, the branch quantities in the current-law equations (a) are currents while the branch quantities in (c) are voltages. The relations (b) are needed in either case to facilitate the substitution of (c) into (a); that is to say, this substitution requires first a conversion from branch currents to branch voltages or vice versa. It is this conversion that is supplied by the relations (b) which depend upon the circuit elements (resistances or conductances in the above example).

The tie set or cut set schedule is thus seen to play a dominant role in either method since it summarizes in compact and readily usable form all pertinent relations except those determined by the element values. The rows of a tie set schedule define an independent set of closed paths, and hence provide a convenient means for obtaining an independent set of Kirchhoff voltage-law equations. Any row of a cut set schedule, on the other hand, represents all of the branches terminating in the subgraph associated with one or more nodes. Since the algebraic sum of currents in such a set of branches must equal zero, the rows of a cut set schedule are seen to provide a convenient means for obtaining an independent set of Kirchhoff current-law equations.

The columns of these same schedules provide the pertinent relations through which the desired variables are introduced. They are useful not only in the process of obtaining the appropriate equilibrium equations, but also in subsequently enabling one to compute any of the branch quantities from known values of the variables.

In situations where the geometry is particularly simple, and where correspondingly straightforward definitions for the variables are appropriate, one may, after acquiring some experience, employ a more direct procedure for obtaining equilibrium equations (as given in Art. 6) which dispenses with the use of schedules.

4. Parameter matrices on the loop and node bases.

It should be observed that the final equilibrium Eqs. 18 and 24 are written in an orderly form in that the variable i_1 (resp. e_1) appears in the first column, the variable i_2 (resp. e_2) in the second column, and so forth. Taking this arrangement for granted, it becomes evident that the essential information conveyed by the Eqs. 18, for example, is contained with equal definiteness but with increased compactness in the array of coefficients

$$[R] = \begin{bmatrix} 35 & -18 & -17 & 16 & 27 & -24 \\ -18 & 19 & 8 & -10 & -18 & 18 \\ -17 & 8 & 22 & 0 & -17 & 8 \\ 16 & -10 & 0 & 19 & 10 & -16 \\ 27 & -18 & -17 & 10 & 31 & -18 \\ -24 & 18 & 8 & -16 & -18 & 24 \end{bmatrix} \qquad (25)$$

known as the loop resistance parameter matrix. The equilibrium equations 24 are similarly characterized by the following node conductance parameter matrix

$$[G] = \begin{bmatrix} 1.142 & -0.976 & 0.643 & 0.500 \\ -0.976 & 2.326 & -1.893 & -0.750 \\ 0.643 & -1.893 & 2.218 & 0.950 \\ 0.500 & -0.750 & 0.950 & 1.061 \end{bmatrix} . \qquad (26)$$

The term matrix is a name given to a rectangular array of coefficients as exemplified by the forms 25 and 26. As will be discussed in later chapters, one can manipulate sets of simultaneous algebraic equations like those given by 18 and 24 in a facile manner through use of a set of symbolic operations known as the rules of matrix algebra. These matters need not concern us at the moment, however, since the matrix concept is at present introduced only to achieve two objectives that can be grasped without any knowledge of matrix algebra whatever, namely:

(a) to recognize that all of the essential information given by the sets of equations 18 and 24 is more compactly and hence more effectively placed in evidence through the rectangular arrays 25 and 26;

(b) to make available a greatly abbreviated method of designating loop- or node-parameter values in numerical examples.

Appreciation of the second of these objectives may better be understood through calling attention first to a common symbolic form in which equations like 18 are written, namely thus

$$\begin{aligned} r_{11} i_1 + r_{12} i_2 + \cdots + r_{1\ell} i_\ell &= 0 \\ r_{21} i_1 + r_{22} i_2 + \cdots + r_{2\ell} i_\ell &= 0 \\ \text{-------}&\text{-------} \\ r_{\ell 1} i_1 + r_{\ell 2} i_2 + \cdots + r_{\ell\ell} i_\ell &= 0 \end{aligned} \qquad (27)$$

Here each coefficient is denoted by a symbol like r_{11}, r_{12}, and so forth. The corresponding matrix reads

$$[R] = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1\ell} \\ r_{21} & r_{22} & \cdots & r_{2\ell} \\ - & - & - & - \\ r_{\ell 1} & r_{\ell 2} & \cdots & r_{\ell\ell} \end{bmatrix} . \tag{28}$$

The general coefficient in this matrix is denoted by r_{sk} in which the indexes s and k can independently assume any integer values from 1 to ℓ. Observe that the first index denotes the row position, and the second one denotes the column position of the coefficient with respect to the array 28.

Analogously, a set of node equations like 24 would symbolically be written

$$\begin{aligned} g_{11} e_1 + g_{12} e_2 + \cdots + g_{1n} e_n &= 0 \\ g_{21} e_1 + g_{22} e_2 + \cdots + g_{2n} e_n &= 0 \\ - - - - - - - - - - &- - \\ g_{n1} e_1 + g_{n2} e_2 + \cdots + g_{nn} e_n &= 0, \end{aligned} \tag{29}$$

with the matrix

$$[G] = \begin{bmatrix} g_{11} & g_{12} \cdots & g_{1n} \\ g_{21} & g_{22} \cdots & g_{2n} \\ - & - & - \\ g_{n1} & g_{n2} \cdots & g_{nn} \end{bmatrix} . \tag{30}$$

Identification of the loop equations 27 in analytic form, with the specific numerical Eqs. 18 would necessitate (without use of the parameter matrix concept) writing

$$r_{11} = 35; \quad r_{12} = -18; \quad r_{13} = -17; \quad \cdots \tag{31}$$

$$- - - - - - - - - - - - - - - -$$

which is clearly an arduous and space-consuming task compared with writing down the numerical matrix 25. Use of the matrix concept takes advantage of the fact that the row and column position of a number identifies it as a specific r_{sk} value; it is no longer necessary to write identifying equations like those given by 31. Similar remarks apply to the numerical identification of parameters on the node basis and the usefulness of the corresponding parameter matrix notation.

5. Regarding the symmetry of parameter matrices.

The parameter matrices 25 and 26 given above have an important and interesting property in common which is described as their symmetry. For example, in the matrix 25 we note that $r_{12} = r_{21} = -18$; $r_{13} = r_{31} = -17$; $r_{14} = r_{41} = 16$; and so forth. More specifically, the matrix 25 is said to possess symmetry about its principal diagonal, the latter being represented by the elements $r_{11} = 35$, $r_{22} = 19$, $r_{33} = 22$, etc. on the diagonal extending from the upper left- to the lower right-hand corner of the array. Elements symmetrically located above and below this diagonal are equal. Symbolically this symmetrical property is expressed by the equation

$$r_{sk} = r_{ks} . \qquad (32)$$

Similar remarks apply to the node-conductance matrix 26.

This symmetry of the parameter matrix is neither accidental nor inherent in the physical property of linear networks. It is the result of having followed a deliberate procedure in the derivation of equilibrium equations that need by no means always be adhered to.

In order to understand the nature of this procedure, let us recall first that the process of deriving equilibrium equations involves predominantly the two sets of relations designated in the summary in Art. 3 as (a) the Kirchhoff law equations and (c) the defining equations for the chosen variables. (The circuit element relations (b) are needed in carrying out the substitution of (c) into (a) but are not pertinent to the present argument.) On the loop basis the variables are loop currents and the Kirchhoff equations are of the voltage-law type; on the node basis the variables are node-pair voltages and the Kirchhoff equations are of the current law type.

The choice of a set of loop-current variables involves the fixing of a set of loops or closed paths (tie sets), either through the choice of a tree and the identification of link currents with loop currents or through the forthright selection of a set of geometrically independent loops. The writing of Kirchhoff voltage-law equations also necessitates the selection of a set of geometrically independent loops, but this set need not be the same as that pertaining to the definition of the chosen loop currents. If the same loops are used in the definition of loop currents and in the writing of the voltage-law equations, then the resulting parameter matrices become symmetrical, but if separate choices are made for the closed paths defining loop currents and those for which the voltage-law equations are written, then the parameter matrices will not become symmetrical.

Thus a more general procedure for obtaining the loop equilibrium equations involves the use of two tie-set schedules. One of these pertains to the definition of a set of loop-current variables (as discussed in Art. 5, Ch. I); the tie sets in the other one serve merely as a basis for writing the voltage-law equations. Instead of using the rows and columns of the same schedule for obtaining the relations (a) and (c) respectively in the summary referred to above, one uses the rows of one schedule and the columns of another. The reader should illustrate these matters for himself by carrying through this revised procedure for the numerical example given above and noting the detailed changes that occur.

Analogously, on the node basis, one must choose a set of geometrically independent node pairs and their associated cut sets for the definition of node-pair voltage variables, and again for the writing of the Kirchhoff current-law equations. The second selection of node pairs and associated cut sets need not be the same as the first, but if they are (as in the numerical example leading to Eqs. 24), then the resulting parameter matrix becomes symmetrical.

Thus a more general procedure for obtaining the node equilibrium equations involves the use of two cut-set schedules. One of these pertains to the definition of a set of node-pair voltage variables (as discussed in Art. 6, Ch. I); the cut sets in the other one are utilized in writing current-law equations. Instead of using the rows and columns of the same schedule, one uses the rows of one schedule and the columns of another.

The significant point in these thoughts is that the choice of variables, whether current or voltage, need have no relation to the process of writing Kirchhoff-law equations. It is merely necessary that the latter be an independent set; the variables in terms of which they are ultimately expressed, may be chosen with complete freedom.

When the same tie sets are used for voltage-law equations and loop-current definitions, or the same cut sets are used for current-law equations and node-pair voltage definitions, then we say that the choice of variables is consistent with the Kirchhoff-law equations. It is this consistency that leads to symmetrical parameter matrices.

The question of symmetry in the parameter matrices is important primarily in that one should recognize the deliberateness in the achievement of this result and not (as is quite common) become confused into thinking that it is an inherent property of linear passive networks to be characterized by symmetrical parameter matrices. We shall, to be sure, follow the usual procedure that leads to symmetry, not only because it obviates two choices being made for a set of loops or node pairs, but also because symmetrical equations are easier to solve, and because a number of interesting network properties are more readily demonstrated. So in the end we follow the customary procedure, but with an added sense of perspective that comes from a deeper understanding of the principles involved.

6. Simplified procedures that are adequate in many practical cases.

We have given the preceding very general approach to the matter of forming the equilibrium equations of networks because, through it as a background, we are now in a position to understand far more adequately and with greater mental satisfaction the following rather restricted but practically very useful procedures applicable to many geometrical network configurations dealt with in practice. Thus, in many situations encountered in engineering work, the network geometry is such that the graph may be drawn on a plane surface without having any branches cross each other. As mentioned in Art. 9, Ch. I such a network is spoken of as being "mappable on a plane," or more briefly as a mappable network. The network whose graph is shown in Fig. 3 is not of the mappable variety, but the one given by the graph in Fig. 1 is.

When the equilibrium equations for a mappable network (such as that shown in Fig. 1) are to be written on the loop basis, it is possible to choose as a geometrically independent set of closed loops the meshes of this network graph (as pointed out in Art. 7 of Ch. I). A simple example of this sort is shown in Fig. 4 in which the meshes are indicated by circulatory arrows.

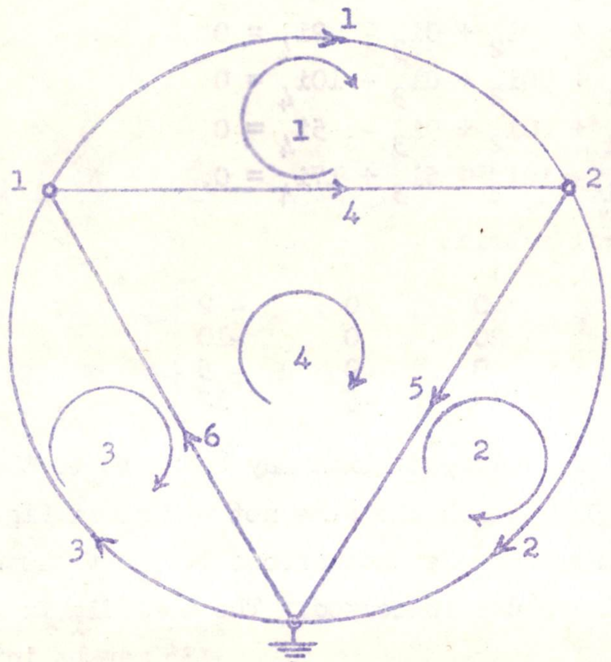


Fig. 4

The corresponding voltage-law equations are

$$\begin{aligned} v_1 - v_4 &= 0 \\ v_2 - v_5 &= 0 \\ v_3 - v_6 &= 0 \\ v_4 + v_5 + v_6 &= 0. \end{aligned} \qquad (33)$$

The branch currents in terms of the loop currents are seen to be given by

$$\begin{aligned} j_1 &= i_1 \\ j_2 &= i_2 \\ j_3 &= i_3 \\ j_4 &= i_4 - i_1 \\ j_5 &= i_4 - i_2 \\ j_6 &= i_4 - i_3. \end{aligned} \qquad (34)$$

Suppose the branch resistance values are

$$r_1 = 5;\ r_2 = 10;\ r_3 = 4;\ r_4 = 2;\ r_5 = 10;\ r_6 = 5. \qquad (35)$$

The equations 34 multiplied respectively by these values yield the corresponding v's by means of which the Eqs. 33 become expressed in terms of the loop currents. After proper arrangement this substitution yields

$$\begin{aligned} 7i_1 + 0i_2 + 0i_3 - 2i_4 &= 0 \\ 0i_1 + 20i_2 + 0i_3 - 10i_4 &= 0 \\ 0i_1 + 0i_2 + 9i_3 - 5i_4 &= 0 \\ -2i_1 - 10i_2 - 5i_3 + 17i_4 &= 0, \end{aligned} \qquad (36)$$

with the symmetrical matrix

$$[R] = \begin{bmatrix} 7 & 0 & 0 & -2 \\ 0 & 20 & 0 & -10 \\ 0 & 0 & 9 & -5 \\ -2 & -10 & -5 & 17 \end{bmatrix}. \qquad (37)$$

A simple physical interpretation may be given to these equations through reference to Fig. 5 in which the same network as in Fig. 4 is redrawn with the branch numbering and reference arrows left off but with the branch resistances and their values indicated. The term $7i_1$ in the first of the Eqs. 36 may be interpreted as the voltage drop caused in mesh 1 by loop current i_1 since the total resistance on the contour of this mesh is 7 ohms; the rest of the terms in this equation represent additional voltage drops caused in mesh 1 by the loop currents i_2, i_3, i_4 respectively. Since no part of the contour of mesh 1 is traversed by the currents i_2 and i_3, these can cause no voltage drop in mesh 1; hence the coefficients of their terms in the first of the Eqs. 36 are zero. The term $-2i_4$ takes account of the fact that loop current i_4, in traversing the 2-ohm

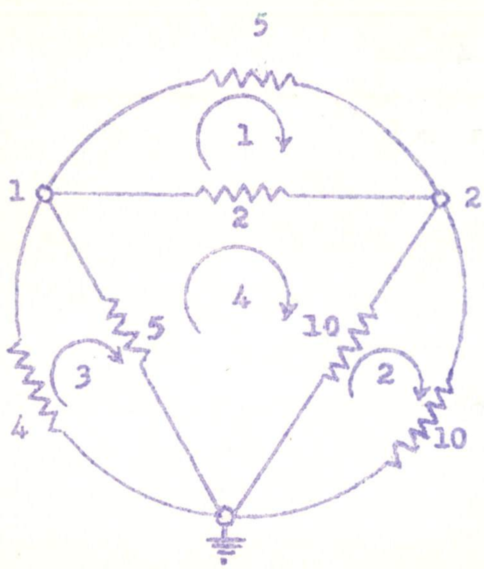


Fig. 5

resistance, contributes to the voltage drop in mesh 1 and that this contribution is negative with respect to the loop reference arrow in mesh 1.

The second of the Eqs. 36 similarly expresses the fact that the algebraic sum of voltage drops caused in mesh 2 by the various loop currents equals zero. Only those terms have nonzero coefficients whose associated loop currents traverse at least part of the contour of mesh 2. The value of any nonzero coefficient equals the ohmic value of the total or partial mesh 2 resistance traversed by the pertinent loop current, and its algebraic sign is plus or minus according to whether the reference direction for this loop current agrees or disagrees respectively with the reference arrow for mesh 2. Analogous remarks apply to the rest of the Eqs. 36.

With this interpretation in mind, one can write the loop-resistance matrix 37 directly. Thus the coefficients on the principal diagonal are respectively the total resistance values on the contours of meshes 1, 2, 3,.... The remaining coefficients are resistances of branches common to a pair of meshes, with their algebraic signs plus or minus according to the confluence or counterfluence of the respective mesh arrows in the pertinent common branch. Specifically, a term r_{sk} in value equals the resistance of the branch common to meshes <u>s</u> and <u>k</u>; its algebraic sign is plus if the mesh arrows have the same direction in this common branch; it is minus if they have opposite directions.

In a mappable network, with the meshes chosen as loops and the loop reference arrows consistently clockwise (or consistently counterclockwise), the algebraic signs of <u>all</u> nondiagonal terms in the loop resistance matrix are <u>negative</u>. It is obvious that this procedure for the derivation of loop equilibrium equations yields a symmetrical parameter matrix ($r_{sk}=r_{ks}$) since a branch common to meshes <u>s</u> and <u>k</u>, whose value determines the coefficient r_{sk}, is at the same time common to meshes <u>k</u> and <u>s</u>.

This simplified procedure for writing down the loop equilibrium equations directly (having made a choice for the loops and loop currents) does not, of course, require mappability of the network, but it is not difficult to appreciate that it soon loses its simplicity and directness when the network geometry becomes random. For, in a random case it may become difficult to continue to speak of meshes as simplified versions of loops; moreover, their choice is certainly no longer straightforward nor is the designation of loop reference arrows as simple to indicate. Any given branch may be common to more than two meshes; the pertinent loop reference arrows may traverse such a branch in random directions, so that the nondiagonal coefficients in the parameter matrix will no longer be consistently negative. Although the simplified procedure may still be usable in some moderately complex nonmappable cases, one will find the more general procedure described earlier preferable when arbitrary network geometries are encountered.

An analogous simplified procedure appropriate to relatively simple geometries may be found for the determination of node equilibrium equations. In this simplified procedure the node-pair voltage variables are chosen as a node-to-datum set, as described in Art. 8 of Ch. I. That is, they are defined as the potentials of the various single nodes with respect to a common (arbitrarily selected) datum node, as illustrated in Ch. I by Fig. 11 for the network graph of Fig. 8. The cut sets (which determine the Kirchhoff current-law equations) are then all given by the groups of branches divergent from the single nodes for which the pertinent node potentials are defined.

With regard to the network of Fig. 4 one may choose the bottom node as the datum or reference, and define the potentials of nodes 1 and 2 respectively as the voltage variables e_1 and e_2. Noting that the pertinent cut sets are the branches divergent from these nodes, the current-law equations consistent with this selection of node-pair voltages are seen to read

$$\begin{aligned} j_1 + j_4 - j_6 - j_3 &= 0 \\ -j_1 - j_4 + j_2 + j_5 &= 0. \end{aligned} \tag{38}$$

The branch voltages in terms of the node potentials are by inspection of Fig. 4

$$\begin{aligned} v_1 &= e_1 - e_2 \\ v_2 &= e_2 \\ v_3 &= -e_1 \\ v_4 &= e_1 - e_2 \\ v_5 &= e_2 \\ v_6 &= -e_1. \end{aligned} \tag{39}$$

The branch conductances corresponding to the resistance values 35 are

$$g_1 = 0.2;\ g_2 = 0.1;,\ g_3 = 0.25;\ g_4 = 0.5;\ g_5 = 0.1;\ g_6 = 0.2. \qquad (40)$$

Equations 39 multiplied respectively by these values yield the corresponding j's in terms of the node potentials. Their substitution into Eqs. 38 results in the desired equilibrium equations, which read

$$\begin{aligned} 1.15e_1 - 0.70e_2 &= 0 \\ -0.70e_1 + 0.90e_2 &= 0, \end{aligned} \qquad (41)$$

with the symmetrical node conductance matrix

$$[G] = \begin{bmatrix} 1.15 & -0.70 \\ -0.70 & 0.90 \end{bmatrix}. \qquad (42)$$

A simple physical interpretation may be given to the node equilibrium Eqs. 41 that parallels the interpretation given above for the loop equations. Thus the first term in the first of the equations 41 represents the current that is caused to diverge from node 1 by the potential e_1 acting alone (that is, while $e_2 = 0$); the second term in this equation represents the current that is caused to diverge from node 1 by the potential e_2 acting alone (that is, while $e_1 = 0$). Since a positive e_2 acting alone causes current to converge upon node 1 (instead of causing a divergence of current) the term with e_2 is numerically negative. The amount of current that e_1 alone causes to diverge from node 1 evidently equals the value of e_1 times the total conductance between node 1 and datum when $e_2 = 0$ (that is, when node 2 coincides with the datum). This total conductance clearly is the sum of the conductances of the various branches divergent from node 1; with reference to Fig. 5 (in which the given parameter values are resistances) this total conductance is $1/5 + 1/2 + 1/5 + 1/4 = 1.15$, thus accounting for the coefficient of the term with e_1 in the first of the Eqs. 41.

The current that e_2 alone causes to diverge from node 1 can traverse only the branches connecting node 1 directly with node 2 (these are the 2 ohm and 5 ohm branches in Fig. 5), and the value of this current is evidently given in magnitude by the product of e_2 and the net conductance of these combined branches. In the present example the pertinent conductance is $1/2 + 1/5 = 0.70$ mho, thus accounting for the value of the coefficient in the second term of the first of the Eqs. 41 (the reason for its negative sign has already been explained). A similar interpretation is readily given to the second of the Eqs. 41.

Thus these equations or their conductance matrix 42 could be written down directly by inspection of Fig. 5, especially if the branch-resistance values are alternately given as branch-conductance values expressed in mhos. The elements on the principal diagonal of $[G]$ are respectively the total conductance values (sums of branch conductances) divergent from nodes 1, 2, ... (in a more general case there will be more than two nodes). The nondiagonal elements of $[G]$ all have negative algebraic signs, for the argument given above in the detailed explanation of Eqs. 41 clearly applies unaltered to all cases in which the node-pair voltage variables are chosen as a node-to-datum set. In magnitude, the nondiagonal elements in $[G]$ equal the net conductance values (sums of branch conductances), for those branches directly connecting the pertinent node pairs. More specifically, the element g_{sk} in $[G]$ equals the negative sum of the conductances of the various branches directly connecting nodes s and k. If these nodes are not directly connected by any branches, then the pertinent g_{sk} value is zero. Note that the consistent negativeness of the nondiagonal terms follows directly from the tacit assumption that any node potential is regarded as positive when it is higher than that of the datum node. This situation parallels the consistent negativeness of the nondiagonal terms in the $[R]$ matrix obtained on the loop basis for a mappable network in which all the mesh reference arrows are chosen consistently clockwise (or consistently counterclockwise), whence in any common branch they are counterfluent.

7. Sources.

When currents and their accompanying voltage drops exist in a resistive network, energy is being dissipated. Since at every instant the rate of energy supply must equal its rate of dissipation, there can be no voltages or currents in a purely resistive or in any "lossy" network unless there are present one or more sources of energy.

Until now the role played by sources has not been introduced into the network picture, and indeed, their presence has nothing whatever to do with the topics discussed so far. Sources were purposely left out of consideration for this reason, since their inclusion would merely have detracted from the effectiveness of the discussion. Now, however, it is time to recognize the significance of sources, their characteristics, and how we are to determine their effect upon the equilibrium equations.

Their most important effect, as already stated, is that without them there would be no response. This fact may clearly be seen for example, from the loop-equilibrium Eqs. 36 for the network of Fig. 5. Since these four equations involving the four unknowns i_1, i_2, i_3, i_4 are independent, and all of the right-hand members are zero, we know according to the rules of algebra that none but the trivial solution $i_1 = i_2 = i_3 = i_4 = 0$ exists. That is to say, in the absence of excitation (which, as we shall see, causes the right-hand members of the equations to be nonzero) the network remains "dead as a doornail."

It was pointed out in the introduction that an electrical network as we think of it in connection with our present discussions is almost always an artificial representation of some physical system in terms of idealized quantities which we call the circuit elements or parameters (the resistance, inductance, and capacitance elements). We justify such an artificial representation through noting: (a) that it can be so chosen as to simulate functionally (and to any desired degree of accuracy) the actual system at any selected points of interest; and (b), that such an idealization is essential in reducing the analysis procedure to a relatively simple and easily understandable form.

Regarding the sources through which the network becomes energized or the physical system derives its motive power, a consistent degree of idealization is necessary. That is to say, the sources, like the circuit elements, are represented in an idealized fashion. We shall see that actual energy sources may thus be simulated through such idealized sources in combination with idealized circuit elements. For the moment we focus our attention upon the idealized sources themselves.

Although the physical function of a source is to supply energy to the system, we shall for the time being find it more expedient to characterize a source as an element capable of providing a fixed amount of voltage or a fixed amount of current at a certain point. Actually it provides both voltage and current, and hence an amount of power equal to their product, but it is analytically essential and practically more realistic to suppose that either the voltage or the current of the source is known or fixed. We culd, of course, postulate a source for which both the voltage and the current are fixed, but such sources would not prove useful in the simulation of physical systems, and we must at all times be mindful of the utility of our methods of analysis.

When we say that the voltage or the current of a source is fixed, we do not necessarily mean that it is a constant, but rather that its value or sequence of values as a continuous function of the time are independent of all other voltages and currents in the entire network. Most important in this connection is the nondependence upon the source's own voltage, if it is a current, or upon its own current if it is a voltage. Thus a so-called idealized voltage source provides at a given terminal pair a voltage function that is independent of the current at that terminal pair; and an idealized current source provides a current function that is independent of the voltage at the pertinent terminal pair.

By way of contrast, it is useful to compare the idealized source as just defined with an ordinary passive resistance or other circuit element. In the latter, the voltage and current at the terminals are related in a definite way which we call the "volt-ampere relationship" for that element. For example, in a resistance the voltage is proportional to the current, the constant of proportionality being what we call the value of the element in ohms. At the terminals of an ideal voltage source, on the other hand, the voltage is whatever we assume for it to be, and it cannot depart one jot from this specification regardless of the current it is called upon to deliver on account of the conditions imposed by its environment. An extreme situation arises if the environment is a short-circuit, for then the source is called upon to deliver an infinite current, yet it does so unflinchingly and without its terminal voltage departing in the slightest from its assigned value. It is, of course, not sensible to place an ideal voltage source in such a situation, for it then is called upon to furnish infinite power. The ideal voltage source is idle when its environment is an open circuit, for then the associated current becomes zero.

Similarly, at the terminals of an ideal current source the current is whatever we assume for it to be, and it cannot depart from this specification regardless of the voltage it is called upon to produce on account of the conditions imposed by its environment. An extreme situation arises in this case if the environment turns out to be an open circuit, for then the source must produce an infinite voltage at its terminals since the terminal current, by definition, cannot depart from its specified value. Like the short-circuited voltage source, it is called upon to deliver infinite power, and hence it is not realistic to place an ideal current source in an open-circuit environment. This type of source is idle when short-circuited, since the associated voltage is then zero.

In the discussion of Kirchhoff's voltage law we found it useful to think of voltage as being analogous to altitude in a mountainous terrain. The potentials of various points in the network with respect to a common reference or datum are thought of as being analogous to the altitudes of various points in a mountainous terrain with respect to sea level as a common reference. Instead of an actual mountainous terrain, suppose we visualize a miniature replica constructed through hanging up a large rubber sheet and suspending from it various weights attached at random places. Since altitude is the analogue of voltage, the problem of finding the altitude of various locations on the sheet (above, say, the floor as a common reference) is analogous to determining the potentials of various nodes in an electrical network with reference to a datum node.

Suppose first that we consider the electrical network to have no sources of excitation; all node potentials are zero. The analogous situation involving the rubber sheet would be to have it lying flat on the floor. To apply a voltage excitation to the network may be regarded as causing certain of its node potentials to be given fixed values. Analogously, certain points in the rubber sheet are raised above the floor to fixed positions and clamped there. As a result, the various nodes in the electrical network whose potentials are not arbitrarily fixed, assume potentials that are consistent with the applied excitation and the characteristics of the network. Analogously, the freely movable portions of the rubber sheet assume positions above the floor level that are consistent with the way in which the sheet is supported at the points where it is clamped (analogous to excitation of the electrical network) and the structural characteristics of the sheet with its system of attached weights.

It is interesting to note from the description of these two analogous situations that electrical excitation by means of voltage sources may be thought of as arbitrarily fixing or clamping the voltage at a certain point or points. A voltage source is thus regarded as an applied constraint, like nailing the rubber sheet to the wall at some point.

Ideal current sources when used to excite an electrical network may likewise be regarded as applied constraints. In any passive network the currents and voltages in its various parts are in general free to assume an array of values subject only to certain interrelationships dictated by the structure of that network, but without any excitation, all voltages and currents remain zero. If we now give to some of these voltages and currents arbitrary nonzero values, we take away their freedom, for they can no longer assume any values except the specified ones, but the remaining voltages and currents whose values are not pegged, now move into positions that are compatible with the network characteristics interrelating all voltages and currents, and with the fixed values of those chosen to play the role of excitation quantities. As more of the voltages and currents are clamped or fixed through the application of sources, fewer remain free to adjust themselves to compatible values. Finally if all voltages and currents were constrained by applied sources, there would be no network problem left, for everything would be known beforehand. In the commonest situation, only a single voltage or current variable is constrained through an applied source; determination of the compatible values of all the others, constitutes the network problem.

Various ways in which sources are schematically represented in circuit diagrams are shown in Fig. 6. Parts (a), (b), and (c) are representations

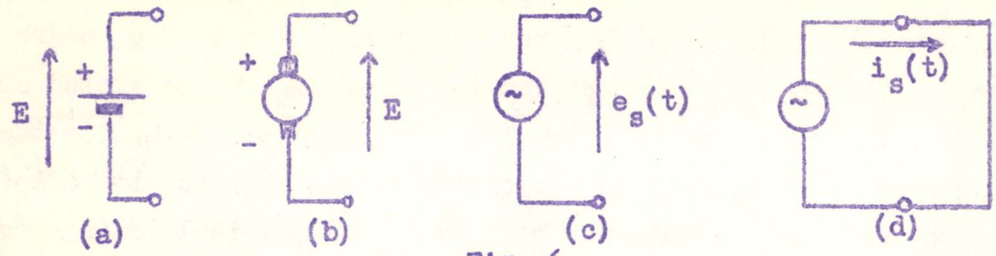


Fig. 6

of voltage sources, while part (d) shows the representation for a current source. Specifically, (a) and (b) are common ways of indicating constant voltage sources, also called "direct current" or "d-c" voltage sources. The schematic (a) simulates a battery, for example, a dry cell in which the zinc electrode (thin line) is positive and the carbon electrode (thick line) is the negative terminal. The d-c source shown in (b) is drawn to resemble the commutator and brushes of a generator. The symbolic representation in (c) is intended to be more general in that the wavy line inside the circle indicates that $e_s(t)$ may be any function of time (not necessarily a sinusoid, although there is an established practice in using this symbol as the representation for a sinusoidal generator). It should be particularly noted that $e_s(t)$ in the symbolic representation of part (c) may be <u>any</u> time function and, in particular, may also be used to denote a constant voltage source (d-c source).

Part (d) of Fig. 6 shows the schematic representation for a current source in which $i_s(t)$ is any time function and hence may be used to denote a constant or d-c source as well as any other.

In all of these source representations it will be noted that a reference arrow is included. This arrow does <u>not</u> imply that the source voltage or current is assumed to act in the indicated direction but only that, if it should at any moment have this direction, it will at that moment be regarded as a positive quantity. The reference arrow establishes a means for telling when the quantity $e_s(t)$ or $i_s(t)$ is positive and when it is negative. A source voltage is said to "act in the direction of the reference arrow" when it is a voltage rise in this direction. The + and - signs of parts (a) and

(b) of Fig. 6 further clarify this statement. In most of the following work the representations shown in parts (c) and (d) will be used.

It should not be overlooked that the representations in Fig. 6 are for ideal sources. Thus the voltage between the terminals in the sketch of part (c) is always $e_s(t)$ no matter what is placed across them. Likewise the current issuing from the terminals in the sketch of part (d) is always $i_s(t)$ no matter what the external circuit may be. An actual physical voltage source may, to a first approximation, be represented through placing a resistance in series with the ideal one so that the terminal voltage decreases as the source current increases. A physical current source may similarly be represented to a first approximation through the ideal one of part (d) with a resistance in parallel with the terminals, thus taking account of the fact that the net current issuing from the terminals of the combination depends upon the terminal voltage, and decreases as this voltage increases. These matters will further be elaborated upon in the applications to come later on.

It is common among students that they have more difficulty visualizing or grasping the significance of current sources than they do in the understanding of voltage sources. A contributing reason for this difficulty is that voltage sources are more commonly experienced. Thus our power systems that supply electricity to our homes and factories are essentially voltage sources in that they have the property of being idle when open-circuited. Sources that are basically of the current variety are far less common. One such source is the photoelectric cell which emits charge proportional to the intensity of the impinging light and hence is definitely a current source; it clearly is idle when short-circuited because it then delivers no energy. Another device that is commly regarded as a current source is the pentode vacuum tube. Its plate current is very nearly proportional to its grid excitation under normal operating conditions, and hence, for purposes of circuit analysis, it is appropriate to consider it as being essentially a current source. In any case it can with very good accuracy be regarded as an ideal current source in parallel with a resistance.

Whether actual sources are more correctly to be regarded as voltage sources or as current sources is, however, a rather pointless argument since we shall soon see that either representation (in combination with an appropriate arrangement of passive circuit elements) is always possible no matter what the actual source really is. Again we must be reminded that circuit theory makes no claim to be dealing with actual things. In fact it very definitely deals only with fictitious things, but in such a way that actual things can thereby be represented. Like all other methods of analysis, circuit theory is merely the means to an end; it lays no claim to being the real thing.

Now as to determing how source quantities enter into the equilibrium equations for a given network, we first make the rather general observation that the insertion of sources into a given passive network is done in either of two ways. One of these is to insert the source into the gap formed by cutting a branch (as with a pliers); the other is to connect the source terminals to a selected node pair (as with a soldering iron). These two methods will be distinguished as the "pliers method" and the "soldering-iron method" respectively. We shall now show that one may consider the pliers method restricted to the insertion of voltage sources and the soldering-iron method to the insertion of current sources. That is to say, the connection of a voltage source across a node pair, or the insertion of a current source in series with a branch, implies a revision of the network geometry with the end result that voltage sources again appear only in series with branches and current sources appear only in parallel with branches (or across node pairs).

For example, in part (a) of Fig. 7 is shown a graph in which a voltage

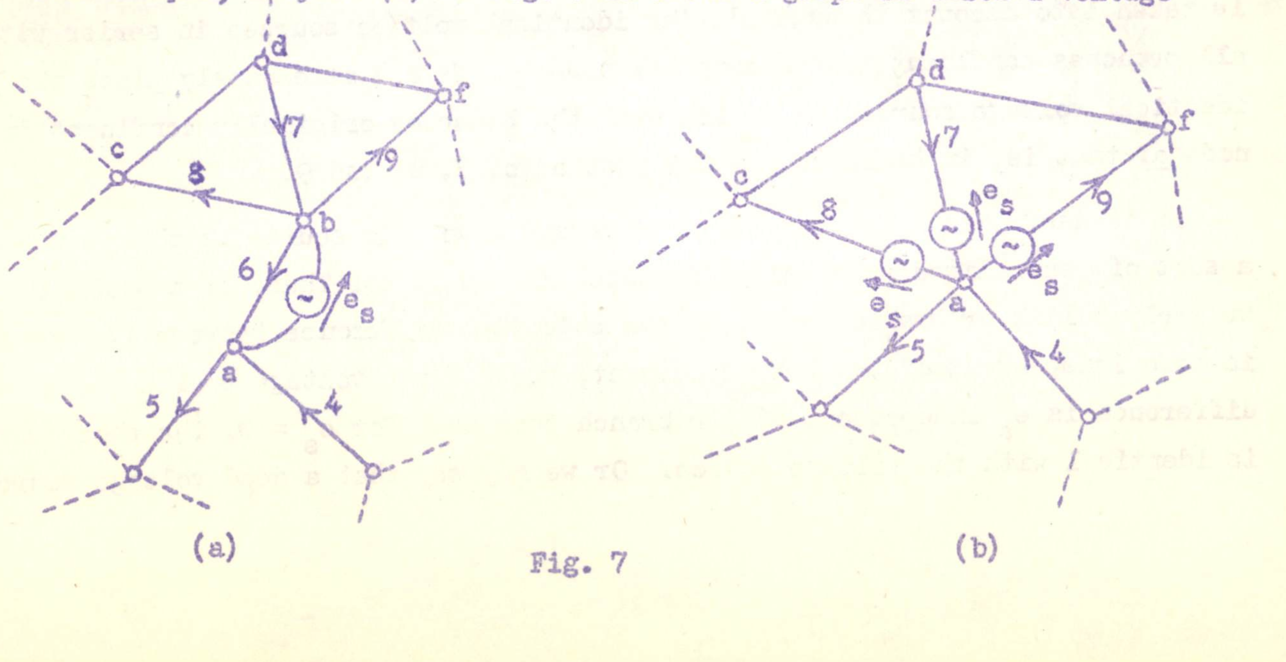


Fig. 7

source e_s appears in parallel with branch 6 of some network, and in part (b) of this figure is shown the resultant change in the network geometry and source arrangement which this situation reduces to. Thus, in considering the given arrangement in part (a), one should first observe that branch 6 is rendered trivial by having e_s placed in parallel with it since the value of v_6 is thus forced to be equal to e_s and hence (along with j_6) is no longer an unknown. That is to say, the determination of the current in branch 6 is rendered trivially simple and independent of what happens in the rest of the network. Therefore we can remove branch 6 from our thoughts and from the rest of the graph so that e_s alone appears as a connecting link between nodes a and b. Next we observe that the potentials of nodes c, d, f, relative to that of node a are precisely the same in the arrangement of part (b) in Fig. ' as they are in part (a). For example, the potential of node c with respect to that of node a is $(e_s - v_8)$ as is evident by inspection of either part (a) or part (b) of this figure. Similarly the potential of node d with respect to that of node a is seen to be $(e_s + v_7)$ in the arrangement of part (a) or of part (b). It thus becomes clear that the branch voltages and currents in the graph of part (b) must be the same as in the graph of part (a), except for the omission of the trivial branch 6.

We may conclude that placing a voltage source across a node pair has the same effect upon the network geometry as does the placing of a short circuit across that node pair Comparing graphs (a) and (b) in Fig. 7 we see, for example, that the voltage source e_s in graph (a) effectively unites nodes a and b in that graph, thus eliminating branch 6, and yielding the revised graph (b). The effect of the voltage source so far as this revised graph is concerned is taken into account through placing identical voltage sources in series with all branches confluent in the original node b. We can alternately place the identical voltage sources in series with the branches originally confluent in node a; that is, in branches 4 and 5 instead of 7, 8, and 9.

It is useful in this connection to regard a voltage source as though it were a sort of generalized short circuit, which indeed it is. Thus, by a short circuit we imply a link or branch for which the potential difference between its terminals is zero independent of the branch current; while for a voltage source the potential difference is e_s independent of the branch current. For $e_s = 0$, the short circuit is identical with the voltage source. Or we may say that a dead voltage source is a short circuit. The preceding discussion shows that the effect of a voltage source upon the network geometry is the same as that of an applied short-circuit constraint.

Analogously, part (a) of Fig. 8 depicts a situation in which a current source i_s appears in series with branch 4 of some network, and part (b) shows the resultant change in geometry and source arrangement which is thereby implied. With reference to the given situation in part (a) it is at once evident that branch 4 becomes trivial since its current is identical with the

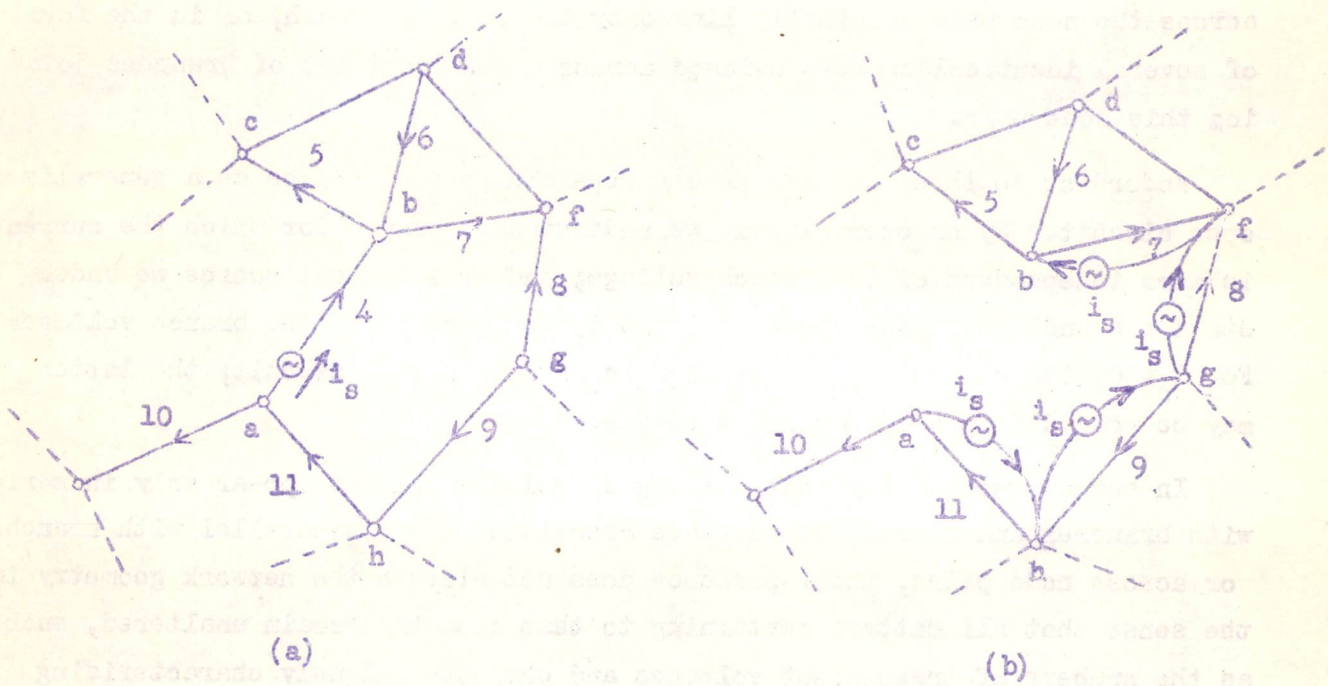


Fig. 8

source current and hence is known. It is also evident that the effect of the current source i_s upon the rest of the network is the same as though there had been no branch linking nodes a and b through which the source is applied. We can, therefore, regard the current source to be bridged across the node pair a-b in a modified graph in which branch 4 is absent.

A further step that results in having all current sources in parallel with branches may be carried out as shown in part (b) of Fig. 8. The equivalence of the four identical current sources i_s bridged across branches

11, 9, 8, 7, with a single source i_s bridged across the node pair a-b is evident by inspection since the same amount of source current still leaves node a and enters node b, while no net source current enters or leaves the nodes f, g, and h.

We may conclude that inserting a current source in series with a branch has the same effect upon the network geometry as does the open-circuiting or the removal of that branch. In this altered network the source appears bridged across the node pair originally linked by the removed branch, or in the form of several identical sources bridged across a confluent set of branches joining this node pair.

According to these results we may regard a current source as a generalized open circuit. By an open circuit we understand a branch for which the current is zero independent of the branch voltage; and by a current source we understand a branch for which the current is i_s independent of the branch voltage. For $i_s = 0$, the current source is identical with an open circuit; the latter may be regarded as a dead current source.

In summary we may say that so long as voltage sources appear only in series with branches, and current sources are associated only in parallel with branches or across node pairs, their presence does not disturb the network geometry in the sense that all matters pertaining to that geometry remain unaltered, such as the numbers of independent voltages and currents uniquely characterizing the state of the network, or their algebraic relations to the branch currents and voltages. In a sense, the open-circuit character of a current source and the short-circuit character of a voltage source become evident here as they do in the reasoning of the immediately preceding paragraphs.

On the other hand, we see that the network geometry is affected whenever a current source is placed in series with a branch or a voltage source in parallel with one. In both cases the branch in question becomes trivial and can be removed, leaving in its place an open circuit if the inserted source is a current, and a short circuit if the inserted source is a voltage. After this revision in the geometry is carried out, the source appears either as a current in parallel with a branch (or with several branches), or as a voltage in series with a branch (or with several branches). These two source arrangements alone, therefore, are all that need to be considered in the following discussion.

Thus we may regard any branch in a network to have the structure shown in Fig. 9. Here the link a-b represents the passive branch without its associated voltage and current sources; that is to say, when the sources are zero (as they usually are for most of the branches in a network), then the branch reduces to this link a-b alone. However, we shall take the attitude at this point that any or all of the branches in a network may turn out to have the associated sources shown in Fig. 9. The network is thus regarded as a geometrical configuration of active instead of passive branches. This turn of events changes nothing with regard to all that has been said previously except the relations between branch voltages and branch currents (designated as the relations (b) in the summary of Art. 3 regarding the formulation of equilibrium equations).

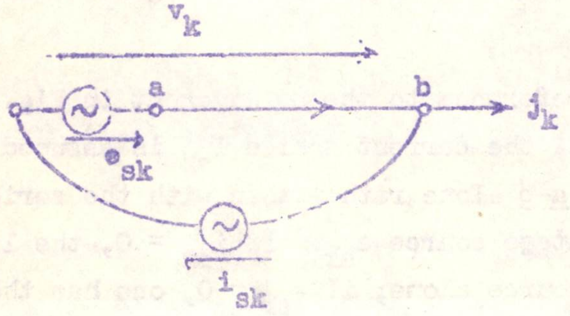


Fig. 9

Since v_k and j_k denote the net voltage drop and the net current in branch k, the voltage drop and current in the passive link a-b (noting the reference arrows in Fig. 9) are $(v_k + e_{sk})$ and $(j_k + i_{sk})$ respectively. These are the quantities that are related by the passive circuit element which the branch represents. If the functional relationship between voltage drop and current in the passive link is formally denoted by $v = z(j)$ or $j = y(v)$, we have for the general active branch of Fig. 9

$$(v_k + e_{sk}) = z(j_k + i_{sk}),$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (43)

$$(j_k + i_{sk}) = y(v_k + e_{sk}).$$

In a resistance branch, the notation z(j) reduces simply to a multiplication of the current j by the branch resistance, and y(v) denotes a multiplication of the voltage drop v by the branch conductance. In capacitive or inductive branches the symbols z(j) and y(v) also involve time differentiation or integration, as will be discussed in detail later on when circuits involving these elements are considered. For the moment it will suffice to visualize the significance of Eqs. 43 with regard to resistance elements alone.

It may be mentioned, with reference to the arrangement in Fig. 9, that the same results are obtained if the current source i_{sk} is assumed to be in parallel with the passive link a-b alone rather than with the series combination of this link and the voltage source e_{sk}. If $i_{sk} = 0$, the link is activated by a series voltage source alone; if $e_{sk} = 0$, one has the representation of a passive branch activated by a current source alone. For $e_{sk} = i_{sk} = 0$, the arrangement reduces to the usual passive branch. Thus the volt-ampere relations 43 are sufficiently general to take care of any functional dependence between net branch voltages and currents that can arise in the present discussions.

The method of including the effect of sources in the derivation of equilibrium equations is now easily stated. Namely, one proceeds precisely as described in the previous articles for the unactivated network except that the relations between branch voltages and branch currents are considered in the form of Eqs. 43, so as to take account of the presence of any voltage or current sources. This statement applies alike to the determination of equilibrium equations on the loop or node basis. Thus, regardless of the nature and distribution of sources throughout the network, the procedure remains straightforward and is essentially the same as for the unexcited network.

8. Summary of the procedures for deriving equilibrium equations.

At this point it is effective to bring together in compact symbolic form the steps involved in setting up equilibrium equations. Thus we have on the **loop basis:**

(a) The Kirchhoff voltage-law equations in terms of branch voltages

$$\Sigma \pm v_k = 0 \tag{44}$$

(b) The relations between branch voltages and branch currents (Eqs. 43)

$$v_k = -e_{sk} + z(j_k + i_{sk}) \tag{45}$$

(c) The branch currents in terms of the loop currents

$$j_k = \Sigma \pm i_r. \tag{46}$$

The rows of a tie set schedule (like 13, for example) place in evidence the Kirchhoff equations 44, while the columns of this schedule yield the branch currents in terms of the loop currents, Eqs. 46. The expressions for the v_k's in terms of the j_k's, Eqs. 45, are obtained from a knowledge of the circuit parameters and the associated voltage and current sources, as illustrated in Fig. 9.

The desired equilibrium equations are the Kirchhoff Eqs. 44 expressed in terms of the loop currents. One accomplishes this end through substituting the j_k's given by Eqs. 46 into Eqs. 45, and the resulting expressions for v_k into Eqs. 44. Noting that the linearity of the network permits one to write $z(j_k + i_{sk}) = z(j_k) + z(i_{sk})$, the result of this substitution among Eqs. 44, 45, 46 leads to

$$\Sigma \pm z(\Sigma \pm i_r) = \Sigma \pm \left\{ e_{sk} - z(i_{sk}) \right\} = e_{s\ell}. \tag{47}$$

Interpretation of this formidable looking result is aided through pointing out that $z(\Sigma \pm i_r)$ represents the passive voltage drop in any branch <u>k</u> due to the superposition of loop currents i_r in that branch, and that the left-hand side of Eq. 47 is the algebraic summation of such passive branch voltage drops around a typical closed loop ℓ. The right-hand side, which is abbreviated by the symbol $e_{s\ell}$, is the net apparent source voltage acting in the same loop. It is given by an algebraic summation of the voltage sources present in the branches comprising this closed contour (tie set) and the additional voltages induced in these branches by current sources that may simultaneously be associated with them. The latter voltages, which are represented by the term $-z(i_{sk})$, must depend upon the circuit parameter relations in the same way as do the passive voltage drops caused by the loop currents except that their algebraic signs are reversed because they are rises.

Thus the resulting equilibrium Eqs. 47 state the logical fact that the net passive voltage drop on any closed contour must equal the net active voltage rise on that contour. If we imagine that the loops are determined through selecting a tree and identifying the link currents with loop currents, then we can interpret the source voltages $e_{s\ell}$ as equivalent link voltages in the sense that if actual voltage sources having these values are placed in the links and all original current and voltage sources are removed, the resulting loop currents remain the same. Or we can say that if the negatives of the voltages $e_{s\ell}$ are placed in the links, then the effect of all other sources becomes neutralized, and the resulting network response is zero; that is, the loop currents or link currents are zero, the same as they would be if all links were opened.

Hence we have a physical interpretation of the $e_{s\ell}$ in that they may be regarded as the negatives of the voltages appearing across gaps formed by opening all the links. In many situations to which the simplified procedure discussed in Art. 6 is relevant, this physical interpretation of the net excitation quantities $e_{s\ell}$ suffices for their determination by inspection of the given network.

An entirely analogous procedure and corresponding process of physical interpretation applies to the derivation of equilibrium equations on the node basis. Here one has:

(a) The Kirchhoff current-law equations in terms of branch currents

$$\sum \pm j_k = 0 \tag{48}$$

(b) The relations between the branch currents and branch voltages (Eqs. 43)

$$j_k = -i_{sk} + y(v_k + e_{sk}) \tag{49}$$

(c) The branch voltages in terms of the node-pair voltages

$$v_k = \sum \pm e_r . \tag{50}$$

The rows of a cut-set schedule (like 20, for example) place in evidence the Kirchhoff equations 48, while the columns of this schedule yield the branch voltages in terms of the node-pair voltages, Eqs. 50. The expressions for the j_k's in terms of the v_k's, Eqs. 49, are obtained from a knowledge of the circuit parameters and the associated voltage and current sources, as illustrated in Fig. 9.

The desired equilibrium equations are the Kirchhoff Eqs. 48 expressed in terms of the node-pair voltages. One obtains this end through substituting the v_k's given by Eqs. 50 into Eqs. 49, and the resulting expressions for j_k into Eqs. 48. Noting that the linearity of the network permits one to write $y(v_k + e_{sk}) = y(v_k) + y(e_{sk})$, the result of this substitution among Eqs. 48, 49, 50 leads to

$$\Sigma \pm y(\Sigma \pm e_r) = \Sigma \pm \left\{ i_{sk} - y(e_{sk}) \right\} = i_{sn}. \qquad (51)$$

Interpretation of this formidable looking result is aided through recognizing that $y(\Sigma \pm e_r)$ represents the passive current in any branch k due to the algebraic sum of node-pair voltages e_r acting upon it, and hence the left-hand side of Eq. 51 is the summation of such branch currents in all branches of a typical cut set, for example, the set of branches divergent from a given node n if the node pair voltages are chosen as a node-to-datum set.

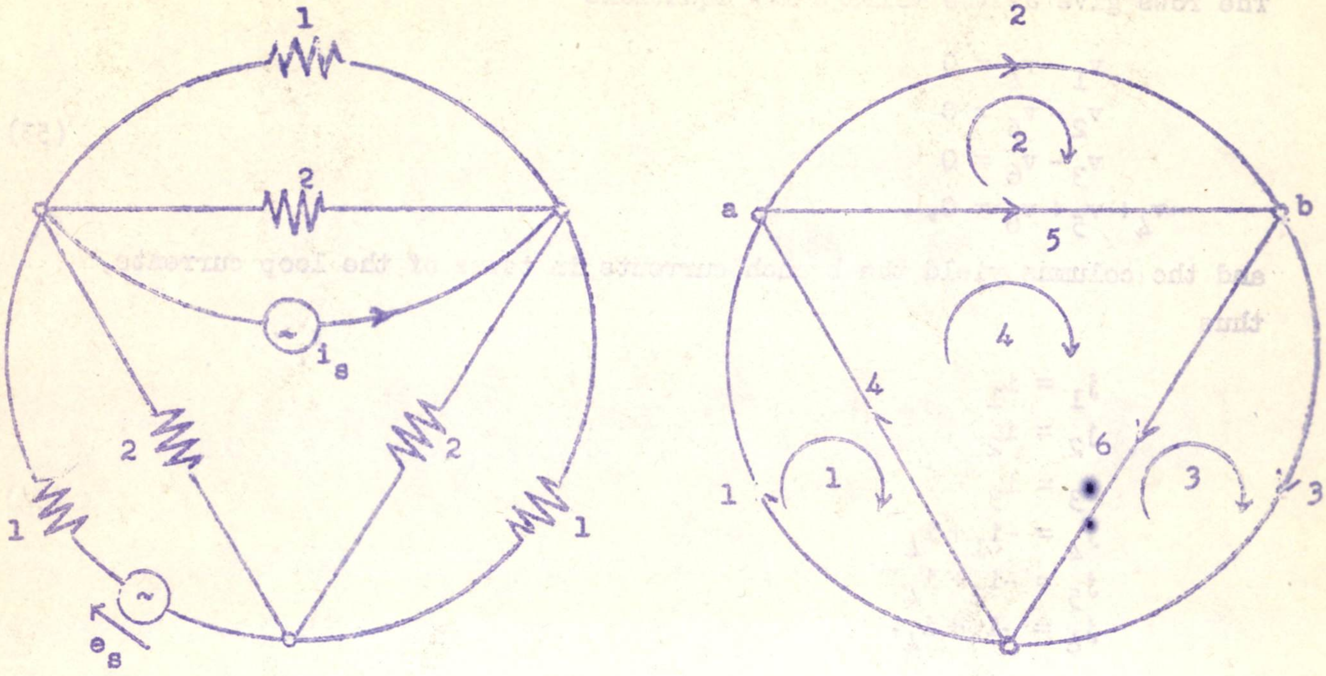
The right-hand side of Eq. 51, which is abbreviated by the symbol i_{sn}, is the net apparent source current for this cut set, for example, it is the net apparent source current entering node n in a node-to-datum situation. The net source current is given by an algebraic summation of the current sources associated with the branches comprising the pertinent cut set and the additional currents induced in these branches by voltage sources that may simultaneously be acting in them. The latter currents, which are represented by the term $-y(e_{sk})$, must depend upon the circuit parameter relations in the same way as do the passive currents caused by the node-pair voltages except that their algebraic signs are reversed because they represent a flow of charge <u>into</u> the cut set rather than out of it.

Thus the resulting equilibrium Eqs. 51 state the logical fact that the net current in the several branches of a cut set must equal the total source current feeding this cut set. If we imagine that the cut sets have been determined through selecting a tree and identifying the tree-branch voltages with node-pair voltages, then we can interpret the source currents i_{sn} as equivalent sources bridged across the tree branches in the sense that if actual current sources having these values are placed in parallel with the tree branches and all original current and voltage sources are removed, the resulting node-pair voltages remain the same. Or we can say that if the negatives of the currents i_{sn} are placed across the tree branches, then the effect of all other sources becomes neutralized, and the resulting network response is zero; that is, the node pair voltages or tree-branch voltages are zero, the same as they would be if all tree branches were short-circuited.

Hence we have a physical interpretation of the i_{sn} in that they may be regarded as the negatives of the currents appearing in short circuits placed across all the tree branches. In a node-to-datum choice of node pairs, the i_{sn} may be regarded as the negatives of the currents appearing in a set of short circuits placed across these node pairs, and a node-to-datum set of current sources having these values can be used in place of the original voltage and current sources in computing the desired network response. In many situations to which the simplified procedure discussed in Art. 6 is relevant, this physical interpretation of the net excitation quantities i_{sn} suffices for their determination by inspection of the given network.

9. Examples

The complete procedure for setting up equilibrium equations will now be illustrated for several specific examples. Consider first the resistance network of Fig. 10. The element values in part (a) are in ohms, and the source values are : i_s = 10 amperes, e_s = 5 volts (both constant). In part (b) of the same figure is shown the graph with its branch numbering and a choice of meshes to define loop currents.



(a) (b)

Fig. 10

The tie-set schedule corresponding to this choice reads

ℓ \ b	1	2	3	4	5	6
1	1	0	0	-1	0	0
2	0	1	0	0	-1	0
3	0	0	1	0	0	-1
4	0	0	0	1	1	1

(52)

The rows give us the voltage-law equations

$$\begin{aligned} v_1 - v_4 &= 0 \\ v_2 - v_5 &= 0 \\ v_3 - v_6 &= 0 \\ v_4 + v_5 + v_6 &= 0, \end{aligned} \qquad (53)$$

and the columns yield the branch currents in terms of the loop currents, thus

$$\begin{aligned} j_1 &= i_1 \\ j_2 &= i_2 \\ j_3 &= i_3 \\ j_4 &= -i_1 + i_4 \\ j_5 &= -i_2 + i_4 \\ j_6 &= -i_3 + i_4. \end{aligned} \qquad (54)$$

These correspond respectively to the Eqs. 44 and 46 in the above summary.

With regard to the Eqs. 45 relating branch voltages to branch currents, we observe that if we associate the current source with branch 5 (we could alternately associate it with branch 2), then all branches except 1 and 5 are passive and no special comment is needed for them. The net voltage drop in branch 1 is $v_1 = -e_s + j_1$, and the net current in the arrow direction in branch 5 is $j_5 = i_s + (v_5/2)$, the term $(v_5/2)$ being the current in the 2-ohm resistance which is the passive part of this branch. Noting the source values given above, the relations expressing net branch voltage drops in terms of net branch currents read

$$\begin{aligned} v_1 &= j_1 - 5 \\ v_2 &= j_2 \\ v_3 &= j_3 \\ v_4 &= 2j_4 \\ v_5 &= 2j_5 - 20 \\ v_6 &= 2j_6. \end{aligned} \qquad (55)$$

The relations involving the active branches are seen to contain terms that are independent of current.

The desired equilibrium equations are found through substitution of Eqs. 54 into 55, and the resulting expressions for the v's into the voltage-law equations 53. After proper arrangement this gives

$$\begin{aligned} 3i_1 + 0i_2 + 0i_3 - 2i_4 &= 5 \\ 0i_1 + 3i_2 + 0i_3 - 2i_4 &= -20 \\ 0i_1 + 0i_2 + 3i_3 - 2i_4 &= 0 \\ -2i_1 - 2i_2 - 2i_3 + 6i_4 &= 20. \end{aligned} \tag{56}$$

These are readily solved for the loop currents. One finds

$$i_1 = 5;\ i_2 = -10/3,\ i_3 = 10/3,\ i_4 = 5, \tag{57}$$

whence substitution into Eqs. 54 yields all the branch currents

$$j_1 = 5,\ j_2 = -10/3,\ j_3 = 10/3,\ j_4 = 0,\ j_5 = 25/3,\ j_6 = 5/3. \tag{58}$$

The value of j_5 is the net current in branch 5. That in the passive part of this branch is smaller than j_5 by the value of the source current, and hence is $(25/3) - 10 = -5/3$.

Now let us solve the network given in Fig. 10 by the node method, choosing as node-pair voltages the potentials of nodes a and b respectively, with the bottom node as a reference. The appropriate cut set schedule reads

n \ b	1	2	3	4	5	6
1	-1	1	0	-1	1	0
2	0	-1	1	0	-1	1

(59)

The rows give us the current-law equations

$$\begin{aligned} -j_1 + j_2 - j_4 + j_5 &= 0 \\ -j_2 + j_3 - j_5 + j_6 &= 0, \end{aligned} \tag{60}$$

and the columns yield the branch voltages in terms of the node-pair voltages, thus

$$
\begin{aligned}
v_1 &= -e_1 \\
v_2 &= e_1 - e_2 \\
v_3 &= e_2 \\
v_4 &= -e_1 \\
v_5 &= e_1 - e_2 \\
v_6 &= e_2.
\end{aligned} \tag{61}
$$

These correspond respectively to Eqs. 48 and 50 in the above summary.

Regarding Eqs. 49 relating the branch currents to the branch voltages, we note as before that $j_1 = v_1 + e_s$ and $j_5 = i_s + 0.5v_5$, so that the complete set of these equations reads

$$
\begin{aligned}
j_1 &= v_1 + 5 \\
j_2 &= v_2 \\
j_3 &= v_3 \\
j_4 &= 0.5v_4 \\
j_5 &= 0.5v_5 + 10 \\
j_6 &= 0.5v_6,
\end{aligned} \tag{62}
$$

which are simply the inverse of Eqs. 55.

The desired equilibrium equations are found through substitution of Eqs. 61 into 62, and the resulting expressions for the j's into the current-law equations 60. After proper arrangement one finds

$$
\begin{aligned}
3e_1 - 1.5e_2 &= -5 \\
-1.5e_1 + 3e_2 &= 10.
\end{aligned} \tag{63}
$$

The solution is readily found to be

$$e_1 = 0, \; e_2 = 10/3, \tag{64}$$

and the branch voltages are then computed from Eqs. 61 to be

$$v_1 = 0, \; v_2 = -10/3, \; v_3 = 10/3, \; v_4 = 0, \; v_5 = -10/3, \; v_6 = 10/3. \tag{65}$$

With regard to branch 1 it must be remembered that the value of v_1 is for the total branch, including the voltage source. The drop in the passive part, therefore, is 5 volts.

As a second example we shall consider the network graph shown in Fig. 11a. The sources in series with the branches are voltages having the

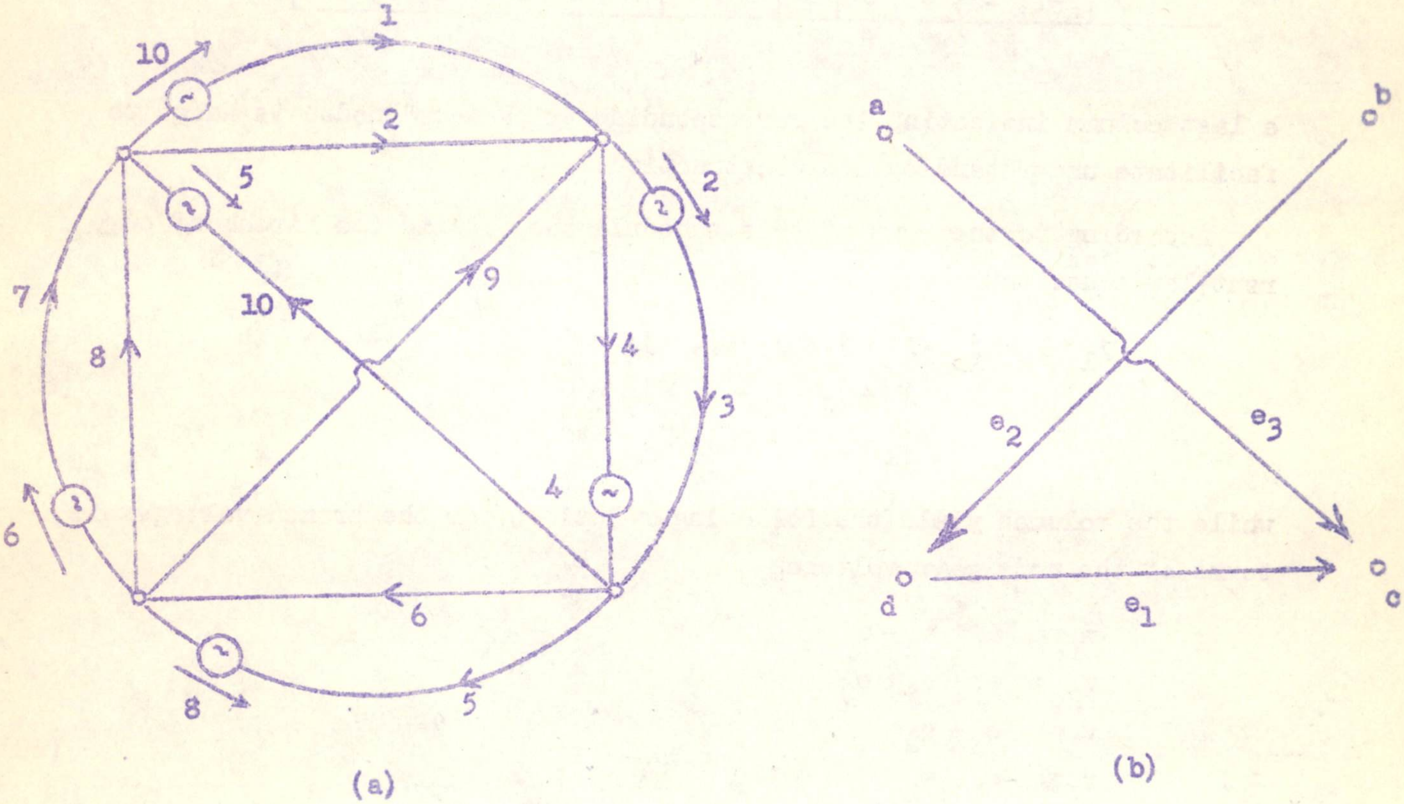


Fig. 11

values indicated. Since for this graph, $b = 10$, $n = 3$, and $\ell = 7$, it will be advantageous to choose the node method. A geometrical specification of node-pair voltages is shown in part (b) of the same figure. In the following cut-set schedule pertaining to this choice of node pairs

n \ b	1	2	3	4	5	6	7	8	9	10	picked-up nodes
1	1	1	-1	-1	1	1	-1	-1	0	0	a, c
2	1	1	-1	-1	0	0	0	0	1	0	a, c, d
3	-1	-1	0	0	0	0	1	1	0	1	b, c, d

(66)

a last column indicating the corresponding "picked-up" nodes is added to facilitate understanding its construction.

According to the rows of this schedule one obtains the Kirchhoff current-law equations

$$\begin{aligned} j_1 + j_2 - j_3 - j_4 + j_5 + j_6 - j_7 - j_8 &= 0 \\ j_1 + j_2 - j_3 - j_4 + j_9 &= 0 \\ -j_1 - j_2 + j_7 + j_8 + j_{10} &= 0, \end{aligned} \tag{67}$$

while the columns yield the following relations for the branch voltages in terms of the node pair voltages

$$\begin{aligned} v_1 &= e_1 + e_2 - e_3 \\ v_2 &= e_1 + e_2 - e_3 \\ v_3 &= -e_1 - e_2 \\ v_4 &= -e_1 - e_2 \\ v_5 &= e_1 \\ v_6 &= e_1 \\ v_7 &= -e_1 + e_3 \\ v_8 &= -e_1 + e_3 \\ v_9 &= e_2 \\ v_{10} &= e_3. \end{aligned} \tag{68}$$

The branches are again considered to be resistive. Let us assume for their conductances the following values in mhos

$$g_1 = 2,\ g_2 = 2,\ g_3 = 1,\ g_4 = 3,\ g_5 = 4,\ g_6 = 5,$$
$$g_7 = 1,\ g_8 = 3,\ g_9 = 2,\ g_{10} = 6. \tag{69}$$

The relations expressing the branch currents in terms of the net branch-voltage drops are then readily found through noting the appropriate expression for the drop in the passive part of each branch and multiplying this by the corresponding conductance. For example, the voltage drop in the passive part of branch 1 is $v_1 + 10$; in branch 3 it is $v_3 + 2$; in branch 5 it is $v_5 - 8$; and so forth. Thus we see that

$$\begin{aligned} j_1 &= 2v_1 + 20 \\ j_2 &= 2v_1 \\ j_3 &= v_3 + 2 \\ j_4 &= 3v_4 - 12 \\ j_5 &= 4v_5 - 32 \\ j_6 &= 5v_6 \\ j_7 &= v_7 + 6 \\ j_8 &= 3v_8 \\ j_9 &= 2v_9 \\ j_{10} &= 6v_{10} - 30. \end{aligned} \tag{70}$$

Substitution of the v's from Eqs. 68 into Eqs. 70, and the resulting expressions for the j's into Eqs. 67 gives the desired equilibrium equations. After proper arrangement these read

$$\begin{aligned} 21e_1 + 8e_2 - 8e_3 &= 8 \\ 8e_1 + 10e_2 - 4e_3 &= -30 \\ -8e_1 - 4e_2 + 14e_3 &= 44. \end{aligned} \tag{71}$$

Their solution yields

$$e_1 = 3.49,\quad e_2 = -4.22,\quad e_3 = 3.93, \tag{72}$$

from which the net branch-voltage drops may readily be computed using Eqs. 68, and the branch currents are then found from Eqs. 70.

CHAPTER X

Generalization of Circuit Equations and Energy Relations

1. Use of matrix algebra

Matrix algebra is a kind of shorthand that enables one to write algebraic relations involving systems of simultaneous equations in a very compact form. Its principal value lies in the circumspection that results from this compactness, and in the facility with which one is thus enabled to carry out and visualize the significance of more elaborate sequences of related algebraic operations. In numerical problems its usefulness lies solely in the systematization that it injects into the computations; it provides no short cuts. It, therefore, is primarily a tool for facilitating analytical manipulations, but as such its usefulness easily justifies the small amount of time and attention required on the part of the uninitiated reader to understand the basic principles and rules involved.

As has previously been pointed out, the so-called matrix corresponding to the set of linear equations

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n &= y_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n &= y_2 \\ &\;\;\vdots \\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n &= y_n, \end{aligned} \tag{1}$$

is written

$$[A]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \multicolumn{4}{c}{\text{- - - - - - - -}} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \tag{2}$$

and represents merely the array of coefficients a_{sk} in the same order regarding their row and column positions as they appear in the related set of systematically written equations. Unlike the determinant

$$A = \begin{bmatrix} a_{11} a_{12} \cdots a_{1n} \\ a_{21} a_{22} \cdots a_{2n} \\ - - - - - - - - \\ a_{n1} a_{n2} \cdots a_{nn} \end{bmatrix}, \tag{3}$$

which is a rational function of its elements a_{sk} and has definite numerical values for given values of these elements, the matrix [A] has no "value" other than the pictorial value that is provided by its outward appearance and structure in relation to the associated equations to which it belongs.

Usually the number of equations equals the number of unknowns (the quantities $x_1, x_2, \ldots x_n$), in which case the matrix has as many rows as it has columns. That is to say, the associated matrix is composed of a <u>square</u> array, and the number of its rows or columns is referred to as the <u>order</u> of the matrix. This circumstance is not necessarily always encountered. For example, in Ch. I where we discuss the relations between branch currents and loop currents or between branch voltages and node-pair voltages, we encounter sets of equations with nonsquare matrices. In this respect a matrix again differs from a determinant, since the latter must always involve a square array of coefficients.

An extreme example of a nonsquare matrix is one with only a single row or a single column. Thus we may write the sets of quantities $x_1 \ldots x_n$ and $y_1 \ldots y_n$, appearing in the Eqs. 1, as matrices

$$x] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad y] = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{4}$$

These are referred to as <u>column</u> matrices.

In terms of the matrices 2 and 4, the shorthand known as matrix algebra enables one to write the set of Eqs. 1 in the abbreviated form

$$[A] \cdot x] = y]. \tag{5}$$

In order to show how this expression may be regarded as the equivalent of 1, we must provide an appropriate interpretation for each of two things or questions: (a) What is meant by the equality of matrices? (b) How must one define the product of two matrices, like [A] . x]?

Regarding the first of these questions, it follows from what has been said about a matrix that two can be equal only if all of their corresponding elements are equal. A necessary (though not sufficient) condition for equality, therefore, is that both matrices have the same number of rows and the same number of columns. Since in 5, the right-hand matrix has only one column, it must turn out that the product [A] . x] be a matrix with a single column. More specifically, if 1 and 5 are to be equivalent then it becomes clear that we must have

$$\begin{bmatrix} a_{11} a_{12} \cdots a_{1n} \\ a_{21} a_{22} \cdots a_{2n} \\ - - - - - - - \\ a_{n1} a_{n2} \cdots a_{nn} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) \\ (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) \\ - - - - - - - - - - - - - \\ (a_{n1}x_1 + a_{n2}x_2 \cdots + a_{nn}x_n) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad (6)$$

in which the sums of terms in parentheses in the intermediate matrix are single elements, so that this is a column matrix like x] or y] in 4 (although its appearance at first glance does not suggest this fact). Equating elements in this matrix with corresponding ones in y] evidently yields the Eqs. 1.

The rule for matrix multiplication made evident in Eq. 6 may be described by saying that one multiplies the elements in the rows of [A] by the respective ones in the column of x] and adds the results; and that the first, second, etc. elements in the resultant (column) matrix involve the first, second, etc. rows of [A].

Stated in more general terms, one may say that in forming the product of two matrices [A] and [B], resulting in the matrix [C], one multiplies the rows of [A] by the columns of [B] in a manner that is most easily understood from the following example.

$$\begin{bmatrix} a_{11} a_{12} a_{13} a_{14} \\ a_{21} a_{22} a_{23} a_{24} \end{bmatrix} \times \begin{bmatrix} b_{11} b_{12} b_{13} \\ b_{21} b_{22} b_{23} \\ b_{31} b_{32} b_{33} \\ b_{41} b_{42} b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} c_{12} c_{13} \\ c_{21} c_{22} c_{23} \end{bmatrix} \quad (7)$$

in which one obtains for the elements of the product matrix

$$\begin{aligned} c_{11} &= a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} \\ c_{12} &= a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} \\ c_{13} &= a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33}+a_{14}b_{43}, \end{aligned} \quad (8)$$

$$\begin{aligned} c_{21} &= a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} \\ c_{22} &= a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} \\ c_{23} &= a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33}+a_{24}b_{43}. \end{aligned} \quad (9)$$

Thus the so-called (s,k)-element in the product matrix (the element c_{sk}) is formed through the addition of products of the respective elements in the sth row of [A] and the kth column of [B], a general formula for this element being

$$c_{sk} = \sum_{r=1}^{n} a_{sr}b_{rk}, \quad (10)$$

in which r is regarded as the summation index. The equivalence of 5 and 6, according to this rule of formation, is readily recognized, whereupon the acceptance of the matrix Eq. 5 as being a compact way of writing the set of Eqs. 1, follows without difficulty.

In determinant multiplication, by the way, one does not have to stick to the rule that only rows of the first may be multiplied by columns of the second determinant in the product A.B. One may equally well form elements in the product determinant through multiplying the columns of A by the rows of B, or rows by rows, or columns by columns. Any one of four different schemes may thus be used in determinant multiplication (albeit one must be consistent throughout the evaluation of a given problem). This freedom results from the fact that it is only the value of the product determinant that matters, and this value turns out to be the same with each one of the four schemes although the specific values for the elements in the product determinant are not the same. In matrix multiplication where the result is a matrix, only one rule of formation can apply, since the elements of this matrix are the quantities of interest.

Another point illustrated by the example of Eq. 7 is the fact that in any product [A] x [B], the number of columns in [A] must equal the number of rows in [B] in order that the number of elements in any row of [A] will equal the number of elements in any column of [B], a necessary condition that is obvious from the way in which the row-by-column product is formed. If the given matrices in a product fulfill this condition, then that product is said to be conformable.

It is likewise clear from the example in Eq. 7 that the number of rows in [A] and the number of columns in [B] may be anything; but that the number of rows in the product matrix [C] equals the number of rows in [A], and the number of columns in [C] equals the number of columns in [B]. If we use for the designation of a matrix [A] with p rows and q columns the notation $[a_{pq}]$, and a similar notation for the matrices [B] and [C], then these remarks are summarized in the equation

$$[a_{pq}] \times [b_{qr}] = [c_{pr}]. \qquad (11)$$

Conformability of this product is placed in evidence through the so-called adjacent index q being the same in the two matrices forming the product, while the remaining indexes p and r correlate the rows of $[a_{pq}]$ with the rows of $[c_{pr}]$ and the columns of $[b_{qr}]$ with the columns of $[c_{pr}]$.

Using this notation, a conformable multiple product is, for example, indicated by

$$[a_{pq}] \times [b_{qr}] \times [c_{rs}] \times [d_{st}] = [g_{pt}] . \qquad (12)$$

Conformability is recognized at once from the fact that all adjacent indexes are alike, and the number of rows and columns in the resultant matrix is make evident from the number of rows in the first matrix and the number of columns in the last.

From the restricted nature of the rule for forming a matrix product, it is clear that the commutative law does not apply; that is

$$[A] \times [B] \neq [B] \times [A]. \qquad (13)$$

However, the associative law does hold; which means that, in the multiple product 12, we may group the terms in any way we wish so long as we preserve their relative order. Thus we may begin the multiplication at the right with $[c_{rs}] \times [d_{st}]$ and work towards the left in successive steps, or we may begin at the left and work towards the right. Again, we may first carry out separately the products $[a_{pq}] \times [b_{qr}]$ and $[c_{rs}] \times [d_{st}]$, and then multiply the result of the first of these by the result of the second. Although the final product matrix is the same in all cases, the computational labor involved is not (herein lies one of the finer points of this subject that we shall not pursue further at this time.)

A matrix is said to be symmetrical if its elements fulfill the condition $a_{ik} = a_{ki}$. If both matrices in the product $[A] \times [B]$ are symmetrical, then 13 obviously does not apply since there is no distinction between the respective elements of the kth row and those of the kth column in either [A] or [B].

The so-called <u>inverse</u> of a matrix [A] is written $[A]^{-1}$ and is defined by the relation

$$[A] \times [A]^{-1} = [A]^{-1} \times [A] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = [U], \qquad (14)$$

in which [U], called the unit matrix, has the indicated structure in which the elements on the principal diagonal are unity and all others are zero. Equations like the set 1 having a unit matrix evidently read: $x_1 = y_1$, $x_2=y_2$, $\ldots x_n=y_n$. Multiplication of any given matrix by the unit matrix leaves the given matrix unchanged. Hence if we multiply on both sides of Eq. 5 by $[A]^{-1}$, we have

$$[A]^{-1} \times [A] \times x]=[U]\times x]= x] = [A]^{-1} \times y]. \tag{15}$$

The set of equations corresponding to this result in the manner that 1 and 5 correspond, has the form

$$\begin{aligned} b_{11}y_1+b_{12}y_2+\ldots+b_{1n}y_n &= x_1 \\ b_{21}y_1+b_{22}y_2+\ldots+b_{2n}y_n &= x_2 \\ &- - - - - - - \\ b_{n1}y_1+b_{n2}y_2+\ldots+b_{nn}y_n &= x_n, \end{aligned} \tag{16}$$

with the matrix

$$[B]=[A]^{-1} = \begin{bmatrix} b_{11} & b_{12} & \ldots b_{1n} \\ b_{21} & b_{22} & \ldots b_{2n} \\ - - - & - - - & - - - \\ b_{n1} & b_{n2} & \ldots b_{nn} \end{bmatrix}. \tag{17}$$

The inverse matrix is thus recognized to be the matrix of the inverse set of equations; that is, the equations that represent the solution to the given set. The property of being inverse is evidently a mutual one. Thus we may equally well regard the set 1 as being the inverse of 16, and hence it follows that [A] is the inverse of [B]; that is

$$[A] = [B]^{-1}; [A] \times [B] = [B] \times [A] = [U]. \tag{18}$$

The problem of finding the inverse of a given matrix is the problem of solving a set of simultaneous equations like 1 or 16. Using determinants and Cramer's rule, one can compactly express the elements of the inverse matrix in terms of those of a given matrix. For example, if the determinant of [A] is denoted by A, as in Eq. 3, and its cofactors are written A_{sk}, it follows (see art. 2, Ch. III) that the elements of the inverse matrix [B], Eq. 17, are given by

$$b_{sk} = \frac{A_{ks}}{A} . \tag{19}$$

Conversely, if the determinant of [B] is B, with cofactors B_{sk}, then

$$a_{sk} = \frac{B_{ks}}{B} . \tag{20}$$

Because of 18, the formula 10 for the elements of the product matrix yields in this case

$$\sum_{r=1}^{n} a_{sr} \cdot b_{rk} = \begin{cases} 1 \text{ for } s = k \\ 0 \text{ for } s \neq k \end{cases} \tag{21}$$

because the product matrix is the unit matrix 14.

From the relations 19 and 20 it is clear that the inverse of a matrix exists only if the corresponding determinant is non-zero. Incidentally, there must be a corresponding determinant in the first place, which implies that the given matrix be a square array. A nonsquare matrix possesses no inverse. A matrix that does possess an inverse is said to be nonsingular; one that does not, is called a singular matrix.

When a given matrix has the so-called diagonal form

$$[D] \quad \begin{bmatrix} d_{11} & 0 & 0 & 0 \dots & 0 \\ 0 & d_{22} & & 0 \dots & 0 \\ - - - & - - - & - - - & - - - & - - - \\ 0 \; 0 & 0 & & 0 \dots & d_{nn} \end{bmatrix} \tag{22}$$

which is like the unit matrix except that the diagonal elements are not equal to unity, then the associated equations read $d_{11}x_1 = y$, $d_{22}\ x_2 = y_2, \ldots d_{nn}x_n = y_n$, which can be inverted by inspection. One recognizes that the inverse of [D] is simply

$$[D]^{-1} = \begin{bmatrix} d_{11}^{-1} & 0 & 0 \ldots 0 \\ 0 & d_{22}^{-1} & 0 \ldots 0 \\ - & - & - \\ 0 & 0 & 0 \ldots d_{nn}^{-1} \end{bmatrix} \qquad (23)$$

That is to say, the inverse of a diagonal matrix is again a diagonal matrix with elements on its diagonal that are respectively the reciprocals of the diagonal elements in the given matrix.

Alteration of a matrix through writing its rows as columns, or vice versa, is called transposition, and the result is referred to as the transposed matrix. Thus the transposition of the matrix [A], Eq. 2, yields

$$[A]_t = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \ldots a_{n1} \\ a_{12} & a_{22} & a_{32} & \ldots a_{n2} \\ - & - & - & - \\ a_{1n} & a_{2n} & a_{3n} & \ldots a_{nn} \end{bmatrix}. \qquad (24)$$

Transposition of the column matrices 4 yields row matrices

$$\begin{aligned} x]_t &= \lfloor x \rfloor = [x_1\ x_2 \ldots x_n], \\ y]_t &= \lfloor y \rfloor = [y_1\ y_2 \ldots y_n]. \end{aligned} \qquad (25)$$

Note that the transposed matrix is indicated by the subscript t, and that column and row matrices are distinguished through writing x] and $\lfloor x \rfloor$ respectively.

When matrices have a large number of rows and columns, it may be effective to partition them into smaller sections, called submatrices. Such a partitioned matrix may be written

$$[A] = [a_{mn}] = \left[\begin{array}{c:c} [a_{pr}] & [a_{ps}] \\ \hdashline [a_{qr}] & [a_{qs}] \end{array}\right] . \qquad (26)$$

Thus the m rows and n columns of the matrix [A] are partitioned into groups of p and q rows, and r and s columns. That portion of [A] consisting of the elements in the first p rows and first r columns (the upper left portion) is the submatrix $[a_{pr}]$; that portion involving elements in the first p rows and the last s columns (the upper right portion) is the submatrix $[a_{ps}]$, and so forth.

If a second matrix [B] is likewise partitioned as shown by

$$[B] = [b_{nk}] = \left[\begin{array}{c:c} [b_{rt}] & [b_{ri}] \\ \hdashline [b_{st}] & [b_{si}] \end{array}\right] . \qquad (27)$$

Then the product [A] x [B] may be evaluated as though the submatrices were ordinary elements. That is to say, Eqs. 26 and 27 yield the product

$$[A] \times [B] = \left[\begin{array}{c:c} [a_{pr}] \times [b_{rt}] + [a_{ps}] \times [b_{st}] & [a_{pr}] \times [b_{ri}] + [a_{ps}] \times [b_{si}] \\ \hdashline [a_{qr}] \times [b_{rt}] + [a_{qs}] \times [b_{st}] & [a_{qr}] \times [b_{ri}] + [a_{qs}] \times [b_{si}] \end{array}\right], \qquad (28)$$

which in turn may be written

$$[A] \times [B] = [C] = \left[\begin{array}{c:c} [c_{pt}] & [c_{pi}] \\ \hdashline [c_{qt}] & [c_{qi}] \end{array}\right] \qquad (29)$$

Note that the partitioning of the columns of [A] and of the rows of [B] (both into groups of r and s) must correspond in order that the products of submatrices appearing in 28 all be conformable. The partitioning of the rows of [A] and of the columns in [B] is arbitrary. From the indexes appearing on the submatrices in [A] and in [B] one can tell the number of rows and columns in the submatrices of the product matrix 29. Partitioning thus lends circumspection where detailed manipulations with elaborate matrices must be carried out.

It may happen that a large number of elements in a rather extensive given matrix are zero so that, after partitioning, one encounters submatrices consisting entirely of zeros. A matrix whose elements are all zeros is called a <u>null</u> matrix. A partitioned matrix having such null submatrices may have the form

$$[A] = \left[\begin{array}{c:c:c} [a_{rr}] & 0 & 0 \\ \hdashline 0 & [a_{ss}] & 0 \\ \hdashline 0 & 0 & [a_{tt}] \end{array}\right] , \qquad (30)$$

where we have written the null matrices as though they were simple zeros. Actually the two zeros in the top row of the matrix 30 are null matrices involving r rows and respectively s and t columns. Similar comment applies to the other zeros appearing in this matrix. The matrix [A] in this example is chosen to be a square array consisting of r+s+t rows and a like number of columns.

Regarding the submatrices in 30 for the moment as one would ordinary elements, we would recognize this matrix [A] to be in the diagonal form. It turns out (as may readily be seen) that its inverse, like that of the diagonal matrix 22, is simply given by

$$[A]^{-1} = \left[\begin{array}{c:c:c} [a_{rr}]^{-1} & 0 & 0 \\ \hdashline 0 & [a_{ss}]^{-1} & 0 \\ \hdashline 0 & 0 & [a_{tt}]^{-1} \end{array}\right] , \qquad (31)$$

which has the form of $[D]^{-1}$ in Eq. 23, although the diagonal members must be found by the method of matrix inversion because they are submatrices and not ordinary elements. However, it is useful in analytic work to be aware of this method of indicating the inverse of the matrix 30.

2. <u>Branch parameter matrices and volt-ampere relations</u>.

We wish now to reconsider the problem of setting up the differential equations expressing the equilibrium of a linear passive network, removing restrictions of any sort so that we will arrive at a formulation that is perfectly general. For certain theoretical considerations to be taken up later on, such an unrestricted point of view is essential, as may also be true in some practical situations.

Although the basis for a general procedure is given in Ch. II, it is there discussed specifically with reference to resistance networks. In order to remove this restriction, it is necessary that we show in detail how the volt-ampere relations pertaining to the branches (expressed formally by Eqs. 43 of Art. 7, Ch. II) may be evaluated for inductances and capacitances as well as for resistances. For the sake of convenience, these equations are repeated below

$$(v_k + e_{sk}) = z(j_k + i_{sk}), \tag{32}$$

$$(j_k + i_{sk}) = y(v_k + e_{sk}), \tag{33}$$

and Fig. 9 of Ch. II to which they refer, is reproduced here as Fig. 1. It depicts the most general form that a passive branch with associated voltage and current sources may take. Since j_k and v_k are the net current and voltage drop, the quantities (j_k+i_{sk}) and (v_k+e_{sk}) are seen to pertain to the passive element (resistance, inductance, or capacitance) alone.

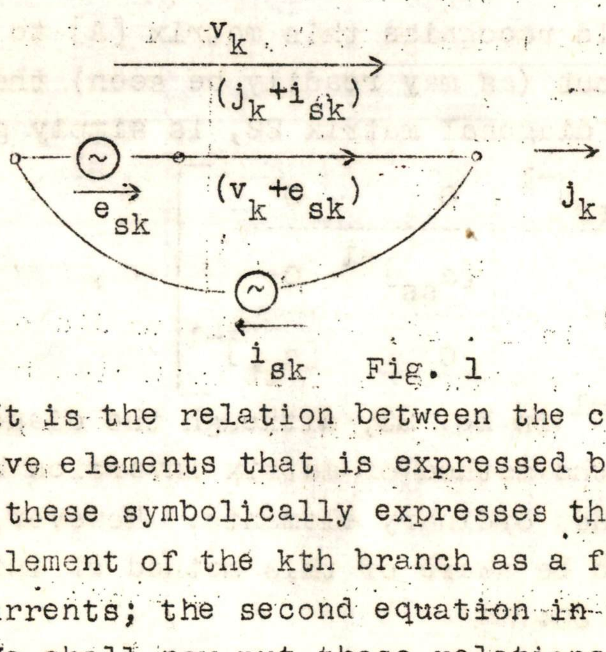


Fig. 1

It is the relation between the currents and voltages in the passive elements that is expressed by the Eqs. 32 and 33. The first of these symbolically expresses the voltage drop in the passive element of the kth branch as a function of the passive branch currents; the second equation in this pair does the reverse. We shall now put these relations into a more explicit form.

The passive branches (single elements) in the network are assumed to be numbered consecutively from 1 to b. Of this total number of b branches let us say that λ are inductive, ρ are resistive, and σ are elastive, $\lambda+\rho+\sigma$ being equal to b. The numbering of the branches, moreover, is carried out in such a fashion that numbers 1 to λ refer to inductances, numbers $\lambda+1$ to $\lambda+\rho$ refer to resistances, and numbers $\lambda+\rho+1$ to $\lambda+\rho+\sigma = b$ refer to elastances.

Since each inductive branch may be mutually coupled with every other branch in this group, the matrix of self and mutual inductance coefficients, according to the discussion in Art. 4 of Ch. VIII, called the <u>branch inductance matrix</u>, has the form

$$[\ell] = \begin{bmatrix} \ell_{11} & \ell_{12} \cdots \ell_{1\lambda} \\ \ell_{21} & \ell_{22} \cdots \ell_{2\lambda} \\ - - - & - - - \\ \ell_{\lambda 1} & \ell_{\lambda 2} \cdots \ell_{\lambda\lambda} \end{bmatrix} \qquad (34)$$

The resistance and elastance branches, on the other hand, cannot be mutually coupled; their parameter matrices must have the diagonal form. Hence the <u>branch resistance matrix</u> is given by

$$[r] = \begin{bmatrix} r_{\lambda+1} & 0 & 0 \cdots 0 \\ 0 & r_{\lambda+2} & 0 \cdots 0 \\ - - - & - - - & - - - \\ 0 & 0 & 0 \cdots r_{\lambda+\rho} \end{bmatrix} \qquad (35)$$

and the <u>branch elastance matrix</u> is written

$$[s] = \begin{bmatrix} s_{\lambda+\rho+1} & 0 & 0 \cdots 0 \\ 0 & s_{\lambda+\rho+2} & 0 \cdots 0 \\ - - - & - - - & - - - \\ 0 & 0 & 0 \quad s_b \end{bmatrix} \qquad (36)$$

In these last two matrices, an element r_i or s_i is simply the resistance in ohms or the elastance in darafs of the single element (passive branch) to which the pertinent subscript refers.

The voltage drops in the passive elements in terms of the currents in these elements are expressed for the inductances by

$$(v_i+e_{si}) = \sum_{k=1}^{\lambda} \ell_{ik}\, p(j_k+i_{sk});\ i = 1, 2, \ldots \lambda\,, \tag{37}$$

for the resistances by

$$(v_i+e_{si}) = r_i(j_i+i_{si});\ i = \lambda + 1, \ldots \lambda + \rho\,, \tag{38}$$

and for the elastances by the equations

$$(v_i+e_{si}) = s_i p^{-1}\,(j_i+i_{si});\ i = \lambda + \rho + 1, \ldots b, \tag{39}$$

in which the abbreviations

$$p = \frac{d}{dt} \text{ and } p^{-1} = \int dt \tag{40}$$

are used to denote the operations of differentiation and integration respectively.

Because of the possibility of mutual coupling between inductive branches, each passive voltage drop (v_i+e_{si}) in one of these (i.e., for $i = 1, 2, \ldots \lambda$) depends in general upon all of the passive currents (j_k+i_{sk}) in these branches. That is why Eqs. 37 involve a summation extending over the inductive branches (reference to Eqs. 30 in Art. 4, Ch. VIII may also be helpful in the interpretation of 37). Each of the Eqs. 38 and 39, in contrast, involves only a single term on the right-hand side because the voltage drop in a resistance or elastance depends upon the current in that branch alone.

The relations 37, 38, and 39 may be combined into a single matrix equation through defining the column matrices

$$[v+e_s] = \begin{bmatrix} v_1+e_{s1} \\ v_2+e_{s2} \\ \vdots \\ v_b+e_{sb} \end{bmatrix} \quad \text{and} \quad [j+i_s] = \begin{bmatrix} j_1+i_{s1} \\ j_2+i_{s2} \\ \vdots \\ j_b+i_{sb} \end{bmatrix} \tag{41}$$

and the branch operator matrix

$$[D] = \left[\begin{array}{c:c:c} [\ell]p & 0 & 0 \\ \hdashline 0 & [r] & 0 \\ \hdashline 0 & 0 & [s]p^{-1} \end{array}\right] \tag{42}$$

in which the branch parameter matrices 34, 35, and 36 are embedded as submatrices (the matrix [D] is written in partitioned form). The scalar operators p and p^{-1} become associated with each element in the matrices $[\ell]$ and [s], since multiplication of a matrix by a scalar, multiplies each element in the matrix by that scalar (as may be seen from the fact that the matrix must vanish if the scalar is zero, and a matrix vanishes only when all elements are zero).

The desired relations expressing the voltage drops in the passive elements in terms of the currents in these elements (equivalent to Eqs. 37, 38, and 39 combined) are given by the single matrix equation

$$[v+e_s] = [D] \times [j+i_s]. \tag{43}$$

Through carrying out the indicated matrix operations one thus obtains an explicit evaluation of the symbolic Eq. 32. (the uninitiated reader should write out the matrix [D] completely and carry through the indicated multiplication in order to understand this result and appreciate its simplicity.)

The inverse relations 33 may similarly be evaluated. To this end the branch parameter matrices are considered in their inverse forms as the reciprocal inductance matrix

$$[\gamma] = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1\lambda} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2\lambda} \\ - & - & - & - \\ \gamma_{\lambda 1} & \gamma_{\lambda 2} & \cdots & \gamma_{\lambda\lambda} \end{bmatrix} = [\ell]^{-1}, \qquad (44)$$

the conductance matrix

$$[g] = \begin{bmatrix} g_{\lambda+1} & 0 & 0 \cdots & 0 \\ 0 & g_{\lambda+2} & 0 \cdots & 0 \\ - & - & - & - \\ 0 & 0 & 0 \cdots & g_{\lambda+\rho} \end{bmatrix} = [r]^{-1} \qquad (45)$$

and the capacitance matrix

$$[c] = \begin{bmatrix} c_{\lambda+\rho+1} & 0 & 0 \cdots & 0 \\ 0 & c_{\lambda+\rho+2} & 0 \cdots & 0 \\ - & - & - & - \\ 0 & 0 & 0 \cdots & c_b \end{bmatrix} = [s]^{-1}. \qquad (46)$$

Since the last two are diagonal matrices, their diagonal elements are simply the reciprocals of the diagonal elements in [r] and [s]. As shown in the preceding article, the elements in $[\gamma]$ are not so simply related to those in $[\ell]$. However, if the determinant of $[\ell]$ be Δ, with cofactors Δ_{sk}, then

$$\gamma_{sk} = \frac{\Delta_{ks}}{\Delta}, \qquad (47)$$

where the indexes s and k on either γ or Δ may be interchanged since the matrices in question are symmetrical.

In algebraic form, the desired inverse relations are expressed by

$$(j_i + i_{si}) = \sum_{k=1}^{\lambda} \gamma_{ik}\, p^{-1}(v_k + e_{sk});\; i = 1, 2, \ldots \lambda, \tag{48}$$

$$(j_i + i_{si}) = g_i(v_i + e_{si});\; i = \lambda + 1, \ldots \lambda + \rho, \tag{49}$$

and

$$(j_i + i_{si}) = c_i p(v_i + e_{si});\; i = \lambda + \rho + 1 \ldots b. \tag{50}$$

In matrix form they may be combined in a single equation through defining the operator matrix (inverse to [D])

$$[D]^{-1} = \left[\begin{array}{c|c|c} [\gamma]p^{-1} & 0 & 0 \\ \hline 0 & [g] & 0 \\ \hline 0 & 0 & [c]\,p \end{array}\right], \tag{51}$$

whereupon the expressions for the currents in the passive elements in terms of the voltage drops in these elements are given by the matrix equation

$$[j + i_s] = [D]^{-1} \times [v + e_s], \tag{52}$$

which is the inverse of 43, and represents the explicit evaluation of the symbolic Eq. 33.

In order to obtain the equilibrium equations we can now follow precisely the same pattern set in Art. 8 of Ch. II for resistance networks as summarized there by Eqs. 44, 45, 46 for the loop basis and by Eqs. 48, 49, 50 for the node basis. The central equation in each of these groups of three, expresses the volt-ampere relations for the branches, which we have just finished putting into the matrix forms 43 and 52. The first and last equation in each group expresses respectively the pertinent Kirchhoff law and the branch variables (current or voltage) in terms of the loop or node variables. These relations involve the tie-set and cut-set schedules. We shall next show how these may conveniently be written as matrix equations and combined through straightforward substitution with Eqs. 43 or 52 to obtain the desired results.

3. Equilibrium equations on the node basis.

The first step is to write the pertinent cut-set schedule in the form of a matrix, thus

$$[a] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1b} \\ a_{21} & a_{22} & \cdots & a_{2b} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nb} \end{bmatrix} \tag{53}$$

The elements in any row of this matrix are the coefficients in a Kirchhoff current-law equation since they pertain to the selection of a cut-set. Their values, therefore, will normally be either ± 1 or zero depending on whether a pertinent branch does or does not belong to the cut set defined by a given row. Each row has b elements; but there are only n rows since this is the number of independent cut sets or node pairs. It should be recalled from the discussion in Ch. I that $n=n_t-1$ where n_t equals the total number of nodes.

It is also shown in Ch. I that the elements in any column of the matrix $[a]$ are coefficients in an equation expressing the pertinent branch-voltage drop in terms of the node-pair voltages that are consistent with cut sets defined by the rows. Therefore if we write column matrices

$$[j] = \begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ j_b \end{bmatrix} \quad \text{and} \quad [v] = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_b \end{bmatrix} \tag{54}$$

for the net branch currents and voltage drops, and express similarly the associated current and voltage sources as

$$[i_s] = \begin{bmatrix} i_{s1} \\ i_{s2} \\ \vdots \\ i_{sb} \end{bmatrix} \quad \text{and} \quad [e_s] = \begin{bmatrix} e_{s1} \\ e_{s2} \\ \vdots \\ e_{sb} \end{bmatrix}, \tag{55}$$

so that the column matrices 41 can be separated as indicated by

$$[j+i_s] = [j] + [i_s]; \quad [v+e_s] = [v] + [e_s]; \tag{56}$$

and if we write the node-pair voltage variables in the form of a column matrix

$$[e] = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \tag{57}$$

then the Kirchhoff-law equations are expressed in matrix form by

$$[\alpha] \times [j] = 0, \tag{58}$$

the volt-ampere relations 52 for the branches can be written

$$[j] + [i_s] = [D]^{-1} \times [v] + [D]^{-1} \times [e_s], \tag{59}$$

and the branch voltages in terms of the node-pair voltages are given by

$$[v] = [\alpha]_t \times [e]. \tag{60}$$

The desired equilibrium equations are the Kirchhoff-law Eqs. 58 expressed in terms of the node-pair voltages as variables. One obtains this result through substituting the expression for [v] from Eq. 60 into Eq. 59, and the ensuing relation for [j] into Eq. 58. After a slight rearrangement of the terms, one finds

$$[\alpha]\times[D]^{-1}\times[\alpha]_t\times[e] = [\alpha]\times([i_s] - [D]^{-1}\times[e_s]) = [i_n], \tag{61}$$

in which the right-hand side, representing the net equivalent current sources feeding the node pairs, is abbreviated as a column matrix

$$[i_n] = \begin{bmatrix} i_{n1} \\ i_{n2} \\ \vdots \\ i_{nn} \end{bmatrix} \tag{62}$$

It is interesting to note the structure of this matrix as expressed by Eq. 61. Thus the term $[i_s]$ within the parenthesis represents current sources associated with the branches, while the term involving $[D]^{-1} \times [e_s]$ represents the transformation of voltage sources associated with the branches (see Fig. 1) into equivalent current sources, the minus sign arising from the opposite reference arrows associated with e_{sk} and i_{sk}. The parenthesis expression, therefore represents net current sources for the branches, and is to be thought of as combining these into a single column matrix. Multiplication of the matrix $[\alpha]$ into this column matrix yields again a column matrix whose elements are algebraic sums of the branch sources according to the groups of branches forming cut sets (these are the branches associated with the pertinent node pairs). The elements in this resultant column matrix $[i_n]$ are thus seen to be equivalent node-pair current sources.

Interpretation of the left-hand side of Eq. 61 is facilitated through an appropriate evaluation of the triple matrix product $[\alpha] \times [D]^{-1} \times [\alpha]_t$. A preliminary step toward achieving this end is the partitioning of the columns in the matrix $[\alpha]$ into groups of λ, ρ, σ, thus

$$[\alpha] = [\alpha_{n\lambda} \;\vdots\; \alpha_{n\rho} \;\vdots\; \alpha_{n\sigma}] \,. \tag{63}$$

The submatrix $[\alpha_{n\lambda}]$ consists of the first λ columns in $[\alpha]$; $[\alpha_{n\sigma}]$ represents the succeeding group of ρ columns, and $[\alpha_{n\sigma}]$ contains the last σ columns. Using the form for $[D]^{-1}$ given in Eq. 51, one then finds

$$[\alpha]\times[D]^{-1}\times[\alpha]_t = [\alpha_{n\lambda} \;\vdots\; \alpha_{n\rho} \;\vdots\; \alpha_{n\sigma}] \times \begin{bmatrix} [\gamma]p^{-1} & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & cp \end{bmatrix} \times \begin{bmatrix} (\alpha_{n\lambda})_t \\ (\alpha_{n\rho})_t \\ (\alpha_{n\sigma})_t \end{bmatrix}$$

$$= [\alpha_{n\lambda}]\times[\gamma] \times [\alpha_{n\lambda}]_t\, p^{-1} + [\alpha_{n\rho}]\times[g]\times[\alpha_{n\rho}]_t + [\alpha_{n\sigma}] \times [c] \times [\alpha_{n\sigma}]_t\, p \tag{64}$$

The expressions

$$[\Gamma] = [\alpha_{n\lambda}] \times [\gamma] \times [\alpha_{n\lambda}]_t$$

$$[G] = [\alpha_{n\rho}] \times [g] \times [\alpha_{n\rho}]_t \qquad (65)$$

$$[C] = [\alpha_{n\sigma}] \times [c] \times [\alpha_{n\sigma}]_t$$

are recognized respectively as the <u>reciprocal inductance</u>, the conductance, and the <u>capacitance</u> parameter matrices pertaining to the node basis. In terms of these and the source matrix 62, the equilibrium equations 61 take the somewhat more familiar form

$$\boxed{([\Gamma]p^{-1} + [G] + [C]p) \times [e] = [i_n]} \qquad (66)$$

The principal results of the present discussion are the evaluation of the resultant source matrix $[i_n]$ as given in Eq. 61, and the expressions 65 for the pertinent parameter matrices. These are given in terms of the branch parameter matrices $[\gamma]$, $[g]$, and $[c]$ through triple matrix products with appropriate portions (and their transpositions) of the α-matrix characterizing the cut-set schedule. The formation of these key quantities appearing in the equilibrium Eqs. 66 is thus in every case reduced to a simple, systematic and straightforward procedure. The equations themselves are a set of simultaneous differential equations in terms of the instantaneous values of the variables.

4. <u>Equilibrium equations on the loop basis.</u>

The procedure is in form entirely analogous to that just described. The pertinent tie-set schedule is characterized by the matrix

$$[\beta] = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots\beta_{1b} \\ \beta_{21} & \beta_{22} & \cdots\beta_{2b} \\ \beta_{l1} & \beta_{l2} & \cdots\beta_{lb} \end{bmatrix} \qquad (67)$$

in which the elements of any row are the coefficients in a Kirchhoff voltage-law equation since they correspond to the selection of a tie-set (set of branches forming a closed loop). Their values will normally be either ± 1 or zero depending on whether a pertinent branch does or does not belong to the tie-set defined by a given row. Each row has $\underline{b}$ elements; but there are only ℓ rows since this is the number of independent tie-sets or loops. It should be recalled from the discussion in Ch. I that $\ell = b - n = b - n_t + 1$.

It is also shown in Ch. I that the elements in any column of the matrix $[\beta]$ are coefficients in an equation expressing the pertinent branch-current in terms of the loop-currents that are consistent with the tie-sets defined by the rows (i.e., currents that circulate upon the closed paths defined by the rows). Therefore, if we make use of the column matrices 54 and 55, the relations indicated in Eqs. 56, and write the loop-current variables in the form of a column matrix

$$[i] = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_\ell \end{bmatrix}, \qquad (68)$$

then the Kirchhoff-law equations are expressed in matrix form by

$$[\beta] \times [v] = 0, \qquad (69)$$

the volt-ampere relations 43 for the branches can be written

$$[v] + [e_s] = [D] \times [j] + [D] \times [i_s], \qquad (70)$$

and the branch-currents in terms of the loop-currents are given by

$$[j] = [\beta]_t \times [i]. \qquad (71)$$

The desired equilibrium equations are the Kirchhoff-law Eqs. 69 expressed in terms of the loop-currents as variables. One obtains this result through substituting the expression for $[j]$ from Eq. 71 into Eq. 70, and the ensuing relation for $[v]$ into Eq. 69. After a slight rearrangement of the terms, one finds

$$[\beta] \times [D] \times [\beta]_t \times [i] = [\beta] \times ([e_s] - [D] \times [i_s]) = [e_\ell] \qquad (72)$$

in which the right-hand side, representing the net equivalent voltage sources feeding the loops, is abbreviated as a column matrix

$$[e_\ell] = \begin{bmatrix} e_{\ell 1} \\ e_{\ell 2} \\ \vdots \\ e_{\ell\ell} \end{bmatrix}. \tag{73}$$

It is interesting to note the structure of this matrix as expressed by Eq. 72. Thus the term $[e_s]$ within the parenthesis represents voltage sources associated with the branches, while the term involving $[D] \times [i_s]$ represents the transformation of current sources associated with branches (see Fig. 1) into equivalent voltage sources, the minus sign arising from the opposite reference arrows associated with i_{sk} and e_{sk}. The parenthesis expression, therefore, represents net voltage sources for the branches, and is to be thought of as combining these into a single-column matrix. Multiplication of the matrix $[\beta]$ into this column matrix yields again a column matrix whose elements are algebraic sums of the branch sources according to the groups of branches forming tie-sets (these are the branches associated with pertinent loops). The elements in this resultant column matrix $[e_\ell]$ are thus seen to be equivalent loop voltage sources.

Interpretation of the left-hand side of Eq. 72 is facilitated through an appropriate evaluation of the triple matrix product $[\beta] \times [D] \times [\beta]_t$. A preliminary step toward achieving this end is the partitioning of the columns in the matrix $[\beta]$ into groups of λ, ρ, σ, thus

$$[\beta] = [\beta_{\ell\lambda} \;\vdots\; \beta_{\ell\rho} \;\vdots\; \beta_{\ell\sigma}]. \tag{74}$$

The submatrix $[\beta_{\ell\lambda}]$ consists of the first λ columns in $[\beta]$; $[\beta_{\ell\rho}]$ represents the succeeding group of ρ columns, and $[\beta_{\ell\sigma}]$ contains the last σ columns. Using the form for $[D]$ given in Eq. 42, one then finds

$$[\beta] \times [D] \times [\beta]_t = [\beta_{\ell\lambda} \;\vdots\; \beta_{\ell\rho} \;\vdots\; \beta_{\ell\sigma}] \times \begin{bmatrix} [\ell]\,p & 0 & 0 \\ 0 & [r] & 0 \\ 0 & 0 & [s]p^{-1} \end{bmatrix} \times \begin{bmatrix} [\beta_{\ell\lambda}]_t \\ [\beta_{\ell\rho}]_t \\ [\beta_{\ell\sigma}]_t \end{bmatrix}$$

$$= [\beta_{\ell\lambda}] \times [\ell] \times [\beta_{\ell\lambda}]_t p + [\beta_{\ell\rho}] \times [r] \times [\beta_{\ell\rho}]_t + [\beta_{\ell\sigma}] \times [s] \times [\beta_{\ell\sigma}]_t p^{-1} \quad . \qquad (75)$$

The expressions

$$\begin{aligned} [L] &= [\beta_{\ell\lambda}] \times [\ell] \times [\beta_{\ell\lambda}]_t \\ [R] &= [\beta_{\ell\rho}] \times [r] \times [\beta_{\ell\rho}]_t \\ [S] &= [\beta_{\ell\sigma}] \times [s] \times [\beta_{\ell\sigma}]_t \end{aligned} \qquad (76)$$

are recognized respectively as the <u>inductance</u>, the <u>resistance</u>, and the <u>elastance</u> parameter matrices pertaining to the loop basis. In terms of these and the source matrix 73, the equilibrium Eqs. 72 take the somewhat more familiar form

$$([L]p + [R] + [S]\,p^{-1}) \times |i| = [e_\ell]. \qquad (77)$$

The principal results of the present discussion are the evaluation of the resultant source matrix $[e_\ell]$ as given in Eq. 72, and the expressions 76 for the pertinent parameter matrices. These are given in terms of the branch parameter matrices $[\ell]$, $[r]$, and $[s]$ through triple matrix products with appropriate portions (and their transpositions) of the β-matrix characterizing the tie-set schedule. The formation of these key quantities appearing in the equilibrium Eqs. 77 is thus in every case reduced to a simple, systematic and straightforward procedure. The equations themselves are a set of simultaneous differential equations in terms of the instantaneous values of the variables.

5. <u>Remarks and examples</u>.

In Ch. II it is pointed out that symmetry of the parameter matrices on the node or loop basis comes about if the definitions of the node-pair voltages or loop currents are chosen to be consistent with the Kirchhoff-law equations, a condition that is commonly met but is by no means necessary. The procedures given in the two preceding articles are, therefore, not completely general, for they satisfy the conditions leading to symmetry. In the node method, for example, the Eq. 58 expressing Kirchhoff's current law and Eq. 60 defining the node-pair voltage variables are consistent, for they involve the same α matrix (cut-set schedule). Similarly for the loop method, the Kirchhoff voltage-law Eq. 69, and Eq. 71 defining the loop-currents are consistent, for they are based upon the same tie-set schedule (β matrix).

What we should recognize is the fact that one may, on the node basis choose one set of node-pairs for the current-law equations and an altogether different one for the definition of the node-pair voltages; or on the loop basis one may choose one set of loops for the voltage-law equations and another as the circulatory paths for the loop currents. Specifically, The Eqs 58 and 60 may involve two different α-matrices or cut-set schedules; and the Eqs. 69 and 71 may involve two entirely different β-matrices or tie-set schedules (so long as the schedules used pertain to the same network, of course). In most instances, however, it is advantageous to adhere to the consistency conditions and obtain symmetrical parameter matrices. Therefore the relations as given in the preceding articles are almost always appropriate and can readily be generalized if desired.

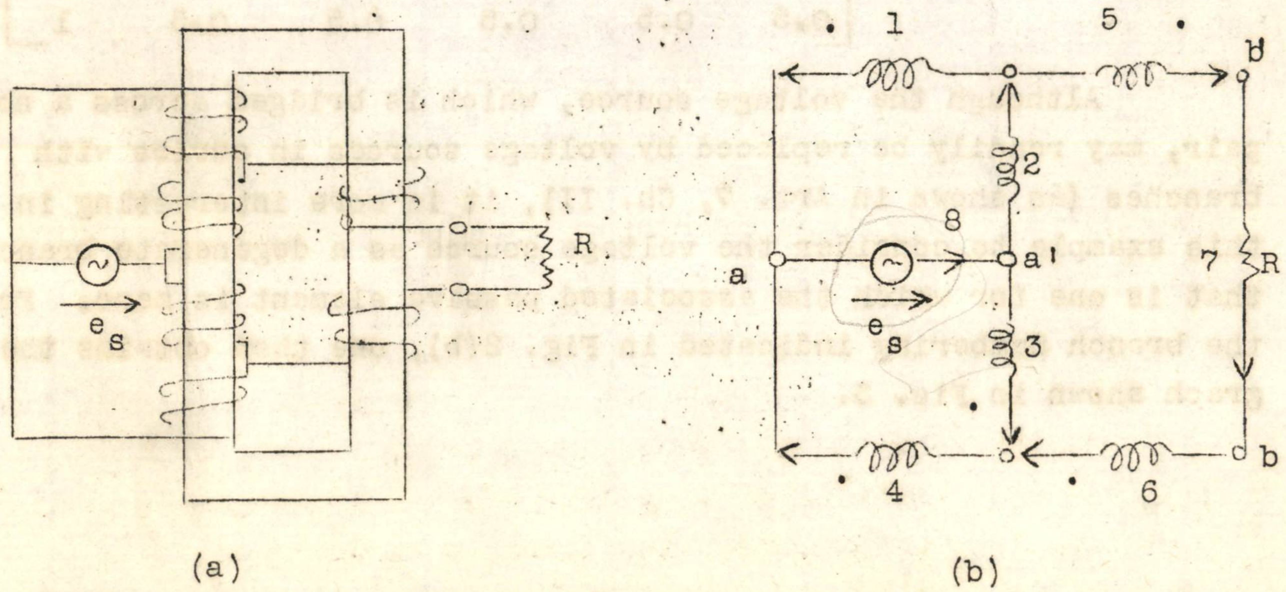


Fig. 2

As an illustrative example of the procedure given in the preceding article, suppose we consider the circuit of Fig. 2(a) which involves six mutually coupled windings on the same magnetic core. An appropriate schematic diagram is shown in part (b) of the same figure. It is easily recognized (according to the discussion in Art. 4, Ch. VIII) that the scheme for indicating relative polarities of the coils by means of dots is applicable to this simple arrangement, and that the system of dots in the schematic of part (b) is consistent with the physical arrangement indicated in part(a) of Fig. 2.

The reference arrows on the inductive branches of this circuit diagram are all chosen so that the tips of these arrows are at the dot-marked ends. As a result of this simple expedient, all mutual inductances between branches become numerically positive. If we assume that all six coils have self inductances equal to unity, and that all mutual inductances are equal to one-half (there is no sense in using more arbitrary numbers in this example), one has the branch inductance matrix

$$[\ell] = \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 1 \end{bmatrix} \qquad (78)$$

Although the voltage source, which is bridged across a node pair, may readily be replaced by voltage sources in series with branches (as shown in Art. 7, Ch. II), it is more interesting in this example to consider the voltage source as a degenerate branch; that is one for which the associated passive element is zero. For the branch numbering indicated in Fig. 2(b), one then obtains the graph shown in Fig. 3.

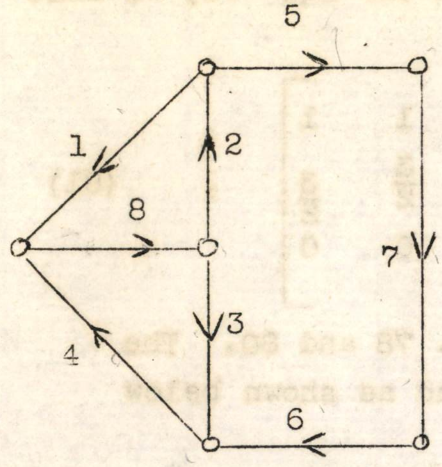


Fig. 3

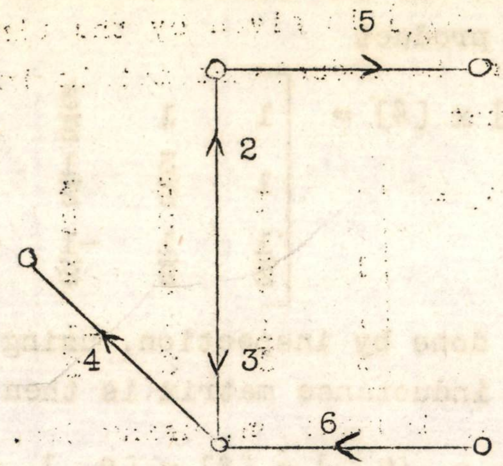


Fig. 4

Since in the given problem we are particularly interested in the current through the resistance R as a function of the source voltage, it is expedient to choose a tree in such a way that branches 7 and 8 become links. The tree shown in Fig. 4, for which branches 1,7,8 are links, is a satisfactory choice, and the identification of link-currents with loop currents is given by

$$\begin{aligned} i_1 &= j_8 \\ i_2 &= j_7 \\ i_3 &= j_1 \end{aligned} \tag{79}$$

seems best because it associates i_1 with the source and i_2 with the load.

The matrix of the resulting tie-set schedule is now recognized by inspection to be

$$[\beta] = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{80}$$

and the submatrix $[\beta_{\ell\lambda}]$ is given by the first six columns. Noting the expression for the inductance matrix in the Eqs. 76, we next evaluate the product

$$[\beta_{\ell\lambda}] \times [\ell] = \begin{bmatrix} 1 & 1 & \frac{3}{2} & \frac{3}{2} & 1 & 1 \\ 1 & \frac{3}{2} & \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix}, \qquad (81)$$

which can be done by inspection, using Eqs. 78 and 80. The desired loop inductance matrix is then found as shown below

$$[L] = [\beta_{\ell\lambda}] \times [\ell] \times [\beta_{\ell\lambda}]_t$$

$$= \begin{bmatrix} 1 & 1 & \frac{3}{2} & \frac{3}{2} & 1 & 1 \\ 1 & \frac{3}{2} & \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & \frac{3}{2} & -1 \\ \frac{3}{2} & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix}. \qquad (82)$$

The equilibrium equations on the loop basis are thus seen to be given by

$$\begin{bmatrix} 3p & (3/2)p & -p \\ (3/2)p & (4p+R) & p \\ -p & p & 2p \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} e_s \\ 0 \\ 0 \end{bmatrix} \qquad (83)$$

where the abbreviation $p = d/dt$ is still used. Since we are only interested in i_1 and i_2, it is a good idea to eliminate i_3 immediately. To do this we can consider the so-called augmented matrix

$$\begin{bmatrix} 3p & (3/2)p & -p & e_s \\ (3/2)p & (4p+R) & p & 0 \\ -p & p & 2p & 0 \end{bmatrix} \qquad (84)$$

and carry out linear combinations of its rows (equivalent to making linear combinations of the Eqs. 83) in such a way as to produce zeros for all elements of the third column except the last.

Thus if we add the (1/2)-multiplied elements of the third row to the respective ones of the first row; and then add the (-1/2)-multiplied elements of the third row to the respective ones of the second row, there results

$$\begin{bmatrix} (5/2)p & 2p & 0 & e_s \\ 2p & (7/2)p+R & 0 & 0 \\ -p & p & 2p & 0 \end{bmatrix} . \qquad (85)$$

The third row is now trivial, and the first two yield the equations

$$\begin{bmatrix} (5/2)p & 2p \\ 2p & (7/2)p+R \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} e_s \\ 0 \end{bmatrix} \qquad (86)$$

By inspection one can recognize that the circuit of Fig. 5 involving only self-inductances with the values indicated, has the same loop equilibrium equations. One may, therefore, regard this simple circuit as the equivalent of the one given in Fig. 2 so far as the terminal-pairs a-a' and b-b' are concerned.

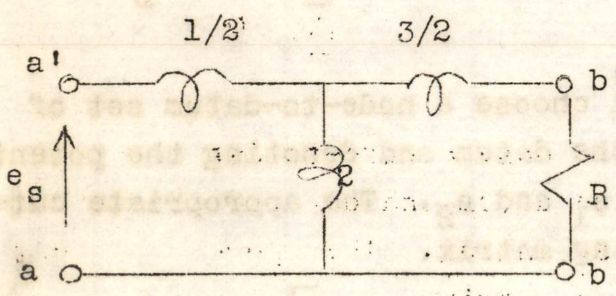


Fig. 5

As an example of the node method, we will consider the circuit of Fig. 6 in which the three inductances are mutually coupled, and for the indicated reference arrows are characterized by the branch inductance matrix

$$[\ell] = \begin{bmatrix} 2 & 4 & -5 \\ 4 & 9 & -11 \\ -5 & -11 & 14 \end{bmatrix} \quad (87)$$

with the inverse

$$[\Upsilon] = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}. \quad (88)$$

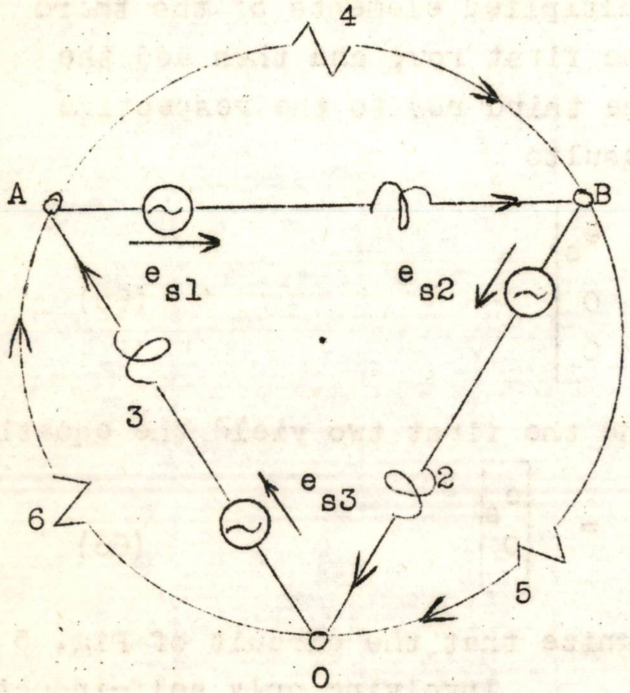


Fig. 6

The resistive branches have conductance values in mhos as indicated in the following branch conductance matrix for the branch numbering in Fig. 6

$$[g] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \quad (89)$$

Ror the variables we will choose a node-to-datum set of voltages, taking the point 0 as the datum and denoting the potentials of nodes A and B respectively by e_1 and e_2. The appropriate cut-set schedule then has the following matrix

$$[a] = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \end{bmatrix}, \quad (90)$$

and

$$[a_{n\lambda}] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}; \quad [a_{n\rho}] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}. \quad (91)$$

The three voltage sources we will assume to be steady sinusoids with values as indicated in the column matrix

$$[e_s] = \begin{bmatrix} 7\cos t \\ 2\sin t \\ .34\cos t \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{92}$$

The matrix $[i_s]$ according to 55 is identically zero because there are no current sources associated with any of the branches.

To evaluate $[i_n]$ in Eq. 61 we must form $[D]^{-1}$ which by Eq. 51 becomes

$$[D]^{-1} = \begin{bmatrix} 5p^{-1} & -p^{-1} & p^{-1} & 0 & 0 & 0 \\ -p^{-1} & 3p^{-1} & 2p^{-1} & 0 & 0 & 0 \\ p^{-1} & 2p^{-1} & 2p^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{93}$$

Actually only the first three rows and columns enter into the evaluation of

$$[D]^{-1}x[e_s] = \begin{bmatrix} 69\sin t + 2\cos t \\ 61\sin t - 6\cos t \\ 75\sin t - 4\cos t \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{94}$$

as the reader may readily verify by inspection, noting incidentally that $p^{-1}\cos t = \sin t$ and $p^{-1}\sin t = -\cos t$. One thus obtains for the matrix representing the net equivalent current sources feeding the nodes

$$[i_n] = -[\alpha]x[D]^{-1}x[e_s] = \begin{bmatrix} 6\sin t - 6\cos t \\ 8\sin t + 8\cos t \end{bmatrix} = \begin{bmatrix} 6\sqrt{2}\sin(t-45^\circ) \\ 8\sqrt{2}\sin(t+45^\circ) \end{bmatrix} \tag{95}$$

Straight-forward substitution into Eqs. 65 next yields for the parameter matrices

$$[\Gamma] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 5 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -7 & 10 \end{bmatrix} \qquad (96)$$

and

$$[G] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 10 \end{bmatrix}, \qquad (97)$$

of which the last could have been written down by inspection of Fig. 6. Thus the equilibrium equations for this network become

$$\begin{bmatrix} (5p^{-1}+5) & -(7p^{-1}+2) \\ -(7p^{-1}+2) & (10p^{-1}+10) \end{bmatrix} \times \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 6\sqrt{2}\,\sin(t-45°) \\ 8\sqrt{2}\,\sin(t+45°) \end{bmatrix}. \qquad (98)$$

The computations shown in Eqs. 93, 94, and 95 suggest that it might be a good idea to decompose the matrices $[i_n]$ and $[e_s]$ in a manner consistent with the partitioning of $[D]^{-1}$ and $[\alpha]$ as given in Eqs. 51 and 63, with the object of making the pertinent expressions less unwieldy. In order to carry out this thought both for the loop and node bases, we partition the source matrices 55 pertaining to the branches as indicated by

$$[i_s] = \begin{bmatrix} i_{s\lambda} \\ \hdashline i_{s\rho} \\ \hdashline i_{s\sigma} \end{bmatrix} \quad \text{and} \quad [e_s] = \begin{bmatrix} e_{s\lambda} \\ \hdashline e_{s\rho} \\ \hdashline e_{s\sigma} \end{bmatrix}. \qquad (99)$$

Here the submatrices $[i_{s\lambda}]$ and $[e_{s\lambda}]$ contain the first λ elements in the column matrices $[i_s]$ and $[e_s]$; the succeeding ρ elements are represented by the submatrices $[i_{s\rho}]$ and $[e_{s\rho}]$, and the last σ elements are combined in the portions $[i_{s\sigma}]$ and $[e_{s\sigma}]$.

The source matrices $[i_n]$ and $[e_\ell]$, Eqs. 62 and 73, pertaining to node-pairs and loops are appropriately resolved into <u>additive</u> components as indicated by

$$[i_n] = [i_n]_\lambda + [i_n]_\rho + [i]_\sigma \qquad (100)$$

$$[e_\ell] = [e_\ell]_\lambda + [e_\ell]_\rho + [e_\ell]_\sigma\,. \qquad (101)$$

We then find that the expression for $[i_n]$ given in Eq. 61 permits the detailed representation

$$[i_n]_\lambda = [\alpha_{n\lambda}] \times ([i_{s\lambda}] - [\gamma] p^{-1} \times [e_{s\lambda}]), \qquad (102)$$

$$[i_n]_\rho = [\alpha_{n\rho}] \times ([i_{s\rho}] - [g] \times [e_{s\rho}]), \qquad (103)$$

$$[i_n]_\sigma = [\alpha_{n\sigma}] \times ([i_{s\sigma}] - [c] p \times [e_{s\sigma}]), \qquad (104)$$

while the expression for $[e_\ell]$ given in Eq. 72 may be decomposed into

$$[e_\ell]_\lambda = [\beta_{\ell\lambda}] \times ([e_{s\lambda}] - [\ell] p \times [i_{s\lambda}]), \qquad (105)$$

$$[e_\ell]_\rho = [\beta_{\ell\rho}] \times ([e_{s\rho}] - [r] \times [i_{s\rho}]), \qquad (106)$$

$$[e_\ell]_\sigma = [\beta_{\ell\sigma}] \times ([e_{s\sigma}] - [s] p^{-1} \times [i_{s\sigma}]). \qquad (107)$$

These components of the matrices $[i_n]$ and $[e_\ell]$ that are distinguished by the subscripts λ , ρ , σ (not to be confused with submatrices) are those additive portions of these source matrices that are contributed by actual current and voltage sources associated with the inductive, the resistive, and the elastive branches respectively. The separate expressions for these components are far less unwieldy than those for $[i_n]$ and $[e_\ell]$. Moreover, one may have sources associated only with one kind of element (inductance, resistance, or capacitance) in which case two of the components are obviously zero and need not confuse the calculations. Thus in the last example above, there are no sources associated with the resistive branches, and so three-fourths of the space occupied by the matrix (93) and half the space occupied by 94 could be saved.

Another point worth emphasizing is concerned with the way in which the voltage source in the problem of Fig. 2 is dealt with. Wherever we encounter a voltage source that is not in series with a passive element or a current source that is not in parallel with a passive element, we have the choice either of revising the circuit according to the discussions given in Art. 7 of Ch. II, or of considering the source as a degenerate branch, as is done in the examples above. The latter scheme preserves the given geometry of the network, a feature that may be important to the subsequent interpretation of the analysis. It is well, therefore, to be aware of the possibility of treating sources in these various ways.

6. Energy functions

Denoting the elements in the loop parameter matrices [L], [R], [S] by L_{ik}, R_{ik}, S_{ik} where the indexes i and k may assume any integer values from 1 to ℓ, one may write the equilibrium Eqs. 77 in the following more explicit algebraic form

$$\sum_{k=1}^{\ell} \left(L_{ik}\frac{d}{dt} + R_{ik} + S_{ik}\int dt\right) i_k = e_{\ell i}; \quad i=1,2,\ldots\ell. \qquad (108)$$

Since this is still a rather compact form for these equations, the reader will better understand the notation involved through writing them out completely on a large sheet of paper. Thus for i = 1 he should write out the terms in the left-hand sum that correspond successively to $k = 1,2,\ldots\ell$ and equate these to $e_{\ell 1}$. He should then do the same for i = 2, i = 3, and so forth down to the equation for $i = \ell$. He will thus understand with ease and clarity how this system of equations appears in detailed form and how the notation in Eq. 108 is to be interpreted.

He should next carry through the detailed evaluation of these same equations starting from the equivalent matrix form 77, writing out all of the matrices explicitly and carrying out the indicated matrix additions and multiplications. A facile understanding of the equivalence of the matrix and the algebraic forms of these equations and their individual detailed interpretation thus gained (and not achievable through any less painful method!) is essential in developing one's ability to comprehend without difficulty the following discussion.

If we interpret the quantities $e_{\ell 1}$, $e_{\ell 2}$, $\ldots e_{\ell\ell}$ as being actual voltage sources acting in the links of the network and regard the loop currents i_1, i_2, $\ldots i_\ell$ as the currents in these same links (which is appropriate because the conditions leading to symmetrical parameter matrices are fulfilled), it is clear that the expression

$$P = e_{\ell 1}\, i_1 + e_{\ell 2} i_2 + \ldots + e_{\ell\ell} i_\ell \qquad (109)$$

represents the total instantaneous power delivered to the network by the sources. In order to study this flow of energy throughout the network we need to construct the expression 109 using the Eqs. 108. Visualizing these equations written out as suggested above, we may form the desired result through multiplying the equations (on both sides) successively by $i_1, i_2, \ldots i_\ell$, and adding all of them.

Through use of the summation sign we can indicate this set of operations very compactly if we multiply on both sides of Eq. 108 by i_i and then sum over the index i from 1 to ℓ. On the left we obtain a double summation, since Eq. 108 already involves a sum with respect to the index k. The result is written

$$P = \sum_{i,k=1}^{\ell} \left(L_{ik}\, i_i \frac{di_k}{dt} + R_{ik} i_i i_k + S_{ik} i_i \int i_k dt \right) = \sum_{i=1}^{\ell} e_{\ell i}\, i_i, \tag{110}$$

where it is important to note that the time-differentiation and integration do not affect i_i.

If we now define three functions as

$$F = \frac{1}{2} \sum_{i,k=1}^{\ell} R_{ik}\, i_i\, i_k, \tag{111}$$

$$T = \frac{1}{2} \sum_{i,k=1}^{\ell} L_{ik}\, i_i\, i_k, \tag{112}$$

$$V = \frac{1}{2} \sum_{i,k=1}^{\ell} S_{ik}\, q_i q_k, \tag{113}$$

where q_k in the last of these is used to denote the <u>loop charge</u> or indefinite time integral of the loop current as indicated in

$$q_k = \int i_k dt \quad \text{or} \quad i_k = \frac{dq_k}{dt}, \tag{114}$$

we observe that Eq. 110 is equivalent to

$$P = 2F + \frac{d}{dt}(T+V) = \sum_{i=1}^{\ell} e_{\ell i}\, i_i. \tag{115}$$

For, through straightforward differentiation we see that

$$\frac{dT}{dt} = \frac{1}{2} \sum_{i,k=1}^{\ell} L_{ik} \frac{d}{dt}(i_i i_k), \tag{116}$$

since summation and differentiation are interchangeable operations, and by the rule for the derivative of a product

$$\frac{d}{dt}(i_i i_k) = i_i \frac{di_k}{dt} + i_k \frac{di_i}{dt}, \tag{117}$$

so that

$$\frac{dT}{dt} = \frac{1}{2} \sum_{i,k=1}^{\ell} L_{ik}\, i_i \frac{di_k}{dt} + \sum_{i,k=1}^{\ell} L_{ik}\, i_k \frac{di_i}{dt}. \tag{118}$$

If in the second of these two sums we interchange the summation indexes i and k (which is permissible since each independently assumes all integer values from 1 to ℓ), and make use of the symmetry condition $L_{ik} = L_{ki}$, it becomes clear that the two sums are identical. Hence

$$\frac{dT}{dt} = \sum_{i,k=1}^{\ell} L_{ik}\, i_i \frac{di_k}{dt}. \tag{119}$$

Analogously we find

$$\frac{dV}{dt} = \sum_{i,k=1}^{\ell} S_{ik}\, q_k \frac{dq_i}{dt} = \sum_{i,k=1}^{\ell} S_{ik}\, i_i \int i_k dt, \tag{120}$$

and thus the equivalence of Eqs. 110 and 115 is established.

Comparison with Eq. 32 of Ch. VII reveals that our present Eqs. 111, 112, 113, and 115 represent a generalization of the previous result pertaining to a simple RLC-circuit, and that the functions 2F, T, and V are respectively the total instantaneous rate of energy dissipation, the instantaneous value of the energy stored in the magnetic fields associated with the inductances, and the instantaneous value of the energy stored in the electric fields associated with the capacitances. Thus Eq. 115 expresses the conservation of energy through showing that the time-rate of energy supplied by the sources equals the sum of the rate of energy dissipation in the circuit resistances and the time-rate-of-change of the net stored energy.

Although F dimensionally is power while T and V represent energy, we speak of the three functions F, T, and V as the <u>energy functions</u> associated with the network. Specifically, T and V are referred to as stored energy functions, while F is also called the <u>loss function</u> (first introduced into this sort of analysis by Lord Rayleigh). In order to preserve homogeneity in form for the three functions, a factor 1/2 is written before the summation in the expression for F as well as in those for T and V, and for this reason it is 2F not F that represents the total instantaneous rate of energy dissipation.

Until the reader has become assustomed to the notation used in Eqs. 111, 112, 113, it is well occasionally to write out an expression of this sort more fully, as for example

$$
\begin{aligned}
2T = {} & L_{11}i_1i_1 + L_{12}\, i_1i_2 + \dots + L_{1\ell}i_1i_\ell \\
& + L_{21}i_2i_1 + L_{22}\, i_2i_2 + \dots + L_{2\ell}i_2i_\ell \\
& + - - - - - - - - - - - - - - - - - - - \\
& + L_{\ell 1}i_\ell i_1 + L_{\ell 2}i_\ell i_2 + \dots + L_{\ell\ell}i_\ell i_\ell .
\end{aligned}
\qquad (121)
$$

Thus the first line in this expression is a summation on the index k from 1 to ℓ while the index i remains constant at the value 1; the second line similarly is a summation on k from 1 to ℓ while i = 2, and so forth. Since one may just as well add the terms by columns, we see incidentally that one can alternately sum on i from 1 to ℓ holding k constant at 1, and then at 2, and so forth. In other words, the double summation is carried out through letting the indexes i and k <u>independently</u> assume all integer values from 1 to ℓ, as stated above.

The expression 121 (compare also with Eq. 154 of Ch. III and with Eqs. 39 and 44 of Ch. VIII) is thus seen to be homogeneous and quadratic (all terms are quadratic) in the loop-current variables. As mentioned previously, a function of this sort is called by mathematicians a quadratic form. The energy functions F, T and V characterizing a linear passive network are thus seen to be quadratic forms in terms of the network variables.

One may alternately derive the energy functions in terms of the node-pair voltages as variables. Starting from the Eqs. 66 pertaining to the node basis, and denoting the elements in the matrices $[\Gamma]$, $[G]$, $[C]$ by Γ_{ik}, G_{ik}, C_{ik}, the equivalent algebraic form for these equations reads

$$\sum_{k=1}^{n}\left(C_{ik}\frac{d}{dt}+G_{ik}+\Gamma_{ik}\int dt\right)e_k=i_{ni};\ i=1,2,\ldots n, \qquad (122)$$

in which $e_1 \ldots e_n$ are the node-pair voltages and $i_{n1} \ldots i_{nn}$ are the equivalent node-pair current sources. Since both sets of quantities refer to the same node pairs, the total instantaneous power supplied to the network by an actual set of current sources feeding the pertinent node pairs is given by

$$P = i_{n1}e_1+i_{n2}e_2+\ldots+i_{nn}e_n. \qquad (123)$$

This expression is formed through multiplying the successive equations in the set 122 respectively by $e_1, e_2, \ldots e_n$ and adding. The result is compactly written

$$P = \sum_{i,k=1}^{n}\left(C_{ik}e_i\frac{de_k}{dt}+G_{ik}e_ie_k+\Gamma_{ik}e_i\int e_k dt\right)= \sum_{i=1}^{n} i_{ni}e_i. \qquad (124)$$

In terms of the energy functions

$$F = \frac{1}{2}\sum_{i,k=1}^{n} G_{ik}e_ie_k \qquad (125)$$

$$V = \frac{1}{2} \sum_{i,k=1}^{n} C_{ik} e_i e_k \tag{126}$$

$$T = \frac{1}{2} \sum_{i,k=1}^{n} \Gamma_{ik} \psi_i \psi_k \tag{127}$$

where ψ_k in the last of these is used to denote the node-pair <u>flux linkages</u> or indefinite time-integrals of the node-pair voltages as indicated in

$$\psi_k = \int e_k dt \quad \text{or} \quad e_k = \frac{d\psi_k}{dt}, \tag{128}$$

we again find that Eq. 124 is an expression of the conservation of energy, for it is equivalent to

$$P = 2F + \frac{d}{dt}(V+T) = \sum_{i=1}^{n} i_{ni} e_i, \tag{129}$$

the detailed justification for this conclusion being entirely similar to that given for the loop basis.

Although initially a given network may be thought of as excited by voltage sources located in the links, we can subsequently assume the sources to be currents having the values of the resulting link currents without altering in any way the rest of the voltages and currents throughout the network. The state of the network, however, is regarded as characterized in terms of loop-current variables or in terms of node-pair voltage variables according to whether the sources are considered to be a set of voltages or currents respectively. In a situation of this sort, the functions 111,112,113 have values that are identical with those obtained from Eqs. 125, 126, 127. The former express the values for the associated energies in terms of loop currents and loop charges as variables while the latter express these same values in terms of node-pair voltages and flux linkages as variables.

So far, the variables and the sources are any time functions. Let us now assume that they are steady sinusoids as they are in many practical applications. For the loop basis we then write

$$i_k = \frac{1}{2}\left(I_k e^{j\omega t} + \bar{I}_k e^{-j\omega t}\right), \qquad (130)$$

$$q_k = \int i_k dt = \frac{1}{2j\omega}\left(I_k e^{j\omega t} - \bar{I}_k e^{-j\omega t}\right), \qquad (131)$$

where the bar signifies the conjugate value, Preparatory to making substitutions into the relations 111, 112, 113 for F, T and V, we compute

$$i_i i_k = \frac{1}{4}\left(I_i e^{j\omega t} + \bar{I}_i e^{-j\omega t}\right)\left(I_k e^{j\omega t} + \bar{I}_k e^{-j\omega t}\right), \qquad (132)$$

which yields

$$i_i i_k = \frac{1}{4}\left\{I_i I_k e^{j2\omega t} + \bar{I}_i \bar{I}_k e^{-j2\omega t} + I_i \bar{I}_k + \bar{I}_i I_k\right\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left[I_i \bar{I}_k\right] + \frac{1}{2}\,\mathrm{Re}\left[I_i I_k e^{j2\omega t}\right] \qquad (133)$$

where Re is a symbol for "real part of" as used in previous discussions. Similarly we find from Eq. 131 that

$$q_i q_k = \frac{1}{2\omega^2}\,\mathrm{Re}\left[I_i \bar{I}_k\right] - \frac{1}{2\omega^2}\,\mathrm{Re}\left[I_i I_k e^{j2\omega t}\right], \qquad (134)$$

and substitution into Eqs. 111, 112, 113 then gives for the energy functions in the sinusoidal steady-state

$$F = \frac{1}{4}\sum_{i,k=1}^{\ell} R_{ik} I_i \bar{I}_k + \frac{1}{4}\,\mathrm{Re}\left[e^{j2\omega t}\sum_{i,k=1}^{\ell} R_{ik} I_i I_k\right], \qquad (135)$$

$$T = \frac{1}{4}\sum_{i,k=1}^{\ell} L_{ik} I_i \bar{I}_k + \frac{1}{4}\,\mathrm{Re}\left[e^{j2\omega t}\sum_{i,k=1}^{\ell} L_{ik} I_i I_k\right], \qquad (136$$

$$V = \frac{1}{4\omega^2}\sum_{i,k=1}^{\ell} S_{ik} I_i \bar{I}_k - \frac{1}{4\omega^2}\,\mathrm{Re}\left[e^{j2\omega t}\sum_{i,k=1}^{\ell} S_{ik} I_i I_k\right], \tag{137}$$

where it is to be noted that the Re-sign is not needed in each first term. For example

$$\mathrm{Re}\sum_{i,k=1}^{\ell} R_{ik} I_i \bar{I}_k = \sum_{i,k=1}^{\ell} R_{ik} I_i \bar{I}_k; \tag{138}$$

that is to say, this sum is real in spite of its complex variables I_i and I_k. The proof of this statement is readily given through showing that the sum on the right-hand side of Eq. 138 is <u>self-conjugate</u>; that is to say it is its own conjugate. Obviously if a number equals its conjugate, that number must be real. Considering the conjugate of this sum which evidently reads

$$\sum_{i,k=1}^{\ell} R_{ik} \bar{I}_i I_k, \tag{139}$$

its value is not changed if we interchange the summation indexes i and k, since any other two symbols could be used in their place. If we then observe that $R_{ki} = R_{ik}$, it is seen that 138 and 139 are identical.

Each of the energy functions F, T, V according to the Eqs. 135, 136, 137, is given by the sum of a constant term and a double-frequency sinusoid, as is shown in Ch. VIII for the simple RLC-circuit. The constant term in each case is the average value of the energy function. We thus have

$$F_{av} = \frac{1}{4}\sum_{i,k=1}^{\ell} R_{ik} I_i \bar{I}_k , \tag{140}$$

$$T_{av} = \frac{1}{4}\sum_{i,k=1}^{\ell} L_{ik} I_i \bar{I}_k, \tag{141}$$

$$V_{av} = \frac{1}{4\omega^2}\sum_{i,k=1}^{\ell} S_{ik} I_i \bar{I}_k. \tag{142}$$

In Art. 7 of Ch. VIII, expressions equivalent to Eqs. 136 and 137 are given in terms of branch currents instead of loop currents. These are less general than the present results because the relation for T given there does not provide for the possibility of mutual inductive coupling. Since branch currents, unlike loop currents, do not traverse any common paths, and since the possibility of mutual inductive coupling is not considered in the discussion given in Ch. VII, the sums appearing there (see Eqs. 84 and 85 of Ch. VII) involve no cross-product terms as do the ones given here, rather only square terms are present. For this reason it is clear by inspection that the constant term or average value is greater than or at least equal to the amplitude of the oscillatory component, as a physical consideration obviously requires since the instantaneous value of the function would otherwise become negative during some interval, a condition that is physically impossible.

Although, for the more general expressions 135, 136, 137 considered here, it is not as simply obvious algebraically that the constant terms are at least as large as the amplitudes of the pertinent oscillatory components, such an independent purely algebraic proof can readily be given. Considering any one function by itself, one introduces a linear transformation of the variables which eliminates the cross-product terms, whereupon the desired result is again obvious (as pointed out in the second paragraph of Art. 7, Ch. VII). The algebraic details involved in this demonstration are, however, not justified at this point.

It is useful to obtain analogous results in terms of the node-pair voltages as variables. Considering the functions, F, T, V, as given by Eqs. 125, 126, 127 we write for the node-pair voltages and flux linkages

$$e_k = \frac{1}{2}\,(E_k e^{j\omega t} + \bar{E}_k e^{-j\omega t}), \qquad (143)$$

and

$$\psi_k = \int e_k dt = \frac{1}{2j\omega}\,(E_k e^{j\omega t} - \bar{E}_k e^{-j\omega t}). \qquad (144)$$

It then follows that

$$e_i e_k = \frac{1}{2} \text{Re} \left[E_i \bar{E}_k\right] + \frac{1}{2} \text{Re} \left[E_i E_k e^{j2\omega t}\right], \tag{145}$$

and

$$\psi_i \psi_k = \frac{1}{2\omega^2} \text{Re} \left[E_i \bar{E}_k\right] - \frac{1}{2\omega^2} \text{Re} \left[E_i E_k e^{j2\omega t}\right], \tag{146}$$

whence substitution into Eqs. 125, 126, 127 yields

$$F = \frac{1}{4} \sum_{i,k=1}^{n} G_{ik} E_i \bar{E}_k + \frac{1}{4} \text{Re} \left[e^{j2\omega t} \sum_{i,k=1}^{n} G_{ik} E_i E_k\right], \tag{147}$$

$$V = \frac{1}{4} \sum_{i,k=1}^{n} C_{ik} E_i \bar{E}_k + \frac{1}{4} \text{Re} \left[e^{j2\omega t} \sum_{i,k=1}^{n} C_{ik} E_i E_k\right], \tag{148}$$

$$T = \frac{1}{4\omega^2} \sum_{i,k=1}^{n} \Gamma_{ik} E_i \bar{E}_k - \frac{1}{4\omega^2} \text{Re} \left[e^{j2\omega t} \sum_{i,k=1}^{n} \Gamma_{ik} E_i E_k\right]. \tag{149}$$

The Re-sign is not needed in each first term for the reason given in the discussion of the analogous situation on the loop basis. The first terms again are the average values and the second terms are double-frequency sinusoids whose amplitudes cannot (for a passive network) exceed the respective values of the constant terms. These average values in terms of the complex amplitudes of the node-pair voltages are

$$F_{av} = \frac{1}{4} \sum_{i,k=1}^{n} G_{ik} E_i \bar{E}_k, \tag{150}$$

$$V_{av} = \frac{1}{4} \sum_{i,k=1}^{n} C_{ik} E_i \bar{E}_k, \tag{151}$$

$$T_{av} = \frac{1}{4\omega^2} \sum_{i,k=1}^{n} \Gamma_{ik} E_i \bar{E}_k. \tag{152}$$

It is now a simple matter to obtain general expressions for active, reactive, and vector power in the sinusoidal steady-state. To this end, consider the loop equ ilibrium Eqs. 108 for the assumptions

$$e_{\ell i} = E_i e^{j\omega t} \text{ and } i_k = I_k e^{j\omega t}. \tag{153}$$

After cancellation of the exponential factor, one has

$$\sum_{k=1}^{\ell} (L_{ik} j\omega + R_{ik} + \frac{S_{ik}}{j\omega}) I_k = E_i;\ i = 1, 2, \ldots \ell. \tag{154}$$

Forming the conjugate value upon both sides and rearranging the terms slightly we have

$$\bar{E}_i = \sum_{k=1}^{\ell} \left\{ R_{ik} - j \ (L_{ik} - \frac{S_{ik}}{\omega 2}) \right\} \bar{I}_k. \tag{155}$$

The total vector power, according to its fundamental definition given in Art. 4, Ch. VII, is now obtained through multiplication by $I_i/2$ and summation over the index i, thus

$$\frac{1}{2} \sum_{i=1}^{\ell} \bar{E}_i I_i = \frac{1}{2} \sum_{i,k=1}^{\ell} R_{ik} I_i \bar{I}_k - \frac{j\omega}{2} \left\{ \sum_{i,k=1}^{\ell} L_{ik} I_i \bar{I}_k - \frac{1}{\omega 2} \sum_{i,k=1}^{\ell} S_{ik} I_i \bar{I}_k \right\}. \tag{156}$$

In view of the relations 140, 141, 142 this gives

$$\frac{1}{2} \sum \bar{E}_i I_i = P_{av} + jQ_{av} = 2F_{av} + j2\omega \ (V_{av} - T_{av}), \tag{157}$$

whereupon

$$P_{av} = 2F_{av}, \tag{158}$$

$$Q_{av} = 2\omega \ (V_{av} - T_{av}), \tag{159}$$

which agree with the results obtained in Ch. VII for the simple circuit considered there.

Thus the real part of the vector power is the average power dissipated in the resistances, and the imaginary part or the so-called reactive power is proportional to the difference between the average energy stored in the electric fields and that stored in the magnetic fields. When these two average stored energies are equal, the sources are not called upon to take part in an interchange of stored energy, and the net reactive power is zero. As stated in Ch. VII, the reactive power Q_{av} is a measure of the extent to which the sources are called upon to participate in an interchange of stored energy. The actual average power consumed by the network is the so-called active power P_{av}.

It is interesting to interpret also Eq. 115, expressing the conservation of energy, in the sinusoidal steady state through substituting the expressions 135, 136, 137 for F, T, V. If we observe that the last two equations yield

$$\frac{dT}{dt} = \frac{1}{2} \operatorname{Re}\left[e^{j2\omega t} \sum_{i,k=1}^{\ell} j\omega L_{ik} I_i I_k \right], \tag{160}$$

and

$$\frac{dV}{dt} = \frac{1}{2} \operatorname{Re}\left[E^{j2\omega t} \sum \frac{S_{ik}}{j\omega} I_i I_k \right] \tag{161}$$

(since the constant terms in 136 and 137 do not contribute to these time-derivatives), and group the sinusoidal terms in a single sum, this substitution into Eq. 115 gives

$$P = \frac{1}{2} \sum_{i,k=1}^{\ell} R_{ik} I_i \bar{I}_k + \frac{1}{2} \operatorname{Re}\left[e^{j2\omega t} \sum_{i,k=1}^{\ell} (R_{ik} + j\omega L_{ik} + \frac{S_{ik}}{j\omega}) I_k I_k \right]$$

or

$$P = 2F_{av} + \frac{1}{2} \operatorname{Re}\left[e^{j2\omega t} \sum_{1=1}^{\ell} \left\{ \sum_{k=1}^{\ell} (R_{ik} + j\omega L_{ik} + \frac{S_{ik}}{j\omega}) I_k \right\} I_i \right] \tag{162}$$

where we have separated the double sum into two single sums in order to show that the one inside the curved brackets, according to Eq. 154, is simply E_i. Hence we have

$$P = P_{av} + \frac{1}{2} \operatorname{Re} \left[e^{j2\omega t} \sum_{i=1}^{\ell} E_i I_i \right], \tag{163}$$

which shows that the total instantaneous power supplied by the sources equals a constant (the average power dissipated by the circuit) plus a double-frequency sinusoid.

From the expression for the double-frequency sinusoid we can see, for example, that if our network is a balanced polyphase system, this term becomes zero, for the source voltages E_i and the source currents I_i are equal in magnitude and are equally spaced in time phase so that the sum of the products $E_i I_i$ over all sources vanishes. In any balanced polyphase system the total instantaneous power is constant and equal to the average power consumed by the network.

Of particular interest is the result 163 if the network is excited by a single source. Letting this one be E_1, we have in this special case

$$P = P_{av} + \operatorname{Re} \left[\frac{E_1 I_1}{2} e^{j2\omega t} \right]. \tag{164}$$

Taking E_1 as phase reference and denoting the input admittance angle by φ, we have

$$P = P_{av} + \frac{|E_1 I_1|}{2} \cos (2\omega t + \varphi). \tag{165}$$

However, noting Eq. 157,

$$\frac{E_1 I_1}{2} = \frac{|\bar{E}_1 I_1|}{2} = \sqrt{P_{av}^2 + Q_{av}^2}, \tag{166}$$

so that Eq. 165 can be written

$$P = P_{av} + \sqrt{P_{av}^2 + Q_{av}^2} \cos (2\omega t + \varphi), \tag{167}$$

a result which shows that the amplitude of the double-frequency sinusoid equals the magnitude of the vector power.

7. Equivalence of Kirchhoff and Lagrange equations.

In this article we wish to show that Lagrange's equations, which express the equilibrium of a system in terms of its associated energy functions, are identical with the Kirchhoff-law equations so far as the end results are concerned. We need first some preliminary relations which can readily be seen from Eqs. 111, 112, 113 for the functions F, T, V in terms of the loop currents. If we differentiate partially with respect to a particular loop current, we find

$$\frac{\partial F}{\partial i_i} = \sum_{k=1}^{\ell} R_{ik} i_k, \tag{168}$$

$$\frac{\partial T}{\partial i_i} = \sum_{k=1}^{\ell} L_{ik} i_k, \tag{169}$$

$$\frac{\partial V}{\partial q_i} = \sum_{k=1}^{\ell} S_{ik} q_k. \tag{170}$$

These results may most easily be obtained if one considers the pertinent function written out completely as T is in Eq. 121 It is then obvious that a particular loop-current, say i_2, is contained in all terms of the second row and second column, and only in these terms. Hence if we differentiate partially with respect to i_2, no other terms are involved, and we find

$$\frac{\partial}{i_2}(2T) = L_{21} i_1 + 2L_{22} i_2 + L_{23} i_3 + \dots + L_{2\ell} i_\ell$$
$$+ L_{12} i_1 + \qquad + L_{32} i_3 + \dots + L_{\ell 2} i_\ell, \tag{171}$$

where we note that the term with L_{22} yields a factor 2 because the derivative of i_2^2 is involved. However, since $L_{ik} = L_{ki}$, we can rewrite this result as

$$\frac{\partial}{\partial i_2}(2T) = 2\ (L_{21}i_1 + L_{22}i_2 + \ldots + L_{2\ell}i_\ell) \tag{172}$$

from which Eq. 169 follows. Eqs. 168 and 170 are obtained in the same manner. In all three, the summation involved is a simple summation on the index k.

If we differentiate Eq. 169 totally with respect to time, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial i_i}\right) = \sum_{k=1}^{\ell} L_{ik}\frac{di_k}{dt}, \tag{173}$$

and Eq. 170 can be rewritten as

$$\frac{\partial V}{\partial q_i} = \sum_{k=1}^{\ell} S_{ik}\int di_k, \tag{174}$$

so that with Eq. 168 we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial i_i}\right) + \frac{\partial F}{\partial i_i} + \frac{\partial V}{\partial q_i} = \sum_{k=1}^{\ell}\left(L_{ik}\frac{d}{dt} + R_{ik} + S_{ik}\int dt\right) i_k. \tag{175}$$

Reference to the Kirchhoff voltage-law Eqs. 108 now shows that these may alternatively be written

$$\frac{d}{dt}\left(\frac{\partial T}{\partial i_i}\right) + \frac{\partial F}{\partial i_i} + \frac{\partial V}{\partial q_i} = e_{\ell i};\ i = 1, 2, \ldots \ell. \tag{176}$$

This form, in which the voltage equilibrium equations are expressed in terms of the energy functions, is known as the Lagrangian equations. From the way in which they are here obtained, it is clear that they are equivalent to the Kirchhoff-law equations although their outward appearance does not place this fact in evidence.

The Lagrangian equations may alternately be expressed in terms of the node-pair voltages as variables. To obtain this result we begin with the Eqs. 125, 126, 127 for F, V, T and form

$$\frac{\partial F}{\partial e_i} = \sum_{k=1}^{n} G_{ik}e_k, \tag{177}$$

$$\frac{\partial V}{\partial e_i} = \sum_{k=1}^{n} C_{ik} e_k, \tag{178}$$

$$\frac{\partial T}{\partial \psi_i} = \sum_{k=1}^{n} \Gamma_{ik} \psi_k. \tag{179}$$

Differentiating Eq. 178 totally with respect to time and rewriting Eq. 179 gives

$$\frac{d}{dt}\left(\frac{\partial V}{\partial e_i}\right) = \sum_{k=1}^{n} C_{ik} \frac{de_k}{dt}, \tag{180}$$

and

$$\frac{\partial T}{\partial \psi_i} = \sum_{k=1}^{n} \Gamma_{ik} \int e_k dt, \tag{181}$$

from which one has

$$\frac{d}{dt} \frac{\partial V}{\partial e_i} + \frac{\partial F}{\partial e_i} + \frac{\partial T}{\partial \psi_i} = \sum_{k=1}^{n} (C_{ik} \frac{d}{dt} + G_{ik} + \Gamma_{ik} \int dt) e_k. \tag{182}$$

In view of this result, the Kirchhoff-law Eqs. 122 may be rewritten in the form

$$\frac{d}{dt}\left(\frac{\partial V}{\partial e_i}\right) + \frac{\partial F}{\partial e_i} + \frac{\partial T}{\partial \psi_i} = i_{ni};\ i = 1, 2, \ldots n. \tag{183}$$

These again are the Lagrangian equations expressing the network equilibrium in terms of the associated energy functions.

It is significant to observe that Eqs. 176 and 183 are dual forms of the Lagrangian equations just as the Kirchhoff Eqs. 108 and 122 are dual forms. It is interesting in comparing these two equations to note that the functions T and V interchange places, as should be expected from the fact that T and V are duals.

8. Relation to impedance functions.

In terms of the previous results of this article, it is a simple matter to express the driving-point impedance of a network in terms of its associated energy functions. Thus Eq. 157 for a single driving-point (which we can call loop 1) yields

$$\bar{E}_1 I_1 = 4F_{av} + j4\omega\,(V_{av} - T_{av}), \tag{184}$$

or, taking the conjugate of each side of this equation,

$$E_1 \bar{I}_1 = 4F_{av} + j4\omega\,(T_{av} - V_{av}). \tag{185}$$

Dividing both sides of Eq. 184 by $E_1\bar{E}_1 = \left|E_1\right|^2$, or both sides of Eq. 185 by $I_1 I_1 = \left|I_1\right|^2$ yields respectively

$$\frac{I_1}{E_1} = Y_{11}(\omega) = \frac{4F_{av} + j4\omega(V_{av} - T_{av})}{\left|E_1\right|^2} \tag{186}$$

and

$$\frac{E_1}{I_1} = Z_{11}(\omega) = \frac{4F_{av} + j4\omega(T_{av} - V_{av})}{\left|I_1\right|^2} \tag{187}$$

A more effective form for these results reads

$$Y_{11}(\omega) = \Big[4F_{av} + j4\omega(V_{av} - T_{av})\Big]_{E_1 = 1} \tag{188}$$

and

$$Z_{11}(\omega) = \Big[4F_{av} + j4\omega(T_{av} - V_{av})\Big]_{I_1 = 1}. \tag{189}$$

In the interpretation of Eq. 188 one should consider F_{av}, V_{av}, T_{av} to be expressed in terms of <u>voltages</u> as in Eq. 150, 151, 152, for their respective values are then readily recognized to be proportional to the square of the voltage E_1 at the driving-point since all other voltages $E_2 \ldots E_n$ are linearly proportional to E_1. In Eq. 188 we say that F_{av}, V_{av}, T_{av} are regarded as being evaluated per volt at the driving-point.

Analogously, in the interpretation of Eq. 189 we associate with F_{av}, T_{av}, V_{av} the expressions 140, 141, 142 in terms of currents. Since all other loop currents are linearly proportional to I_1, the average energy functions are seen to be proportional to I_1^2. Their values per ampere at the driving-point may be regarded as normalized values. Eq. 189 expresses the driving-point impedance in terms of these normalized values of the average energy functions.

As pointed out in Art. 6 of Ch. VII where these same expressions for the driving-point admittance and impedance functions are derived in terms of the simple RLC-circuit, one recognizes by inspection that a resonance condition is one for which the average stored energies are equal, for the input impedance or admittance then becomes purely real. It is also clear that the stored magnetic energy predominates when the reactive part of the impedance is positive, while a negative reactive part indicates that the stored electric energy predominates. Thus these characteristics of the impedance are more clearly and directly related to the physical properties of the network.

The relations 188 and 189 in a sense permit the same simple correlations between impedance or admittance and the physical network to be made in all cases, that are otherwise only possible with the simple parallel or series RLC-circuit. They are, however, restricted in that the impedance or admittance is expressible only for pure imaginary complex frequencies (also referred to as real frequencies since they correspond to sinusoids with steady amplitudes). In the following we shall remove this restriction through the introduction of a related set of energy functions that have significance for any complex values of the frequency variable $s = \sigma + j\omega$; and simultaneously we shall generalize the result so as to include all possible transfer impedances or admittances as well as the driving-point functions.

Starting again with the loop equilibrium Eqs. 108, let us substitute

$$e_{\ell i} = E_i e^{st} \text{ and } i_k = I_k e^{st} \qquad (190)$$

and obtain after cancellation of the exponential factor

$$\sum_{K=1}^{\ell} (L_{ik}s + R_{ik} + \frac{S_{ik}}{s}) I_k = E_i; \; i = 1, 2, \ldots \ell. \qquad (191)$$

The successive equations that result for i = 1, i = 2, and so forth, we now multiply respectively by $\bar{I}_1, \bar{I}_2, \ldots \bar{I}_\ell$ and add. If we introduce the notation

$$T_o = \sum_{i,k=1}^{\ell} L_{ik}\bar{I}_i I_k = \sum_{i,k=1}^{\ell} L_{ik} I_i \bar{I}_k \qquad (192)$$

$$F_o = \sum_{i,k=1}^{\ell} R_{ik}\bar{I}_i I_k = \sum_{i,k=1}^{\ell} R_{ik} I_i \bar{I}_k \qquad (193)$$

$$V_o = \sum_{i,k=1}^{\ell} S_{ik}\bar{I}_i I_k = \sum_{i,k=1}^{\ell} S_{ik} I_i \bar{I}_k \qquad (194)$$

the result may be written

$$sT_o + F_o + \frac{V_o}{s} = \sum_{i=1}^{\ell} E_i \bar{I}_i. \qquad (195)$$

The equivalence of the forms for T_o, F_o, V_o shown in Eqs. 192, 193, 194 is seen to follow if we interchange the letters i and k, which are merely summation indexes, and then note the symmetry condition $L_{ik} = L_{ki}$ (and so forth) for the parameters. Although T_o is the only one of these three functions that has the dimensions of energy (F_o is dimensionally power, and V_o has the dimensions of the time-rate-of-change of power) we shall for the sake of simplicity refer to all three as energy functions. Their relation to the functions T_{av}, F_{av}, V_{av} is discussed later on.

Our next objective is to extract from Eq. 195 some rather general relations concerning driving-point and transfer impedances. In this regard we observe first of all, that in most cases we are not interested in having sources in all the loops of a network. In order, however, to leave this question in a flexible state, we shall assume that, of the ℓ loops in the network, only p will be considered as being points of access, and permit p to be any integer from 1 to ℓ. These p points of access are the terminal pairs of the voltage sources located in p of the loops. The network as a whole we assume to be enclosed in a box with only the accessible terminal pairs brought out.

Since we are interested only in the currents at these p terminal pairs, we wish to eliminate all other currents involved in the Eqs. 191. In order to indicate specifically how this elimination is done, we may assume, without introducing any restriction, that the points of access correspond to loops 1 to p. Use of the abbreviation

$$\zeta_{ik} = L_{ik}s+R_{ik}+ \frac{S_{ik}}{s} \tag{196}$$

then permits the resulting equilibrium Eqs. 191 to be written

$$\begin{aligned}
&\zeta_{11}I_1+\zeta_{12}I_2+ \dots+\zeta_{1\ell}I_\ell = E_1 \\
&\text{- -} \\
&\zeta_{p1}I_1 +\zeta_{p2} I_2+\dots+\zeta_{p\ell}I_\ell= E_p \\
&\zeta_{p+1,1}I_1 +\zeta_{p+1,2}I_2+\dots+\zeta_{p+1,\ell}I_\ell = 0 \\
&\text{- -} \\
&\zeta_{\ell 1} I_1+ \zeta_{\ell 2}I_2+\dots+\zeta_{\ell\ell}I_\ell = 0,
\end{aligned} \tag{197}$$

thus showing that all but the first p loops have no excitation.

The ζ-matrix of this set of equations we now represent in the partitioned form

$$[\zeta] = \begin{bmatrix} \zeta_{11} & \zeta_{12} & \cdots \zeta_{1\ell} \\ \zeta_{21} & \zeta_{22} & \cdots \zeta_{2\ell} \\ \hline \zeta_{\ell 1} & \zeta_{\ell 2} & \cdots \zeta_{\ell\ell} \end{bmatrix} = \left[\begin{array}{c|c} \zeta_{pp} & \zeta_{pq} \\ \hline \zeta_{qp} & \zeta_{qq} \end{array}\right] . \quad (198)$$

where the submatrix $[\zeta_{pp}]$ contains the elements of the first p rows and p columns, $[\zeta_{pq}]$ contains the elements of the first p rows and the last q columns where $q = \ell - p$, and so forth. The column matrices

$$[I] = \begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ \cdot \\ I_\ell \end{bmatrix} \quad \text{and} \quad [E] = \begin{bmatrix} E_1 \\ E_2 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (199)$$

are correspondingly partitioned as indicated by the forms

$$[I] = \begin{bmatrix} I_p \\ \hline I_q \end{bmatrix} \quad \text{and} \quad [E] = \begin{bmatrix} E_p \\ \hline 0 \end{bmatrix} , \quad (200)$$

in which $[I_p]$ and $[E_p]$ are column submatrices containing the first p elements in [I] and [E]; $[I_q]$ contains the last q elements in [I], and [0] is a column of q zeros. The eqs. 197 may then be written more compactly as

$$[\zeta_{pp}]x[I_p]+[\zeta_{pq}]x[I_q] = [E_p]$$

$$[\zeta_{qp}]x[I_p]+[\zeta_{qq}]x[I_q] = [0]. \quad (201)$$

The second of these equations may be solved for $[I_q]$ giving

$$[I_q] = -[\zeta_{qq}]^{-1} x[\zeta_{qp}]x[I_p], \quad (202)$$

and substitution into the first equation yields

$$([\zeta_{pp}]-[\zeta_{pq}]x[\zeta_{qq}]^{-1}x[\zeta_{qp}]) \; x \; [I_p] \equiv [E_p] \quad (203$$

which symbolizes the desired set of equations involving only the currents at the accessible terminal-pairs.

If we introduce the square impedance matrix of order p

$$[z_{pp}] = [\zeta_{pp}] - [\zeta_{pq}] \times [\zeta_{qq}]^{-1} \times [\zeta_{qp}]$$

$$= \begin{bmatrix} z_{11} & z_{12} & \cdots z_{1p} \\ z_{21} & z_{22} & \cdots z_{2p} \\ - & - & - \\ z_{p1} & z_{p2} & \cdots z_{pp} \end{bmatrix}, \qquad (204)$$

we may write the Eqs. 203 in the equivalent algebraic form

$$\sum_{k=1}^{p} z_{ik} I_k = E_i; \; i = 1, 2, \ldots p. \qquad (205)$$

This is an abridged form of the Eqs. 191 appropriate to the situation in which voltage sources are present in the first p loops only. The coefficients z_{ik} in these equations are determined from the ζ_{ik} (Eq. 196) characterizing the Eqs. 191, in a manner indicated by the matrix Eq. 204. The transition from Eqs. 191 to Eqs. 205 may be described as a process of suppressing or eliminating the inaccessible or unwanted currents.

Although the Eqs. 191 or 205 are derived specifically on the assumption that the E's are sources and the I's are responses, they correctly relate these voltages and currents even though some or all of the I's may be sources and correspondingly some or all of the E's become responses. If the $I_1 \ldots I_p$ in Eqs. 205 are regarded as sources, then the resulting $E_1 \ldots E_p$ at the respective terminal pairs are <u>explicitly</u> given by these equations. Under these circumstances the terminal pairs are all open-circuited, and for this reason z_{ik} are referred to as a set of <u>open-circuit driving-point and transfer impedances</u> characterizing the p terminal-pair network (compare with the analogous quantities described in Art. 7, Ch. III for resistance networks).

Specifically, if I_1 is the only nonzero current source, then

$$E_1 = z_{11}I_1$$
$$E_2 = z_{21}I_1$$
$$------$$
$$E_p = z_{p1}I_1, \qquad (206)$$

or

$$z_{11} = E_1/I_1$$
$$z_{21} = z_{12} = E_2/I_1 \qquad (207)$$
$$z_{p1} = z_{1p} = E_p/I_1.$$

If I_1 is thought of as being equal in value to one reference ampere, then the complex voltage E_1 at terminal-pair 1 is numerically identical with the transfer impedance $z_{12} = z_{21}$; and so forth. In any of these transfer relations like $E_2 = z_{12}I_1$, it is important to observe that I_1 must be the source and E_2 the response; and that this specific relation is invalid if E_2 now is regarded as a source and I_1 as a response because this change of attitude violates the conditions under which the particular relations 206 are extracted from the general relations 205 (namely, I_2 is no longe zero).

Another set of particular relations may be extracted from the Eqs. 205 on the assumption that I_2 is the only nonzero current, namely

$$E_1 = z_{12}I_2$$
$$E_2 = z_{22}I_2.$$
$$------ \qquad (208)$$
$$E_p = z_{p2}I_2,$$

from which we may again obtain relations for the z's like Eqs. 207 which lend themselves to physical interpretation; and once more it must be emphasized that the transfer relations apply only if I_2 is a source.

The driving-point relations like $E_1 = z_{11}I_1$ or $E_2 = z_{22}I_2$, in contrast, are valid regardless of whether the voltage or the current is the source because no restriction is placed upon the nonzero I_1 in Eqs. 206 or upon the nonzero I_2 in Eqs. 208. A driving-point relation always remains valid regardless of which of the quantities E or I is the source and which is the response, while the derivation of a transfer relation invariable involves a restriction which fastens the roles of source and response upon specific ones of the two quantities E and I.

If we now substitute the expression for E_i as given by Eqs. 205 into E Eq. 195, the right-hand summation in the latter is restricted to the first p terms, and we have

$$sT_o + F_o + \frac{V_o}{s} = \sum_{i,k=1}^{p} z_{ik}I_i\bar{I}_k = \sum_{i,k=1}^{p} z_{ik}I_i\bar{I}_k, \tag{209}$$

in which the equivalence of the two double sums is seen to follow from the symmetry condition $z_{ik} = z_{ki}$. As later discussions will show, it is possible to determine all the properties of driving-point and transfer impedances from this result. At this time we shall consider only the specific relation obtained for a single driving-point (p=1) which reads

$$sT_o+F_o+ \frac{V_o}{s} = z_{11}I_1\bar{I}_1 = z_{11}\left|I_1\right|^2 \tag{210}$$

whence

$$z_{11}= \frac{sT_o+F_o+V_o/s}{\left|I_1\right|^2} \tag{211}$$

or

$$z_{11}= (sT_o+F_o+ \frac{V_o}{s})I_1 = 1. \tag{212}$$

With the expressions 192, 193, 194 for T_o, F_o, V_o, this result is the desired generalization of the one given by Eq. 189 to permit the consideration of any complex frequencies. Its usefulness may be attributed to the fact that the functions T_o, F_o, V_o are real and positive in spite of any complex values that the I_k's may have as a result of satisfying Eqs. 191 for an arbitrary complex frequency s. This fact may readily be proved through writing

$$I_i = a_i + jb_i; \; \bar{I}_k = a_k = jb_k \tag{213}$$

in which the a's and b's are real but otherwise arbitrary. Then

$$I_i I_k = (a_i a_k + b_i b_k) + j(a_k b_i - a_i b_k). \tag{214}$$

Suppose we substitute into Eq. 192 for T_o, and consider first the imaginary part which may be written as the difference of two sums, thus

$$\sum_{i,k=1}^{\ell} L_{ik} a_k b_i - \sum_{i,k=1}^{\ell} L_{ik} a_i b_k \,. \tag{215}$$

If in the first of these we interchange the letters i and k (which we can do because they are merely summation indexes) and then note the symmetry condition $L_{ik} = L_{ki}$; it becomes clear that thetwo sums in 215 are identical and hence that their difference vanishes. The first part of our above statement, namely that T_o is real for any complex I_k's, is thus proved.

Substitution of 214 into 192 now yields T_o in the form

$$T_o = \sum_{i,k=1}^{\ell} L_{ik} a_i a_k + \sum_{i,k=1}^{\ell} L_{ik} b_i b_k \tag{216}$$

Reference to Eq. 112 shows that each of these double sums is a quadratic form like T representing the instantaneous value of stored magnetic energy. In Eq. 112 the variables are denoted by the letter i while in the sums 216 the variables involve respectively the letters a and b which, like the instantaneous currents i_k, are real quantities. Since the quadratic form T is related to a passive network, its values cannot become negative no matter what values (positive or negative) are assigned to the i_k's since one can through the insertion of current sources force the currents in a network to have <u>any</u> set of values, and yet the instantaneous stored magnetic energy must always have a positive value.

The property of a quadratic form like T to have only positive values no matter what the values of its variables may be (it is then referred to as a positive definite quadratic form) must clearly be the result of its coefficients L_{ik} having certain relative values; that is to say, it is a property of the matrix [L] characterizing the quadratic form. It follows, therefore, that the quadratic forms in Eq. 216 can have only positive values, and hence T_o can have only positive values.

Since the quadratic forms 193 and 194 for F_o and V_o are identical in form with T_o, the same argument shows that all three functions T_o, F_o, V_o are real and positive for any complex frequency s, and that this result follows from the positive definite character of the instantaneous energy functions T, F, and V as given by Eqs. 111, 112, 113.

The expression 212 for a driving-point impedance is obviously not an explicit one so far as its dependence upon the complex variable s is concerned, for the functions T_o, F_o, V_o, implicitly are functions of s since they depend upon the I_k's which are solutions of the Eqs. 191 for a specific value of s. Nevertheless the representation for z_{11} as given by Eq. 212 enables one to determine all of the properties pertinent to the driving-point impedance of a linear passive network. Such determinations, which are invaluable to devising methods of synthesis for prescribed impedance functions, will be carried out in the discussions appropriate to these topics.

At present we wish rather to show next that analogous results pertinent to admittance functions are obtained through following a procedure which is precisely dual to the one just given. Thus in the node equilibrium Eqs. 122 we introduce the assumptions

$$i_{ni} = I_i e^{st} \quad \text{and} \quad e_k = E_k e^{st} \tag{217}$$

and, after cancellation of the exponential factor, have

$$\sum_{k=1}^{n} \left(C_{ik}s + G_{ik} + \frac{\Gamma_{ik}}{s}\right)E_k = I_i ; \; i = 1, 2, \dots n. \tag{218}$$

The successive equations in this set we multiply by $\overline{E}_1, \overline{E}_2 \ldots \overline{E}_n$, and then add the results. Introducing the notation

$$V_o^* = \sum_{i,k=1}^{n} C_{ik}\overline{E}_i E_k = \sum_{i,k=1}^{n} C_{ik} E_i \overline{E}_k, \tag{219}$$

$$F_o^* = \sum_{i,k=1}^{n} G_{ik}\overline{E}_i E_k = \sum_{i,k=1}^{n} G_{ik} E_i \overline{E}_k, \tag{220}$$

$$T_o^* = \sum_{i,k=1}^{n} \Gamma_{ik}\overline{E}_i E_k = \sum_{i,k=1}^{n} \Gamma_{ik} E_i \overline{E}_k, \tag{221}$$

in which the equivalence of the two sums in each equation follows from the symmetry condition $C_{ik} = C_{ki}$ etc., the result of these operations yields

$$sV_o^* + F_o^* + \frac{T_o^*}{s} = \sum_{i=1}^{n} I_i \overline{E}_i. \tag{222}$$

This relation is dual to Eq. 195 obtained on the loop basis, and the quadratic forms V_o^*, F_o^*, T_o^* are respectively dual to T_o, F_o, V_o. Like the latter their values are real and positive for all complex E_k-values resulting from the solution of Eqs. 218 for a chosen complex s-value. In order to consider these equations appropriate to a p terminal-pair network, we shall assume that nonzero current sources are applied only to the first p node-pairs, and with the abbreviation

$$\eta_{ik} = C_{ik}s + G_{ik} + \frac{\Gamma_{ik}}{s}, \tag{223}$$

write the pertinent Eqs. 218 in the more explicit form

$$\begin{aligned}
\eta_{11}E_1 + \eta_{12}E_2 + \ldots + \eta_{1n}E_n &= I_1 \\
-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,- & \\
\eta_{p1}E_1 + \eta_{p2}E_2 + \ldots + \eta_{pn}E_n &= I_p \\
\eta_{p+1,1}E_1 + \eta_{p+1,2}E_2 + \ldots + \eta_{p+1,n}E_n &= 0 \\
-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,- & \\
\eta_{n1}E_1 + \eta_{n2}E_2 + \ldots + \eta_{nn}E_n &= 0.
\end{aligned} \tag{224}$$

The η-matrix of this set of equations we now represent in the partitioned form

$$[\eta] = \begin{bmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1n} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2n} \\ \hline \eta_{n1} & \eta_{n2} & \cdots & \eta_{nn} \end{bmatrix} = \left[\begin{array}{c|c} \eta_{pp} & \eta_{pq} \\ \hline \eta_{qp} & \eta_{qq} \end{array}\right], \qquad (225)$$

entirely analogous to the partitioning used in Eq. 198 except that here p+q = n instead of ℓ. The column matrices

$$[E] = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix} \quad \text{and} \quad [I] = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ 0 \end{bmatrix}, \qquad (226)$$

are correspondingly partitioned as indicated by

$$[E] = \begin{bmatrix} E_p \\ \hline E_q \end{bmatrix} \quad \text{and} \quad [I] = \begin{bmatrix} I_p \\ \hline 0 \end{bmatrix}, \qquad (227)$$

whereupon one may write the Eqs. 224 in the equivalent matrix form

$$\begin{aligned} [\eta_{pp}] \times [E_p] + [\eta_{pq}] \times [E_q] &= [I_p] \\ [\eta_{qp}] \times [E_p] + [\eta_{qq}] \times [E_q] &= [0]. \end{aligned} \qquad (228)$$

The second of these equations may be solved for $[E_q]$ giving

$$[E_q] = -[\eta_{qq}]^{-1} \times [\eta_{qp}] \times [E_p], \qquad (229)$$

whence substitution into the first equation yields

$$([\eta_{pp}] - [\eta_{pq}] \times [\eta_{qq}]^{-1} \times [\eta_{qp}]) \times [E_p] = [I_p] \qquad (230)$$

which symbolizes the desired set of equations involving only the voltages at the accessible node-pairs.

If we introduce the square admittance matrix of order p

$$[y_{pp}] = [\eta_{pp}] - [\eta_{pq}] \times [\eta_{qq}]^{-1} \times [\eta_{qp}]$$

$$= \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1p} \\ y_{21} & y_{22} & \cdots & y_{2p} \\ - & - & - & - \\ y_{p1} & y_{p2} & \cdots & y_{pp} \end{bmatrix}, \qquad (231)$$

we may write the Eqs. 230 in the equivalent algebraic form

$$\sum_{k=1}^{p} y_{ik} E_k = I_i; \; i = 1, 2, \ldots p. \qquad (232)$$

This is an abridged form of the Eqs. 218 appropriate to the situation in which current sources are feeding only the first p node-pairs. The co-efficients y_{ik} in these equations are determined from the η_{ik} (Eq. 223) characterizing the Eqs. 218, in a manner indicated by the matrix Eq. 231. The transition from Eqs. 218 to 232 may be described as a process of suppressing or eliminating the inaccessible or unwanted node-pair voltages.

Although the Eqs. 218 or 232 are derived specifically on the assumption that the I's are sources and the E's are responses, they correctly relate these currents and voltages even though some or all of the E's may be sources and correspondingly some or all of the I's become responses. If the $E_1 \ldots E_p$ in Eqs. 232 are regarded as sources, then the resulting $I_1 \ldots I_p$ at the respective terminal-pairs are <u>explicitly</u> given by these equations. Under these circumstances the terminal-pairs are all short-circuited, and for this reason the y_{ik} are referred to as a set of <u>short-circuit driving-point and transfer admittances</u> characterizing the p terminal-pair network (compare with the analogous quantities described in Art. 7, Ch. III for resistance networks).

Specifically, if E_1 is the only nonzero voltage source, then

$$\begin{aligned} I_1 &= y_{11}E_1 \\ I_2 &= y_{21}E_1 \\ &\text{- - - - - - -} \\ I_p &= y_{p1}E_1 , \end{aligned} \qquad (233)$$

or

$$\begin{aligned} y_{11} &= I_1/E_1 \\ y_{21} &= y_{12} = I_2/E_1 \\ &\text{- - - - - - - - -} \\ y_{p1} &= y_{1p} = I_p/E_1 . \end{aligned} \qquad (234)$$

If E_1 is thoughtof as being equal in value to one reference volt, then the complex current I_1 at terminal-pair 1 is numerically identical with the driving-point admittance y_{11}; the complex current I_2 at terminal-pair 2 is numerically identical with the transfer admittance $y_{12} = y_{21}$; and so forth. In any of these transfer relations like $I_2 = y_{12}E_1$, it is important to observe that E_1 must be the source and I_2 the response; and that this relation is invalid if I_2 now is regarded as a source and E_1 as a response because this change of attitude violates the conditions under which the particular relations 233 are extracted from the general relations 232 (namely, E_2 is no longer zero).

Another set of particular relations may be extracted from the Eqs. 232 on the assumption that E_2 is the only nonzero voltage source, namely

$$\begin{aligned} I_1 &= y_{12}E_2 \\ I_2 &= y_{22}E_2 \\ &\text{- - - - - - -} \\ I_p &= y_{p2}E_2 , \end{aligned} \qquad (235)$$

from which we may again obtain relations for the y's like Eqs. 234 which lend themselves to physical interpretation; and once more it must be emphasized that the transfer relations apply only if E_2 is a source.

The driving-point relations like $I_1 = y_{11}E_1$, or $I_2 = y_{22} E_2$, in contrast, are valid regardless of whether the voltage or the current is the source because no restriction is placed upon the nonzero E_1 in Eqs. 233 or upon the nonzero E_2 in Eqs. 235. As stated before, a driving-point relation remains valid regardless of which quantity, E or I, is the source and which is the response, while a transfer relation is valid only for a specific assignment of the roles of source and response, namely that one which is implicit in its derivation.

It is significant to mention in passing that if the Eqs. 205 and 232 are derived for the same p terminal-pair network, then they must obviously be inverse sets, and the matrices $[z_{pp}]$ and $[y_{pp}]$, Eqs. 204 and 231, are inverse. Specifically, if these matrices have the determinants Z and Y with cofactors Z_{sk} and Y_{sk}, then

$$z_{sk} = \frac{Y_{ks}}{Y} \text{ and } y_{sk} = \frac{Z_{ks}}{Z} . \qquad (236)$$

If we now substitute the expression for I_i as given by Eqs. 232 into Eq. 222, the right-hand summation in the latter is restricted to the first p terms, and we have

$$sV_o^* + F_o^* + \frac{T_o^*}{s} = \sum_{i,k=1}^{p} y_{ik} \bar{E}_i E_k = \sum_{i,k=1}^{p} y_{ik} E_i \bar{E}_k , \qquad (237)$$

in which the equivalence of the two double sums is seen to follow from the symmetry condition $y_{ik} = y_{ki}$. For p = 1 we obtain the specific result pertinent to a single driving-point.

$$sV_o^* + F_o^* + \frac{T_o^*}{s} = y_{11}E_1\bar{E}_1 = y_{11}\left|E_1\right|^2 , \qquad (238)$$

whence

$$y_{11} = \frac{sV_o^* + F_o^* + T_o^*/s}{|E_1|^2} \tag{239}$$

or

$$y_{11} = \left(sV_o^* + F_o^* + \frac{T_o^*}{s}\right)_{E_1 = 1} . \tag{240}$$

Together with the relations 219, 220, 221 for V_o^*, F_o^*, T_o^*, this result is the desired generalization of the one given by Eq. 188 permitting the consideration of any complex frequency. Like the expression 212 for z_{11}, its usefulness stems from the fact that the functions V_o^*, F_o^*, T_o^* have positive real values for all complex s, the method of proof for this statement being precisely that given in the consideration of T_o, F_o, V_o.

It remains to establish relations between T_o, F_o, V_o with and without asterisk, and T_{av}, F_{av}, V_{av} that apply to $s = j\omega$. If Eqs. 195 and 222 relate to the same network with sources at p points of access, then the right-hand sums are conjugates, and hence the left-hand sides of these equations must likewise be conjugates; that is

$$sT_o + F_o + \frac{V_o}{s} = \bar{s}V_o^* + F_o^* + \frac{T_o^*}{\bar{s}} , \tag{241}$$

and so we have

$$F_o = F_o^* , \; V_o = |s|^2 V_o^*, \; T_o^*/|s|^2 . \tag{242}$$

Comparison of the expressions for these functions with pertinent ones for F_{av}, T_{av}, V_{av}, moreover, shows that for $s=j\omega$ one may make the identifications

$$F_o = 4F_{av} = F_o^*; \; T_o = 4T_{av} = T_o^*/\omega^2; \; V_o^* = 4V_{av} = V_o/\omega^2 . \tag{243}$$

For complex s-values, a physical interpretation of the functions T_o, F_o, V_o, or V_o^* F_o^*, T_o^* is not readily possible, but this fact is of little consequence since it is their mathematical rather than their physical significance that justifies introducing them into the present discussions.