GENERAL ELECTRIC
COMPANY

MEDIUM TRANSFORMER DEPARTMENT

REDMOND CIRCLE . . . . ROME, GEORGIA

August 3, 1960

Mrs. Linda S. Danaceau
Secretary to Mr. Burton Grad
I. B. M.
112 East Post Rd.
White Plains, N. Y.

Dear Mrs. Danaceau:

The presentation on mechanized budgeting, which I made at the Seventeenth National Meeting of the Operations Research Society of America, was designed for the "General Sessions" having as their purpose to illustrate how operations research aids in dealing with business problems. The presentations in these sessions were to be relatively nontechnical and were to emphasize the problem area rather than the methods of analysis used.

I pointed out during this presentation that a report covering the details of the research effort, titled "Mechanized Budget Systems - Project Report #70", written by Mr. R. D. Henderson, was available to those who wished a more comprehensive understanding of this approach to budgeting. Mr. Henderson was responsible for this project while employed on my Operations Research and Synthesis team in the Jet Engine Department of the General Electric Company.

From the nature of your request, I have assumed that you are interested in the detailed Project Report, as opposed to the general paper I presented. I am, therefore, enclosing a copy of the Project Report with this letter.

Very truly yours,

G.E. Walker

Operations Research & Synthesis
Specialist

G. E. WALKER:b
Encl.

TEMPO REPORT

CHARACTERISTICS OF OPTIMUM INVENTORY POLICIES

RM 58TMP-50

TECHNICAL MILITARY PLANNING OPERATION

GENERAL ELECTRIC

SANTA BARBARA, CALIFORNIA

CHARACTERISTICS OF OPTIMUM INVENTORY POLICIES

R.L. Bovaird

RM 58TMP-50

10 November 1958

TECHNICAL MILITARY PLANNING OPERATION
GENERAL ELECTRIC COMPANY
SANTA BARBARA, CALIFORNIA

SUMMARY

This Research Memorandum describes the sensitivity of the optimum inventory policies developed for WS 107A-1 guidance equipment spares to variations in policy-determining parameters.[1] This sensitivity is described in terms of the effect on the total program cost when any one of these parameters which are estimated for the spare is varied. Errors in these estimates lead to adopting a non-optimum policy which costs more than the optimum policy. The added cost caused by parameter estimation errors is calculated and presented as a percent of the cost of the optimum policy.

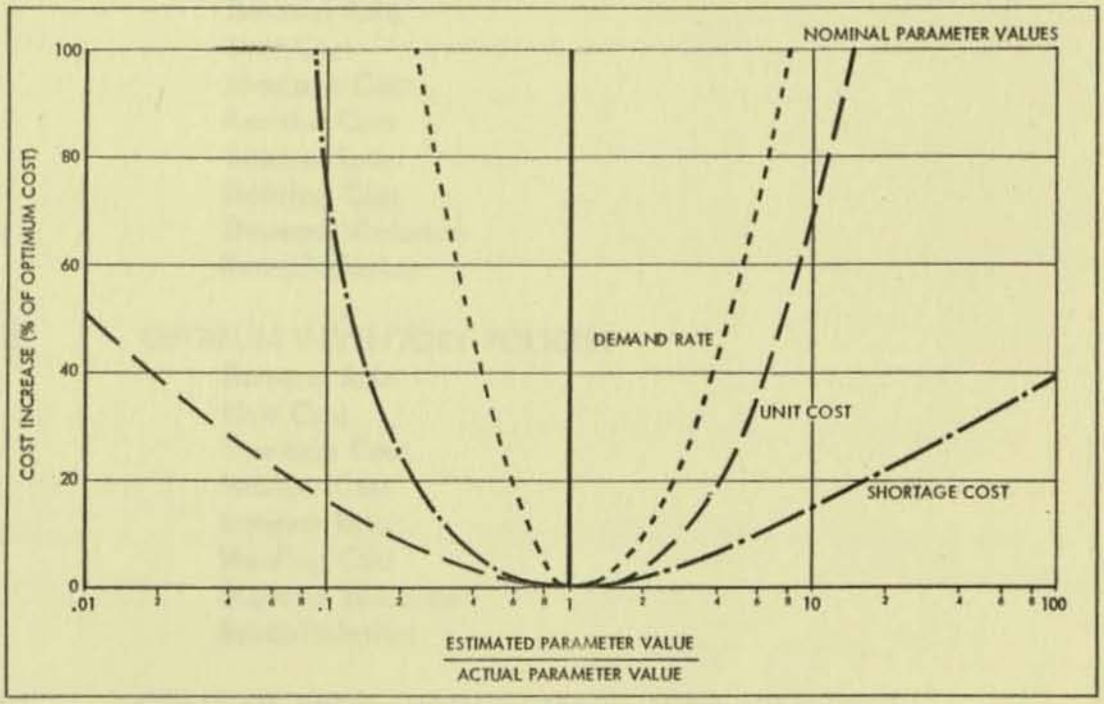

Figure 1. Cost of Parameter Errors

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[1] These policies apply to non-reparable base spares and are described in RM 57TMP-10, Report on Phase A Materiel Management Study, Herbert W. Karr, Technical Military Planning Operation, General Electric Company, Santa Barbara, California, 1 November 1957.

Figure 1 shows the additional cost resulting from parameter errors about nominal values for each of the three most important inventory parameters: Mean demand rate, unit cost, and shortage cost. It is seen that program costs are most sensitive to errors in estimating demand rate. Program costs are also sensitive to overestimation errors in unit cost and underestimation errors in shortage cost. Program costs are less sensitive to overestimation errors in shortage cost and underestimation errors in unit cost.

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INTRODUCTION

This is one of a series of reports resulting from a study project which was established in order to achieve effective supply support for the WS-107A Ground Guidance System. One of the goals of this project was the establishment of optimum inventory policies which will specify initial provisioning quantities, stock control levels, and reorder points for every component of the guidance system. Mathematical decision models capable of generating these optimum inventory policies have been developed and are described in two other reports.[1]

The inventory model described in RM 57TMP-10[1] requires the estimation of several parameters for each spare part in order to determine the inventory policy which is optimum for the part. Since there are normally several thousand different parts in a piece of electronic equipment, the accurate estimation of these parameters for all parts is a formidable task. Accordingly, it is important from an economic point of view that the accuracy of each parameter estimate be commensurate with that parameter's influence on the total cost attributable to the particular spare under consideration. That is, if total costs are influenced very little by a particular parameter, then it need not be known very accurately. On the other hand, if total costs can be significantly reduced by increasing the accuracy of the parameter estimate, then it is worthwhile expending the additional time and effort required to get a more accurate estimate.

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[1] RM 57TMP-10, Report on Phase A Materiel Management Study, Herbert W. Karr, Technical Military Planning Operation, General Electric Company, Santa Barbara, California, 1 November 1957.

RM 58TMP-19, Interim Provisioning Policies for WS-107A Guidance Spares, R. F. McIntosh and L. Fisher, Technical Military Planning Operation, General Electric Company, Santa Barbara, California, 20 June 1958.

The approach to this problem that has been adopted consists of systematically investigating the behavior of total program costs when the parameters describing a spare part are varied one at a time. This provides insight into the relative importance of the various parameters. It also gives understanding of the characteristics of the optimum policies which are produced by the inventory model.

The way in which optimum policies allocate the money spent during a program is of particular interest, for knowledge of this provides the perspective necessary for good judgment of the relative importance of each parameter. This subject is discussed in Section I. Within this frame of reference, the characteristics of optimum policies described in Section II are relatively easy to understand.

It is of interest to compare the program costs resulting from optimum policies with those resulting from non-optimum policies. One type of non-optimum policy is caused by parameter estimation errors. Traditional inventory policies are also frequently non-optimum. The adoption of either type of non-optimum policy results in additional program costs, although the nature of the cost increase differs in the two cases. This is discussed in Section III.

For a given total program cost the supply system effectiveness of a non-optimum policy is less than is that for an optimum policy. However, program costs of non-optimum policies belonging to the family produced by the inventory model have an essentially different relationship to operational effectiveness than do traditional policies. This is discussed in detail in Section IV.

Throughout the report it has been necessary to adopt a device to simplify the presentation. The technique adopted has been to choose a "nominal" set of parameter values about which the parameters are varied one at a time. This is necessary, since seven parameters are varied in Sections I and II. Six functions of these parameters are shown in Section I and two functions in Section II. The behavior of these functions in the neighborhood of the nominal point has been found to be characteristic of that throughout the range of interest with the exception of extreme parameter values. In particular, it is assumed throughout the report that the part has parameter values such that it would actually be purchased as a spare. This means that an initial order is placed for at least one unit. If this is not the case, then the only costs incurred are shortage costs and the policy is very simple: Buy none.

Before going into a detailed study of the characteristics of optimum inventory policies, it will be helpful to point out some general features of the policies and their associated cost. Figure 2 shows a typical optimum inventory policy. It consists of a stock control level and a reorder point for each year in the program. These are determined by the number of years remaining in the program. For the example shown in Figure 2, both the stock control level and reorder point are approximately constant until about six years remain in the program. Thereafter, the stock control level decreases until the end of the program. These portions of the program are said to have periodic and life-of-type policies, respectively. If the program is sufficiently long, the experience of following the above policy will be quite uniform in the periodic region. The reorder point will be passed at reasonably regular time intervals, and the number of units ordered at the next order time will always be about the same. As the program nears completion and the life-of-type region is entered, reorders cease and the remaining units are used up.

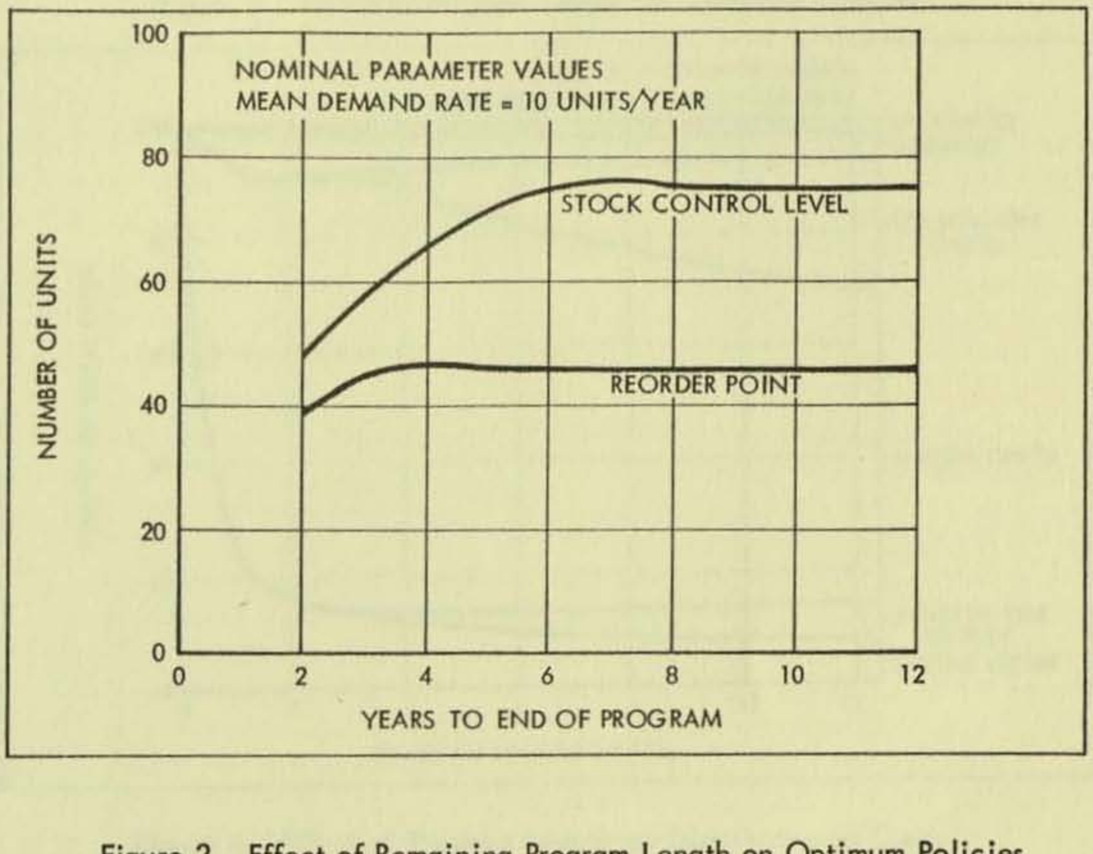

Figure 2. Effect of Remaining Program Length on Optimum Policies

The distribution of total program cost corresponding to the policy of Figure 2 is shown in Figure 3 for various program lengths. As can be seen, for short programs the initial purchase of spares is almost the only cost. As the program length increases the additional cost becomes almost entirely allocated to follow-on orders and units. Increases in holding and shortage costs are relatively insignificant.

The percentage cost distribution is shown in Figure 4. It is seen that the portions of total cost allocated by optimum policies to units and to orders remain constant with program length. The division of these costs between initial and follow-on purchases changes, however, with program length.

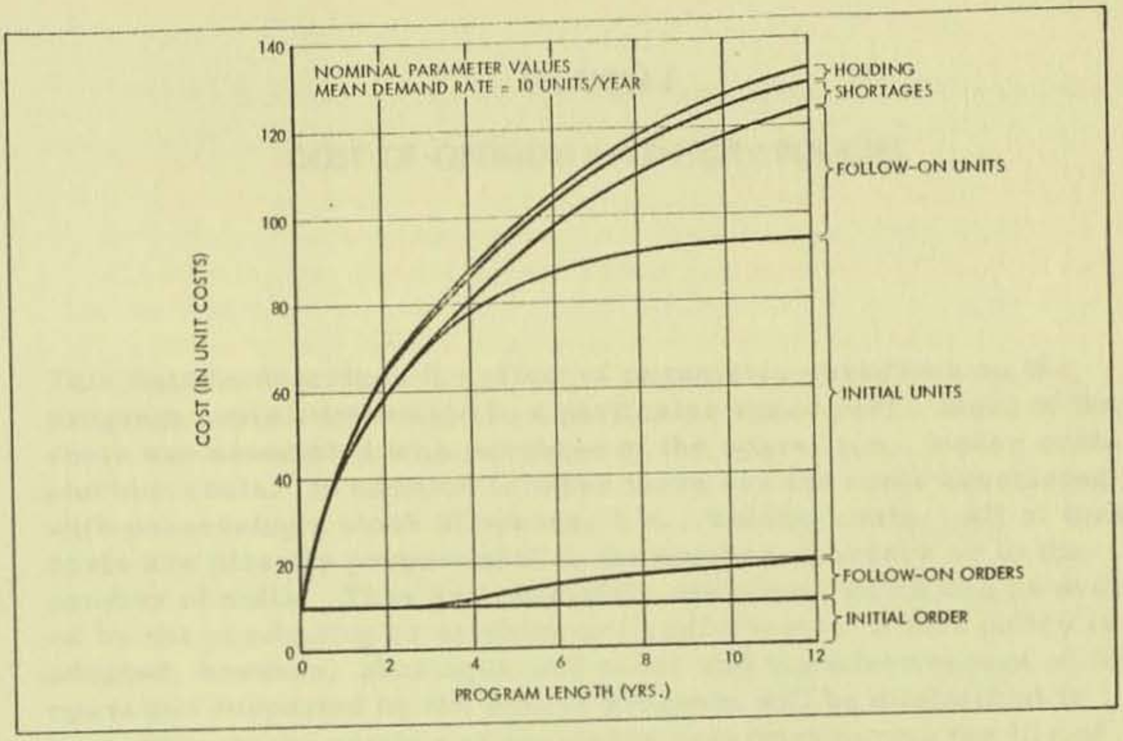

Figure 3. Effect of Program Length on Optimum Policies

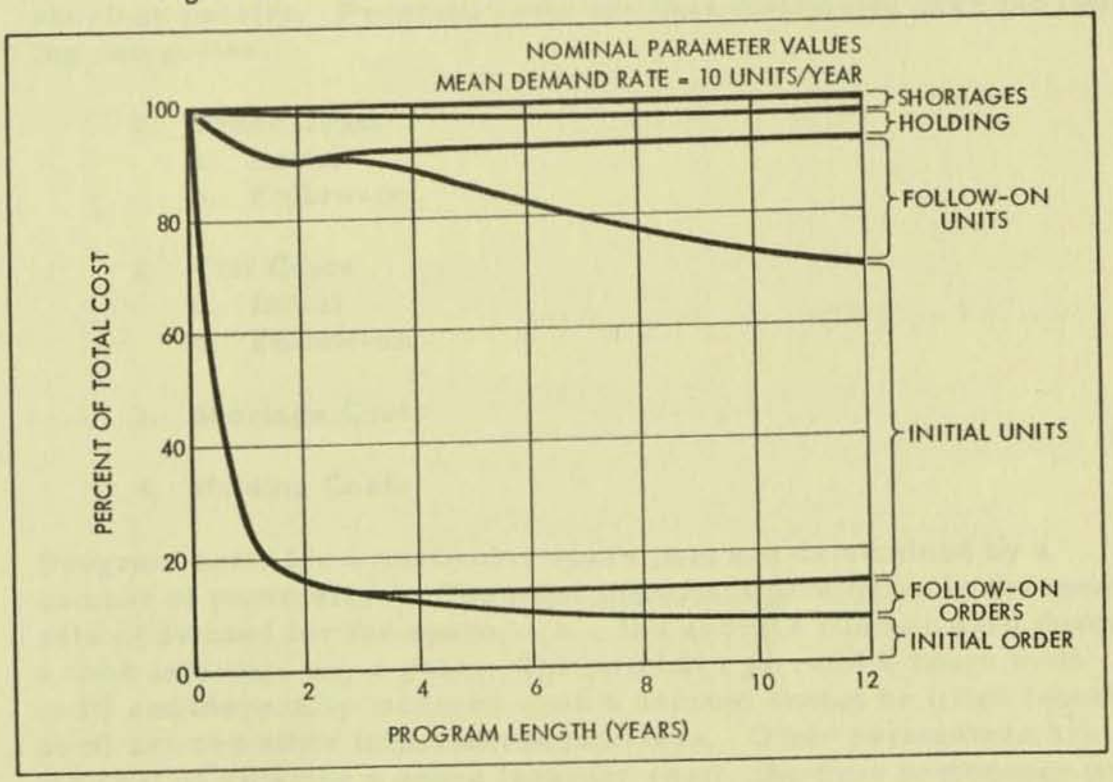

Figure 4. Effect of Program Length on Distribution of Costs

SECTION I

COST OF OPTIMUM INVENTORY POLICIES

This Section describes the effect of parameter variations on the program costs attributable to a particular spare part. Most of these costs are associated with purchase of the spare, i. e., order costs and unit costs. In addition to these there are the costs associated with possessing a stock of spares, i. e., holding costs. All of these costs are directly proportional to the number of orders or to the number of units. They are ultimately real costs which can be avoided by not purchasing or stocking any spare parts. If this policy is adopted, however, shortages will occur and the effectiveness of the operation supported by the spares program will be diminished in proportion to the number of shortages occurring during the life of the operation. The cost of these shortages is determined by the shortage penalty. Program costs are thus distributed over the following categories:

1. Order Costs
   a. Initial
   b. Follow-on

2. Unit Costs
   a. Initial
   b. Follow-on

3. Shortage Costs

4. Holding Costs

Program costs for a particular spare part are determined by a number of parameters. The most important parameter is the mean rate of demand for the spare, i. e., the average number used during a time interval, say a year. The purchase price of a spare (unit cost) and the penalty incurred when a demand cannot be filled (shortage cost) are two other important parameters. Other parameters are the cost of ordering a spare (reorder cost), the time preference for money (interest rate), the cost of carrying a spare in stock for a time unit (holding cost), and the variability of the mean demand rate (demand variance). These parameters are varied one at a time and are discussed in the following order:

1. Demand Rate
2. Unit Cost
3. Shortage Cost
4. Reorder Cost
5. Interest Rate
6. Holding Cost
7. Demand Variance

Unless otherwise noted, all parameter variations shown in the figures are variations about the following nominal values:

1. Demand Rate - 1 unit per year
2. Unit Cost - $1
3. Shortage Cost - 100 unit costs
4. Reorder Cost - 10 unit costs
5. Interest Rate - 0.2 (compounded annually)
6. Holding Cost - 0.01 unit costs per year
7. Demand Variance - 5 mean demand rates

These nominal parameter values correspond approximately to the average WS-107A Guidance System spare part. The data upon which the figures are based is given in tabular form in Appendix I. The method of computation is described in Appendix IV.

DEMAND RATE

The effect of demand rate on program costs is shown in Figure 5. It is seen that total costs are almost directly proportional to demand rate except for the constant initial order cost. Virtually all the increase in cost caused by increasing demand goes for initial units, although follow-on unit and order costs increase as well. The costs allocated to shortages and for holding units remain practically constant throughout the range of demands shown.

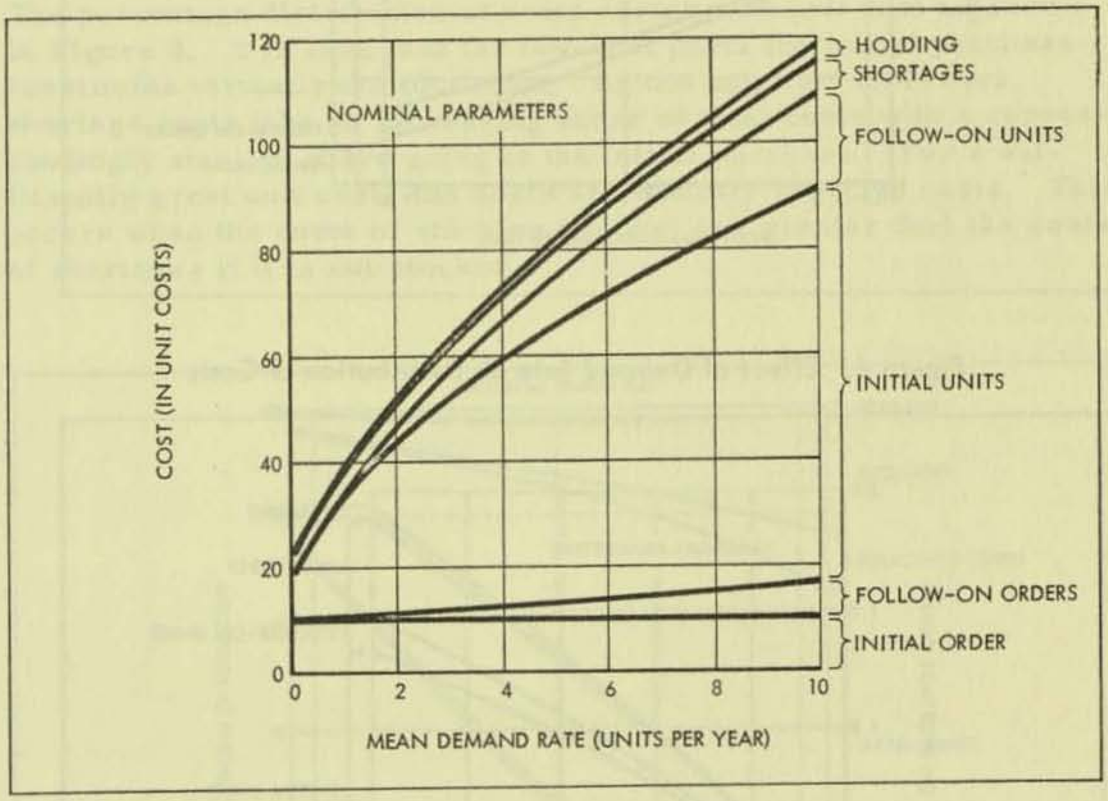

Figure 5. Effect of Demand Rate on Costs of Optimum Policies

The effect of demand rate on the percentage distribution costs is shown in Figure 6. It is seen that as the demand rate increases, unit costs take a larger and larger share of the costs. The percentage allocated to shortages and orders is correspondingly reduced. Most of the costs at the lower demand rates are for the initial purchase while follow-on order and follow-on unit costs become more important as the demand rate increases.

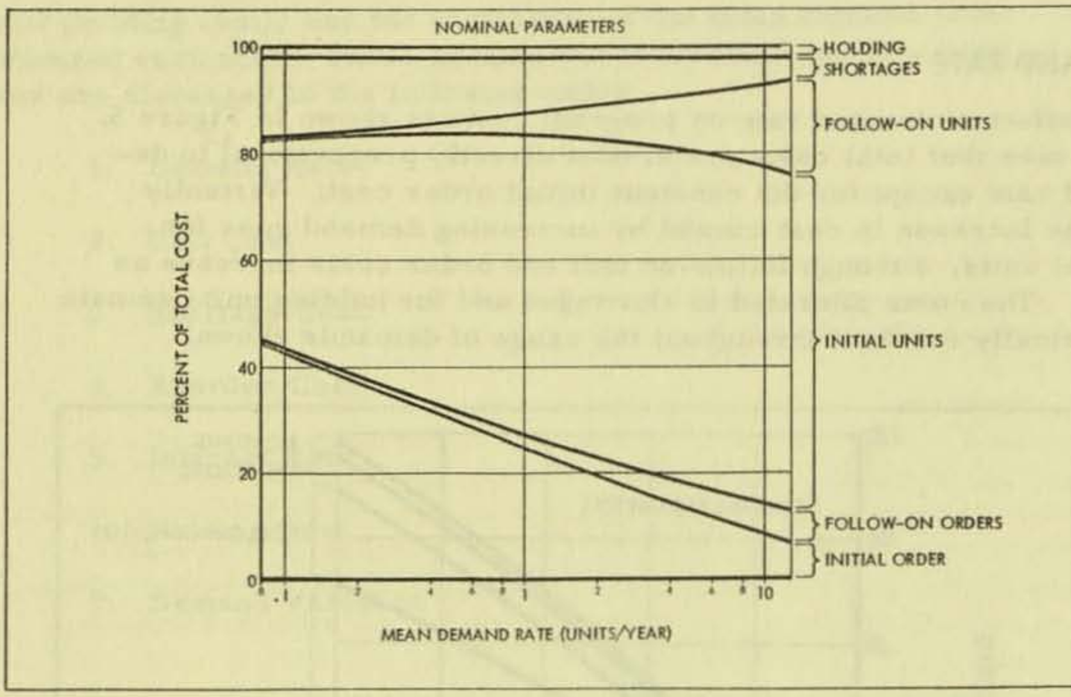

Figure 6. Effect of Demand Rate on Distribution of Costs

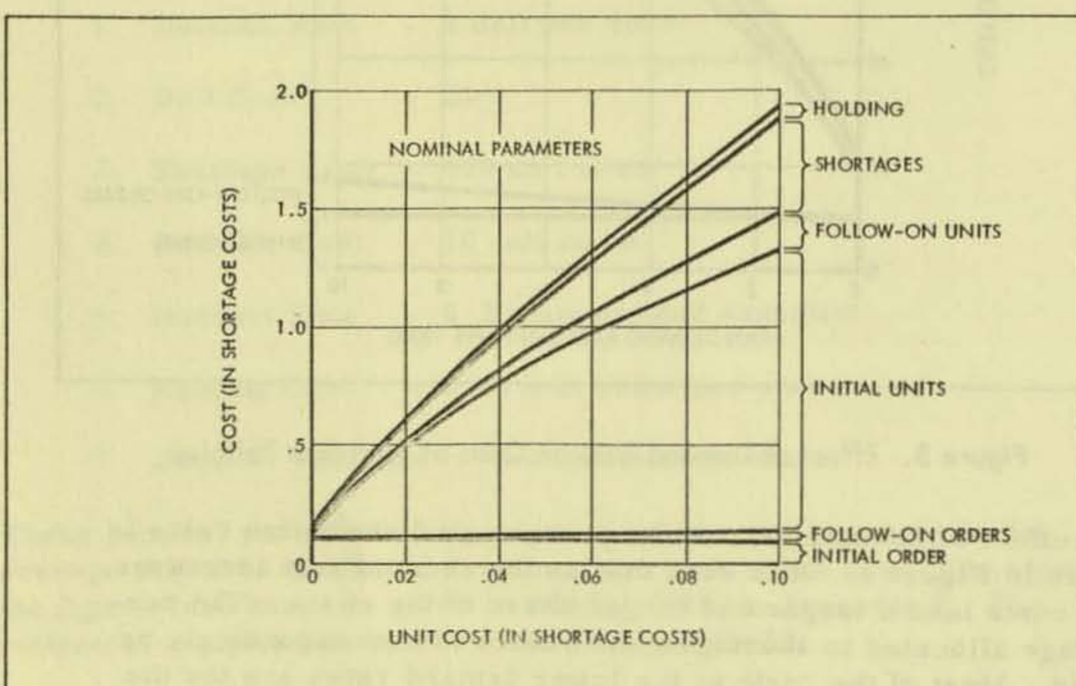

Figure 7. Effect of Unit Cost on Costs of Optimum Policies

Program costs vary with unit cost as shown in Figure 7. As with demand rate, total costs are almost directly proportional to unit cost. Most of the increase is in the cost of initial units, although the cost of shortages is also proportional to unit cost. Holding costs and order costs are almost constant throughout the range of unit costs shown.

The percentage distribution of costs varies with unit cost as shown in Figure 8. It is seen that for low-cost parts the initial purchase constitutes virtually the total cost. As the unit cost increases, shortage costs take an increasing share of total costs with a correspondingly smaller share going to the initial purchase. For a sufficiently great unit cost, the costs are entirely shortage costs. This occurs when the costs of stocking the part are greater than the costs of shortages if it is not stocked.

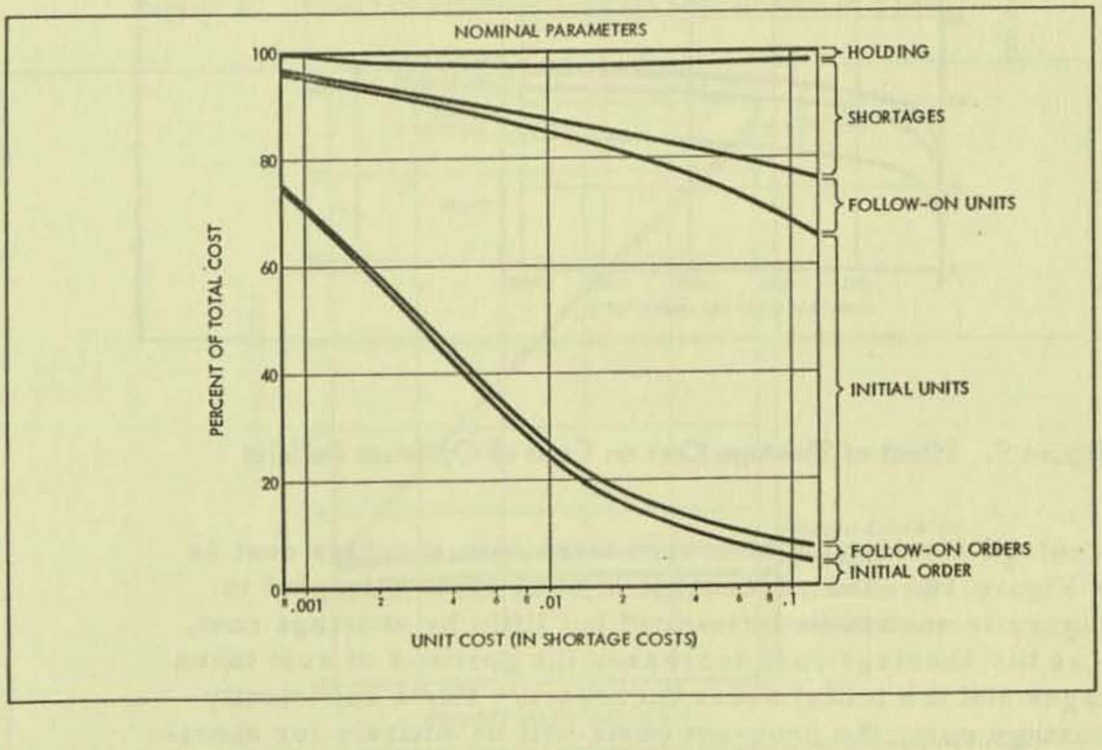

Figure 8. Effect of Unit Cost on Distribution of Costs

SHORTAGE COST

The effect of shortage cost on program costs is shown for two demand rates in Figure 9. It is seen that increasing the shortage cost affects only the unit costs to any appreciable extent. Unit costs rise rapidly with shortage cost initially, but the rate of increase diminishes as the shortage cost becomes greater than about 100 unit costs. The costs of orders, shortages, and holding units are virtually constant throughout the range of shortage costs shown.

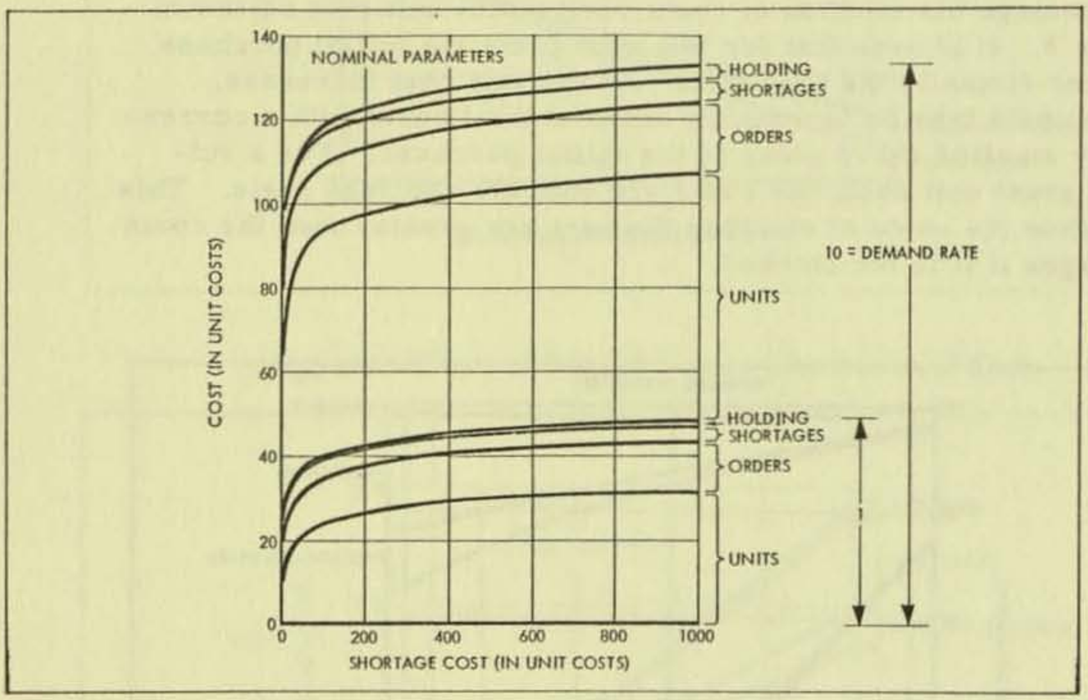

Figure 9. Effect of Shortage Cost on Costs of Optimum Policies

The percentage distribution of costs varies with shortage cost as shown in Figure 10. The percentage of total costs allocated to each category is seen to be influenced but little by shortage cost, although as the shortage cost decreases the portions of cost taken by shortages and the initial order increases. For a sufficiently small shortage cost, the program costs will be entirely for shortages. This happens when the cost of stocking the part exceeds the cost of shortages if it is not stocked.

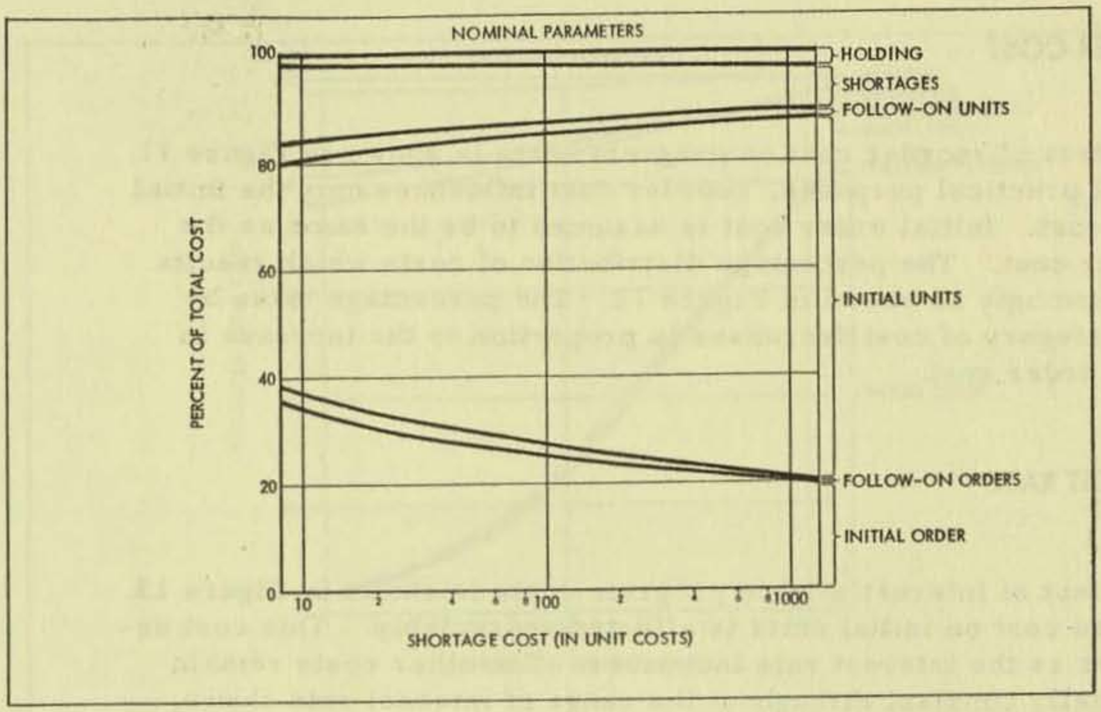

Figure 10. Effect of Shortage Cost on Distribution of Costs

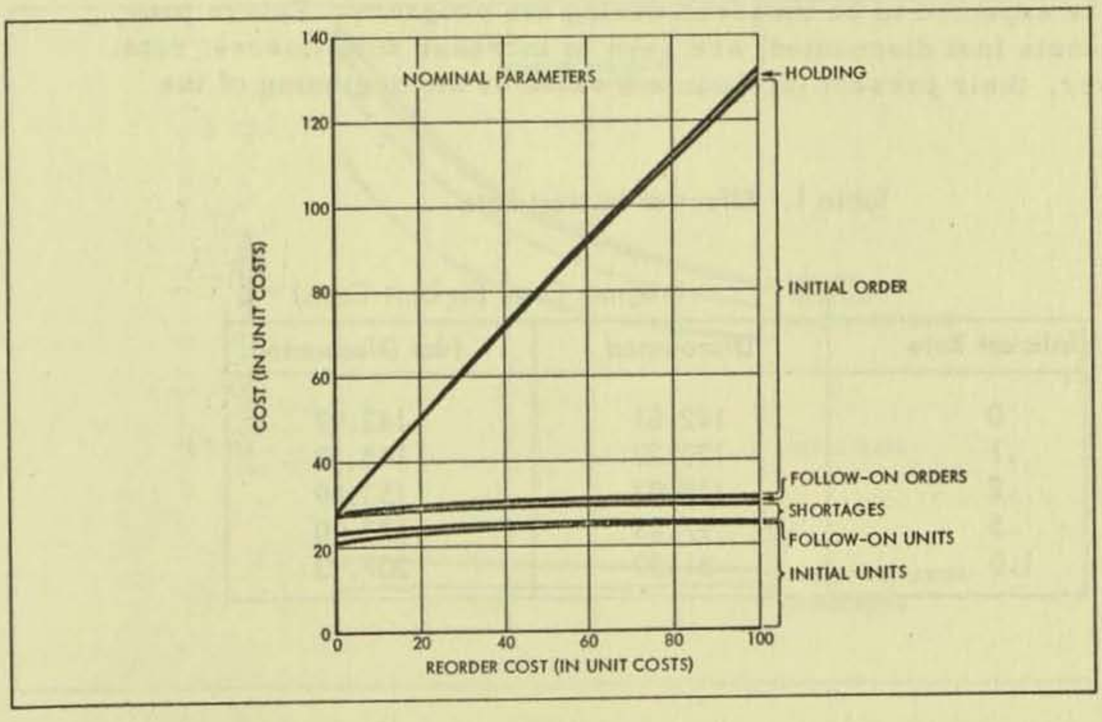

Figure 11. Effect of Reorder Cost on Costs of Optimum Policies

REORDER COST

The effect of reorder cost on program costs is shown in Figure 11. For all practical purposes, reorder cost influences only the initial order cost. Initial order cost is assumed to be the same as the reorder cost. The percentage distribution of costs which results is accordingly as shown in Figure 12. The percentage taken by each category of cost decreases in proportion to the increase in initial order cost.

INTEREST RATE

The effect of interest rate on program costs is shown in Figure 13. Only the cost on initial units is affected appreciably. This cost decreases as the interest rate increases. The other costs remain essentially constant throughout the range of interest rate shown. This perhaps surprising result is easily understood by considering the data shown in Table 1. In the table the discounted total costs shown in Figure 13 are compared to the undiscounted total cost which is expected to be incurred during the program. Future program costs (not discounted) are seen to increase with interest rate. However, their present (discounted) value at the beginning of the

Table 1. Effect of Interest Rate

	Program Costs (in Unit Costs)	
Interest Rate	Discounted	Not Discounted
0	142.81	142.97
.1	130.23	145.58
.2	118.87	153.89
.5	97.13	173.90
1.0	81.39	207.93

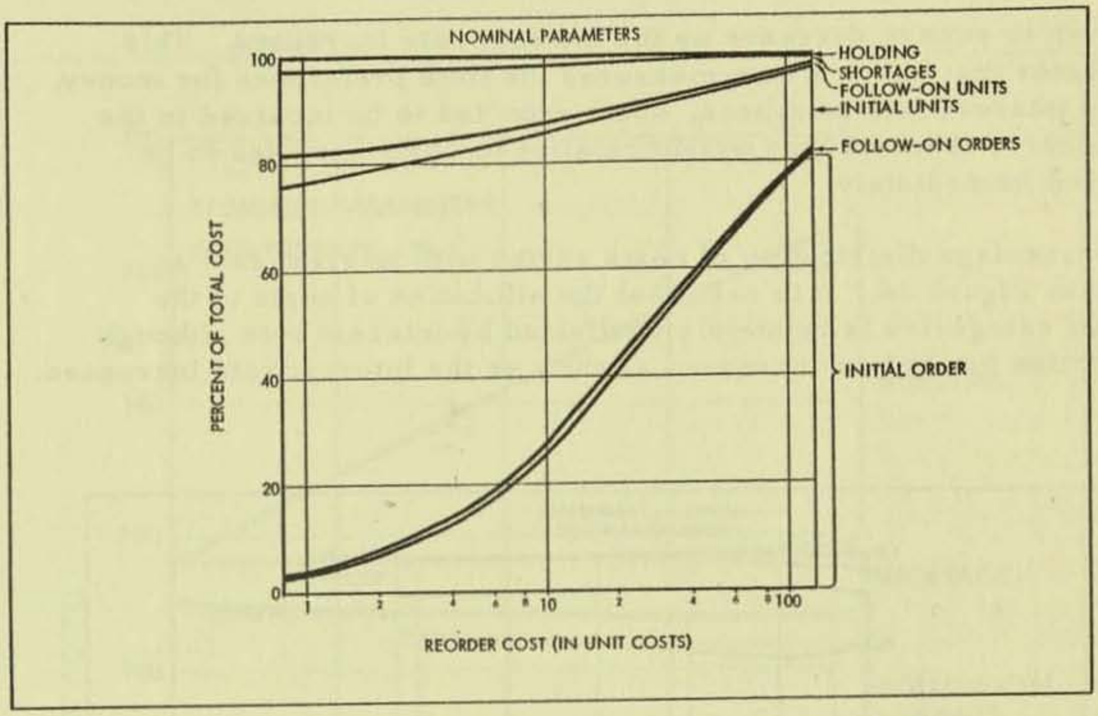

Figure 12. Effect of Reorder Cost on Distribution of Costs

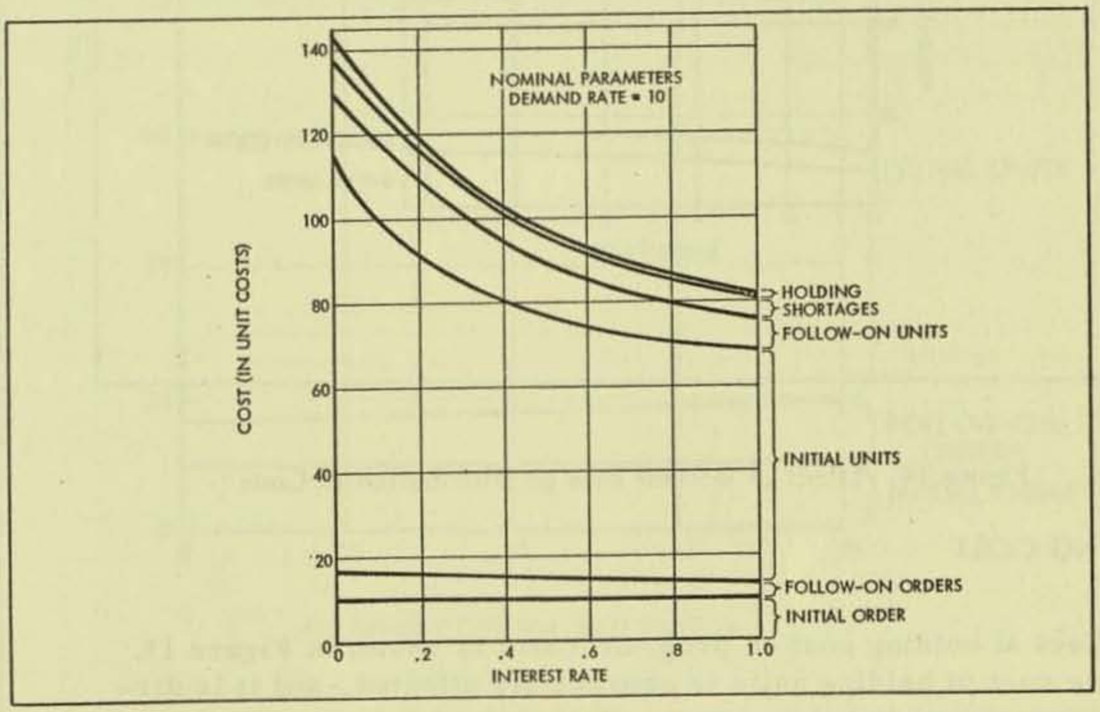

Figure 13. Effect of Interest Rate on Costs of Optimum Policies

program is seen to decrease as the interest rate increases. This illustrates that interest rate measures the time preference for money. As the interest rate increases, costs expected to be incurred in the future carry less and less weight relative to costs expected to be incurred immediately.

The percentage distribution of costs varies with interest rate as shown in Figure 14. It is seen that the allocation of costs to the various categories is relatively unaffected by interest rate although the portion for orders increases slightly as the interest rate increases.

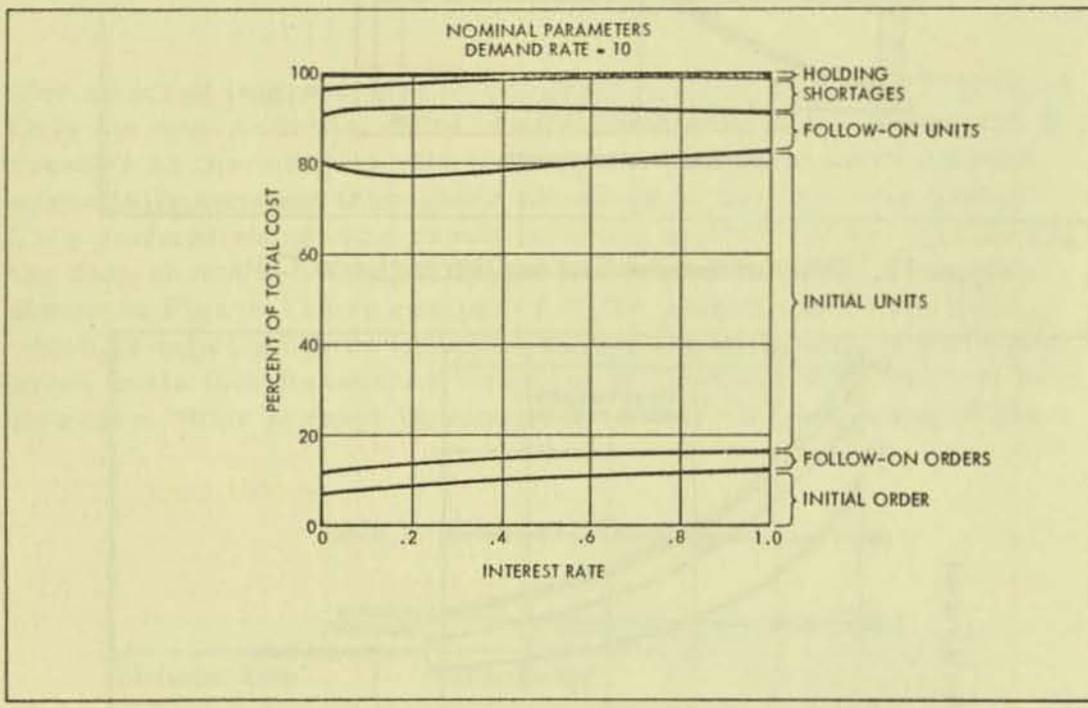

Figure 14. Effect of Interest Rate on Distribution of Costs

HOLDING COST

The effect of holding cost on program costs is shown in Figure 15. Only the cost of holding units is appreciably affected, and it is directly proportional to holding cost. Unit costs decrease slightly and order costs increase slightly as the holding cost increases.

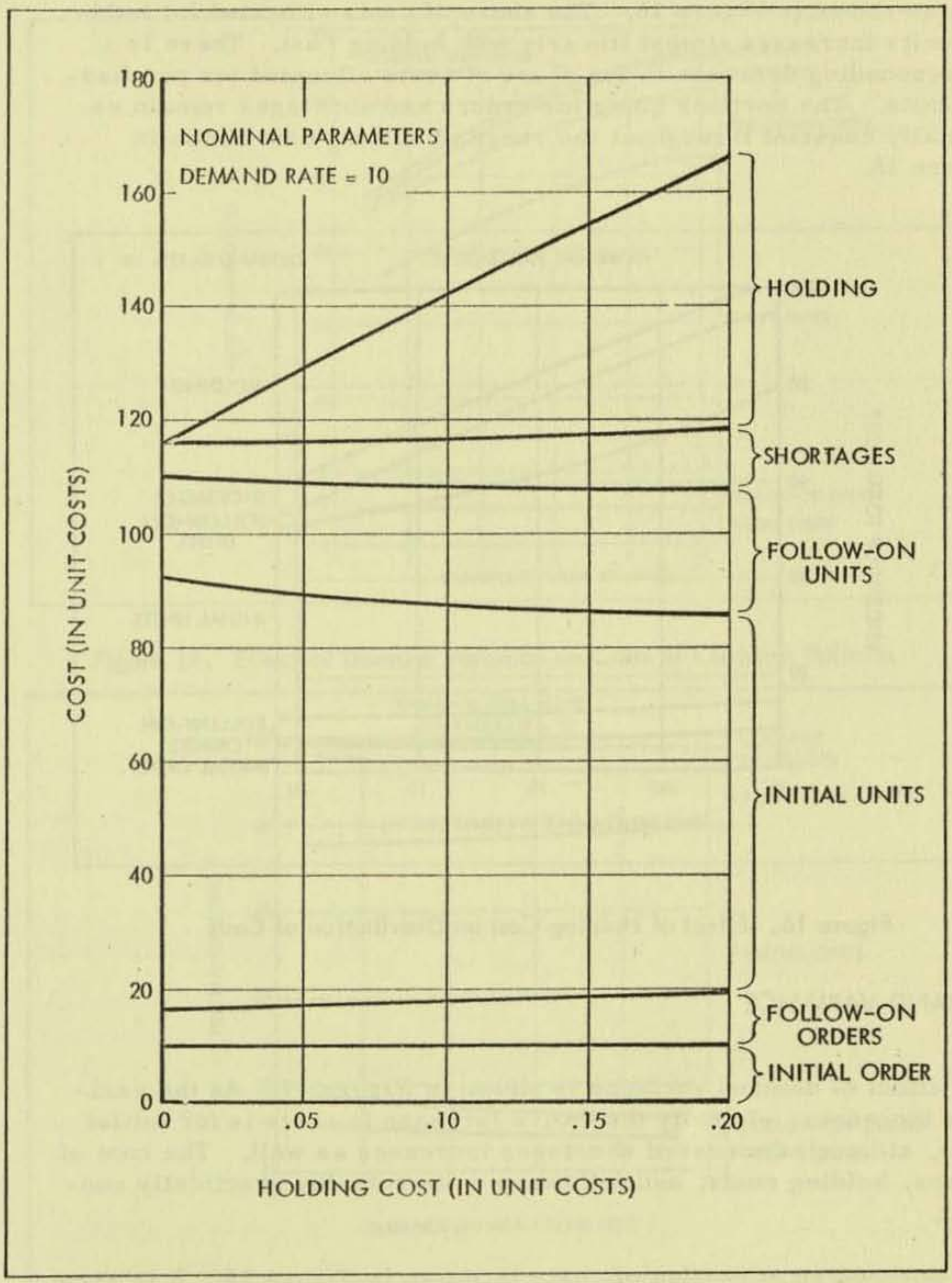

Figure 15. Effect of Holding Cost on Costs of Optimum Policies

The percentage distribution of program costs varies with holding cost as shown in Figure 16. The share of costs allocated for holding units increases almost linearly with holding cost. There is a corresponding decrease in the share of costs allocated for purchasing units. The portions going for orders and shortages remain essentially constant throughout the range of holding cost shown in Figure 16.

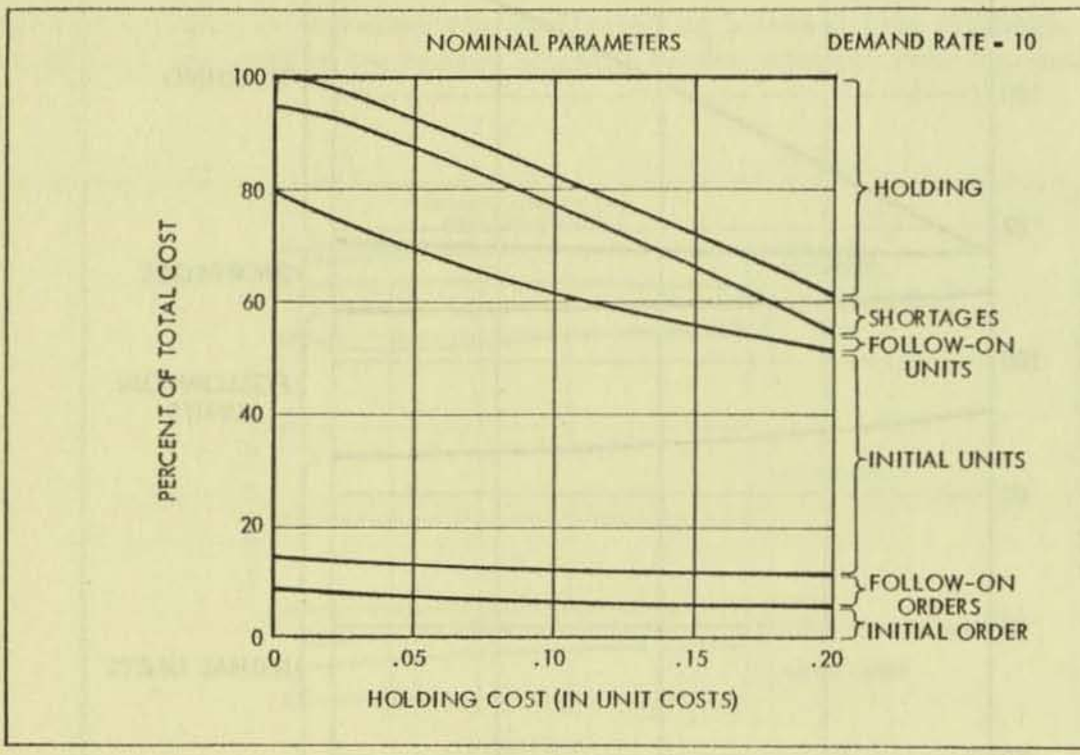

Figure 16. Effect of Holding Cost on Distribution of Costs

DEMAND VARIANCE

The effect of demand variance is shown in Figure 17. As the variance increases, virtually the entire increase in costs is for initial units, although the cost of shortages increases as well. The cost of orders, holding costs, and follow-on units remains practically constant.

The percentage allocation of costs is shown in Figure 18. It is seen to be almost unaffected by demand variance, although the share of costs allocated to shortages and initial units increases slightly at the expense of orders and follow-on units.

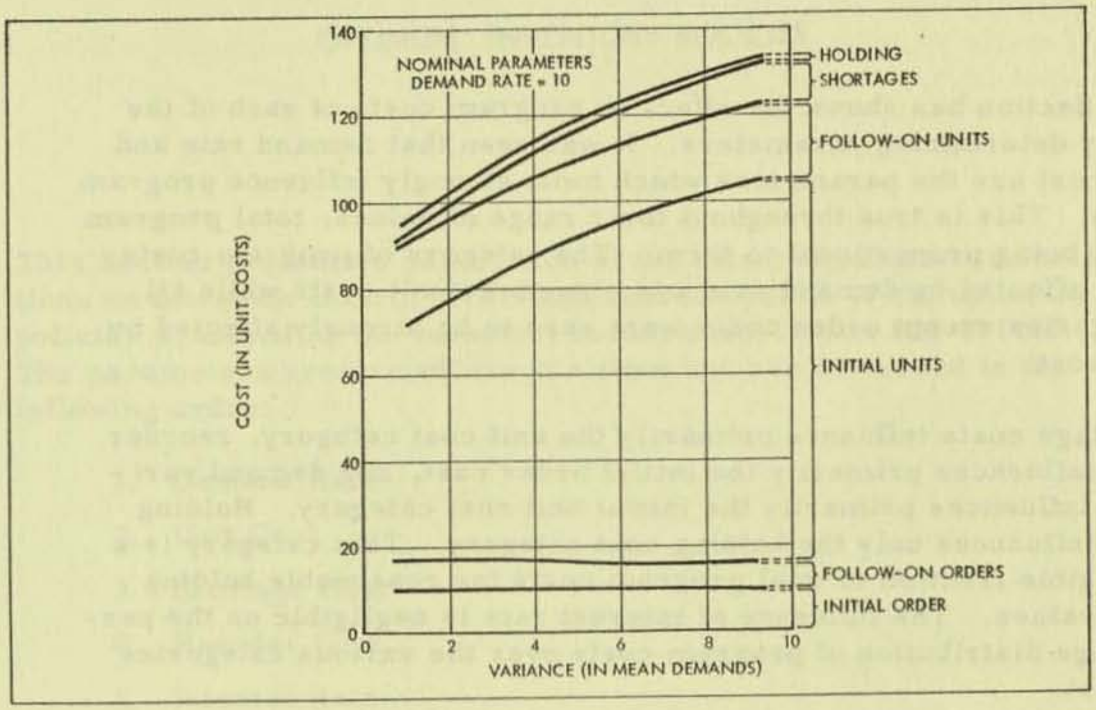

Figure 17. Effect of Demand Variance on Costs of Optimum Policies

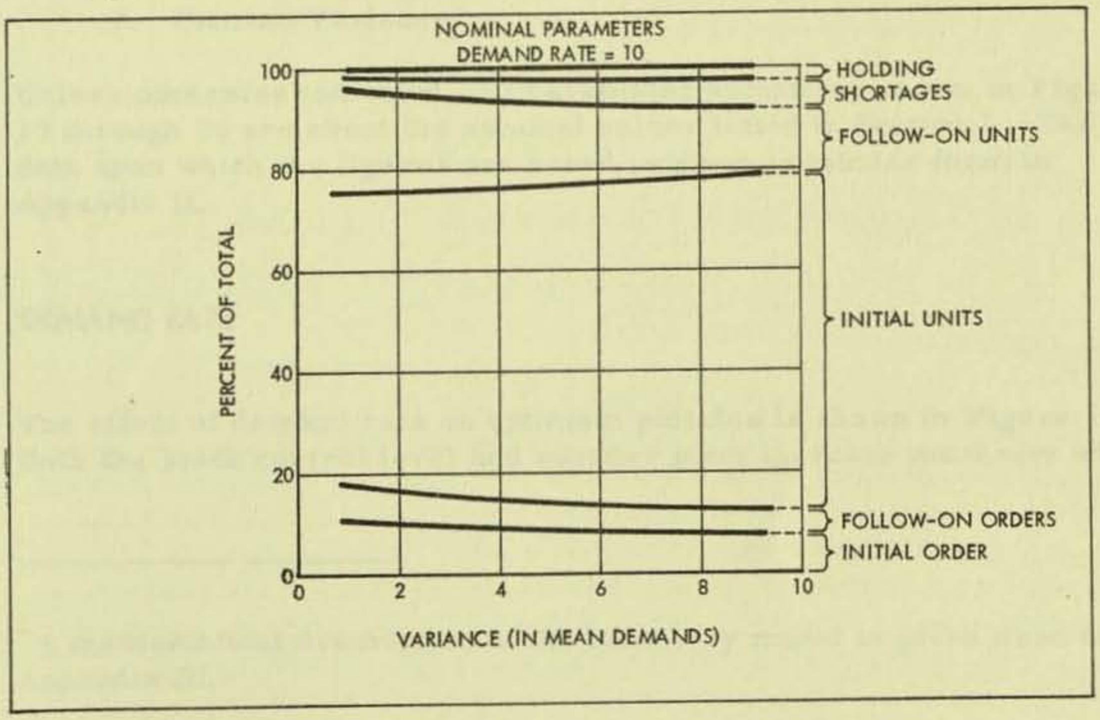

Figure 18. Effect of Demand Variance on Distribution of Costs

RECAPITULATION

This Section has shown the effect on program costs of each of the policy determining parameters. It was seen that demand rate and unit cost are the parameters which most strongly influence program costs. This is true throughout their range of values, total program costs being proportional to them. The category of program costs most affected by demand rate was seen to be unit costs while all categories except order costs were seen to be strongly affected by unit cost.

Shortage costs influence primarily the unit cost category, reorder cost influences primarily the initial order cost, and demand variance influences primarily the initial unit cost category. Holding cost influences only the holding cost category. This category is a negligible fraction of total program costs for reasonable holding cost values. The influence of interest rate is negligible on the percentage distribution of program costs over the various categories of cost.

SECTION II

OPTIMUM INVENTORY POLICIES

This Section presents a description of the effect of parameter variations on the stock control levels and reorder points of the optimum policies produced by the inventory model described in RM 57TMP-10.[1] The parameters are varied one at a time and are discussed in the following order:

1. Demand Rate
2. Unit Cost
3. Shortage Cost
4. Reorder Cost
5. Interest Rate
6. Holding Cost
7. Demand Variance

Unless otherwise indicated, all parameter variations shown in Figures 19 through 25 are about the nominal values listed in Section I. The data upon which the figures are based is given in tabular form in Appendix II.

DEMAND RATE

The effect of demand rate on optimum policies is shown in Figure 19. Both the stock control level and reorder point increase markedly with

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[1] A mathematical description of the inventory model is given here in Appendix III.

demand rate. For larger demands, they are nearly directly proportional to demand rate. The minimum reorder quantity (the difference between the stock control level and reorder point) is also proportional to demand rate. In effect, demand rate is a scale factor on optimum policies, although it is not quite a linear factor.

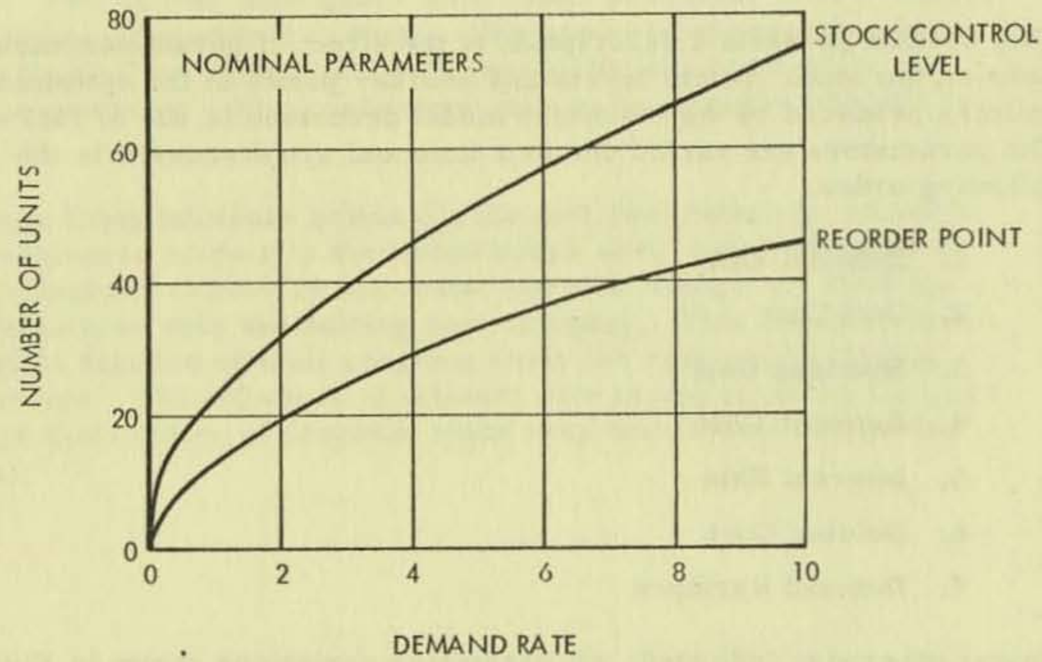

Figure 19. Effect of Demand Rate on Optimum Policies

UNIT COST

The effect of unit cost on optimum policies is shown in Figure 20 for two different demand rates. Both the stock control level and reorder point decrease as unit cost increases, but the stock control level decreases much more rapidly. The reorder point levels out and approaches an asymptote as the unit cost increases. Accordingly, the primary effect of unit cost is on the minimum reorder quantity and thus the stock control level. The reorder quantity for low unit cost items is seen to be quite large relative to that for items with high unit costs. The reorder points differ relatively little.

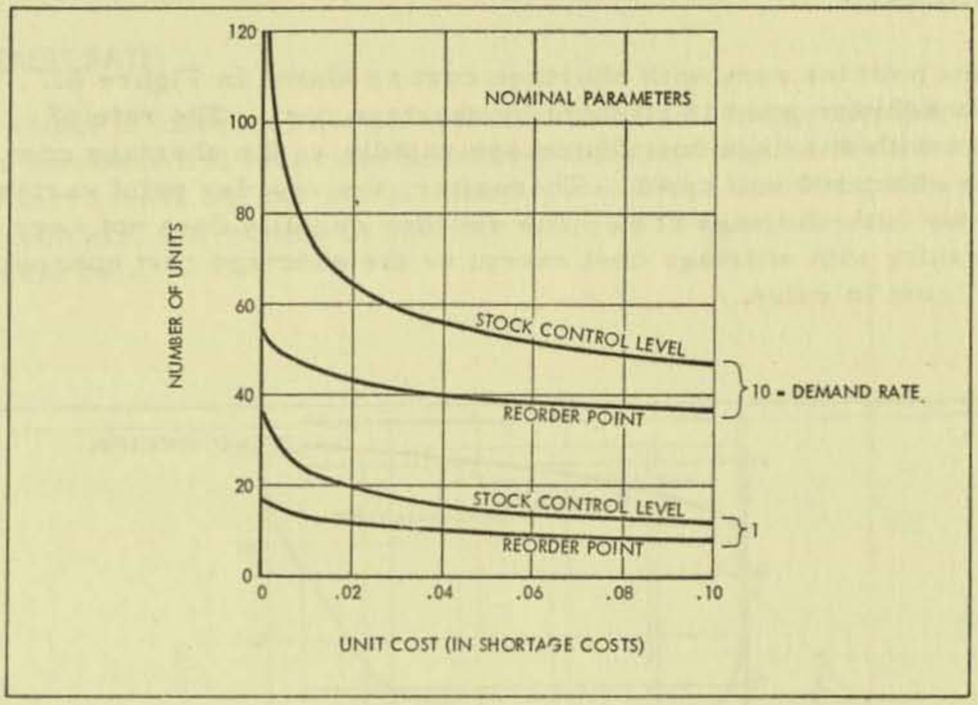

Figure 20. Effect of Unit Cost on Optimum Policies

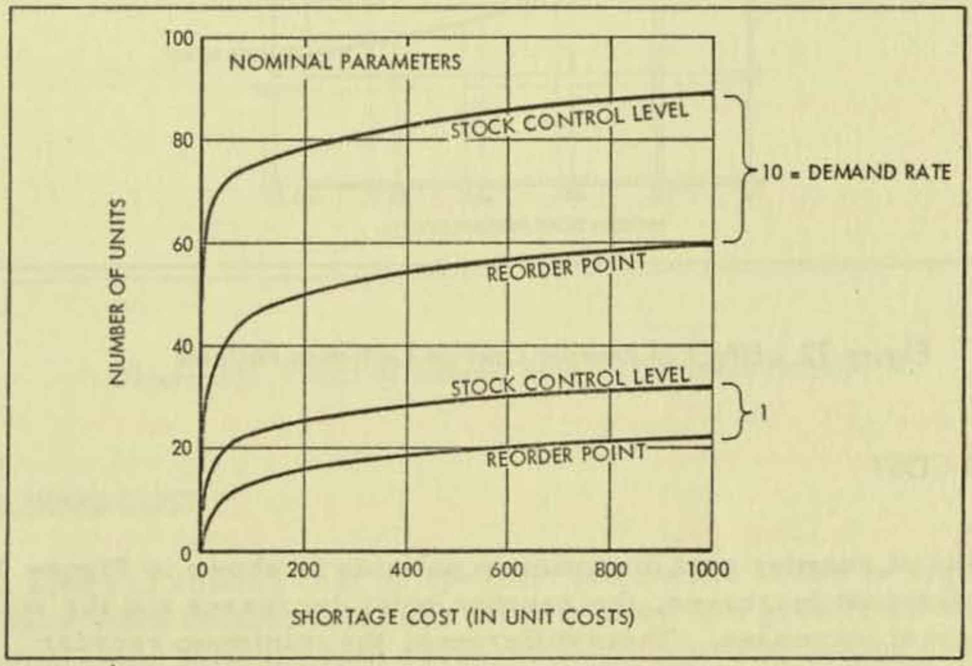

Figure 21. Effect of Shortage Cost on Optimum Policies

SHORTAGE COST

Optimum policies vary with shortage cost as shown in Figure 21. Only the reorder point is affected by shortage cost. The rate of increase with shortage cost decreases rapidly as the shortage cost reaches about 100 unit costs. Thereafter, the reorder point varies but slowly with shortage cost. The reorder quantity does not vary significantly with shortage cost except as the shortage cost approaches the unit cost in value.

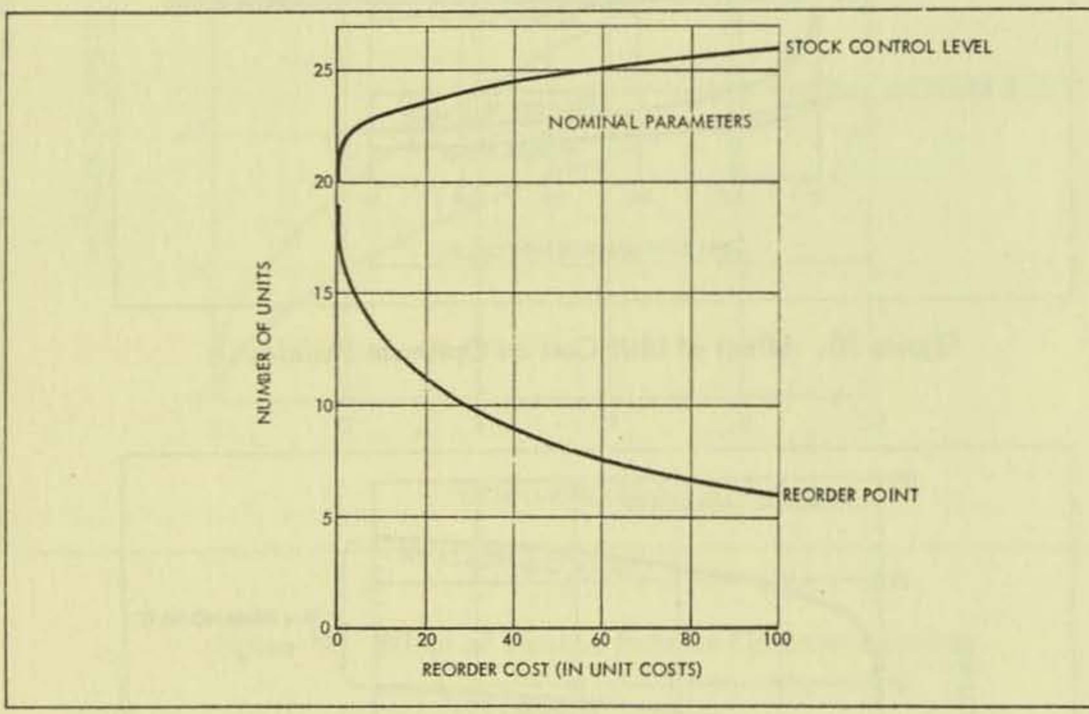

Figure 22. Effect of Reorder Cost on Optimum Policies

REORDER COST

The effect of reorder cost on optimum policies is shown in Figure 22. As reorder cost increases, the reorder point decreases and the stock control level increases. Their difference, the minimum reorder quantity, increases rapidly with reorder cost from unity, its value when the reorder cost is zero.

INTEREST RATE

The effect of interest rate on optimum policies is shown in Figure 23. Both stock control level and reorder point decrease as the interest rate increases, but the stock control level drops more rapidly. Accordingly, the minimum reorder quantity is most affected by interest rate.

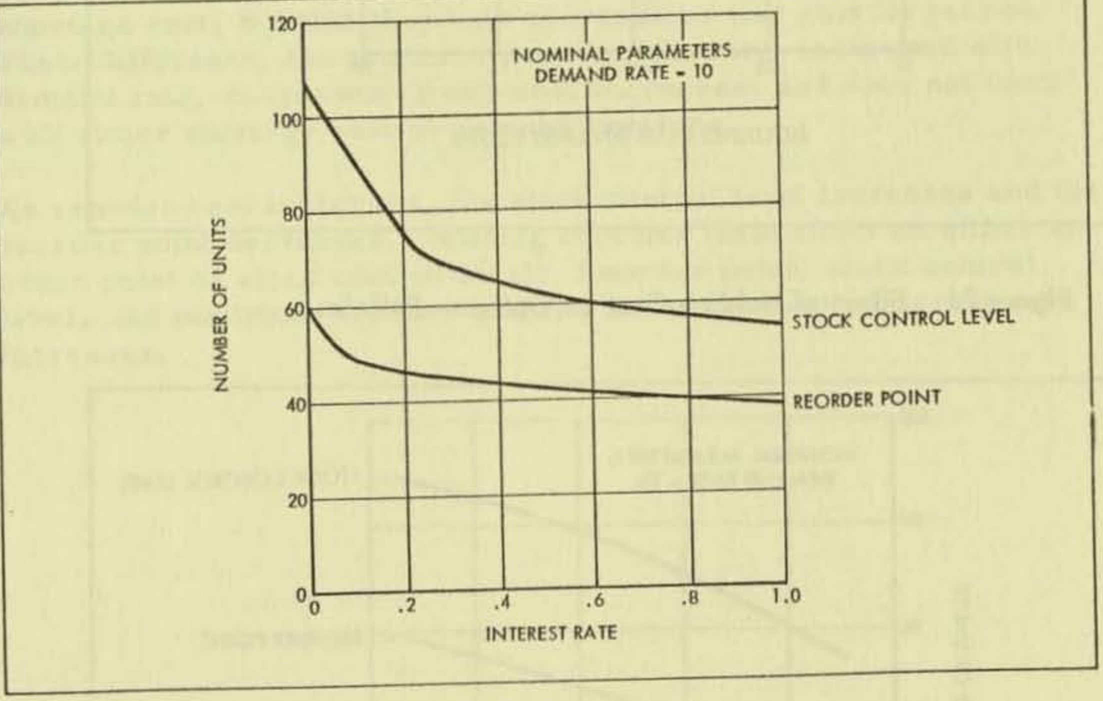

Figure 23. Effect of Interest Rate on Optimum Policies

HOLDING COST

The effect of holding cost on optimum policies is shown in Figure 24. Neither stock control level nor reorder point are greatly affected by holding cost, although both decrease slightly as the holding cost increases. The minimum reorder quantity is most affected since the stock control level drops somewhat more rapidly than the reorder point.

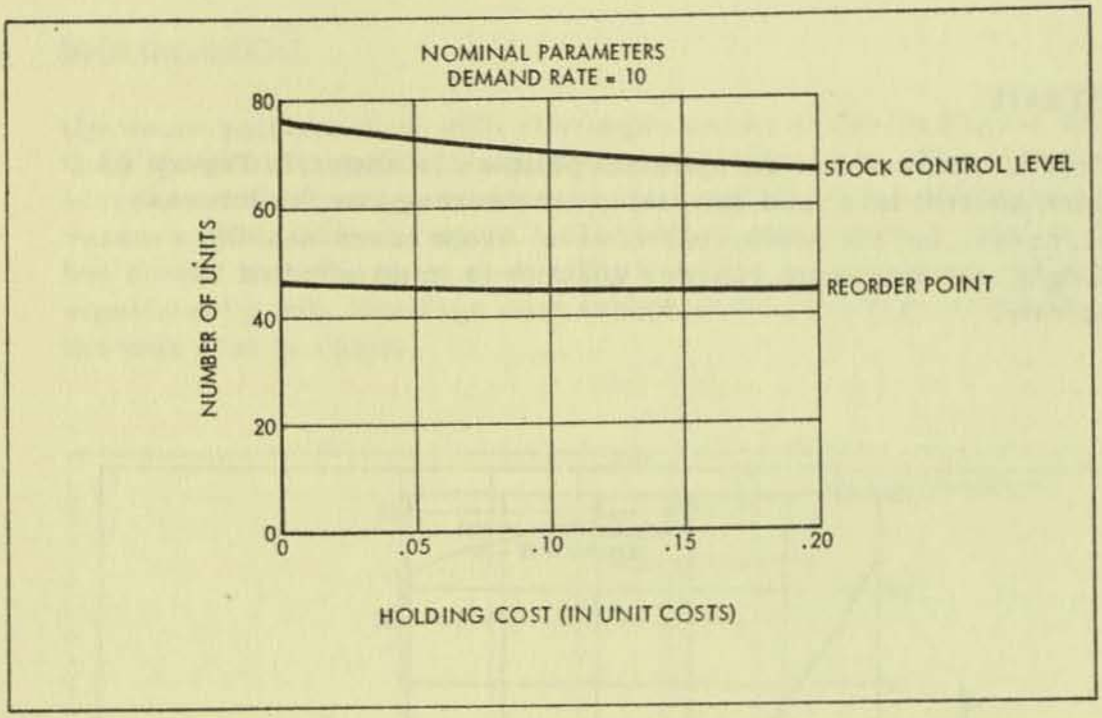

Figure 24. Effect of Holding Cost on Optimum Policies

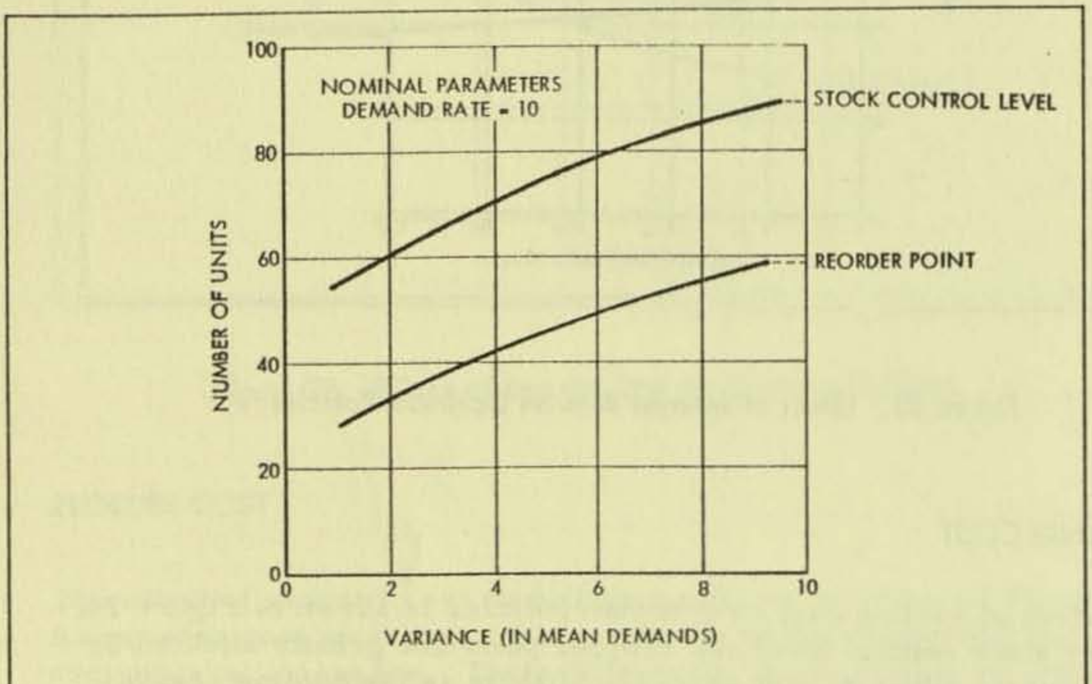

Figure 25. Effect of Demand Variance on Optimum Policies

DEMAND VARIANCE

Optimum policies vary with demand variance as shown in Figure 25. Only the reorder point is affected. It increases nearly linearly from its initial value with demand variance. The minimum reorder quantity is essentially uninfluenced by demand variance.

RECAPITULATION

This Section has shown the effect on optimum inventory policies of varying each parameter. It was seen that the reorder point and stock control level increase with demand rate, demand variance, and shortage cost, but that they both decrease as unit cost increases. Their difference, the minimum reorder quantity, increases with demand rate, decreases as unit cost increases, and does not vary with either shortage cost or demand variance.

As reorder cost increases, the stock control level increases and the reorder point decreases. Holding cost has little effect on either reorder point or stock control level. Reorder point, stock control level, and minimum reorder quantity decrease as the interest rate increases.

SECTION III

COSTS OF NON-OPTIMUM INVENTORY POLICIES

This Section presents a discussion of the costs incurred by selecting a policy for a spare part which is not the optimum one produced for it by the inventory model. One type of non-optimum policy is caused by adopting spare part categories such as is done in preparing a provisioning guide,[1] and following a single policy for all spares belonging to the category, even though the exact parameter values for all spares in a given category are not actually the same. Non-optimum policies may also result from errors in parameter estimation. These two sources of non-optimum policies are discussed jointly. A third type of non-optimum policy results from traditional "benchmark" methods of determining inventory policies. The effect on program costs of these various causes of non-optimum policies is described in this Section in the following order:

1. Parameter Errors
   a. Demand Rate
   b. Unit Cost
   c. Shortage Cost
   d. Reorder Cost
   e. Interest Rate
   f. Holding Cost
   g. Demand Variance

2. Traditional Policies

As in Sections I and II, unless otherwise noted in the figures, all results shown are for the nominal parameter values.

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[1] For a description of the provisioning guide procedure, refer to RM 58TMP-19, Interim Provisioning Policies for WS-107A Guidance Spares, R. F. McIntosh and L. Fisher, Technical Military Planning Operation, General Electric Company, Santa Barbara, California, 20 June 1958.

PARAMETER ERRORS

Demand Rate

The additional cost incurred by adopting demand rate categories is shown in Figure 26. It is seen that a part with a demand rate of about 6.2 may be assigned to either the demand rate category of 1 or 10. In either case, the cost of following the non-optimum policy is about 75 percent greater than that of the optimum policy. The equal cost point between the fixed policy categories for demand rates of 0.1 and 1 falls at a demand rate of about 0.56. The cost incurred by following either non-optimum policy is about 23 percent greater than that of the optimum policy. As the demand rate becomes still lower, the percentage increase over the optimum cost for a given error factor in demand rate becomes still lower.

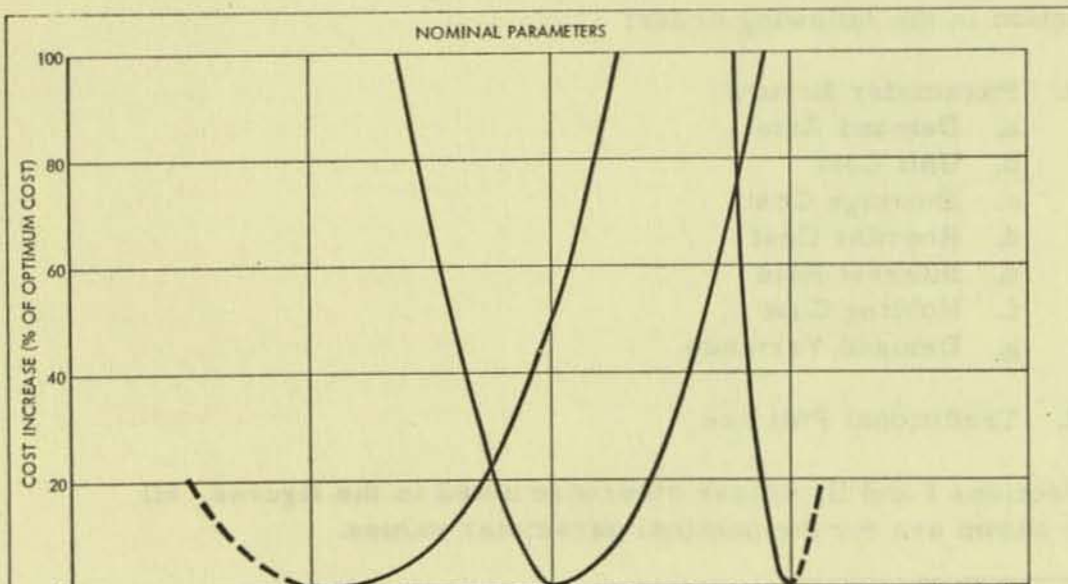

Figure 26. Cost of Following a Fixed Policy with Varying Demand Rate

The increase in program costs caused by errors in demand rate estimates are shown in Figure 27.[1] The sensitivity of program costs to errors in demand rate is seen to increase with demand rate.

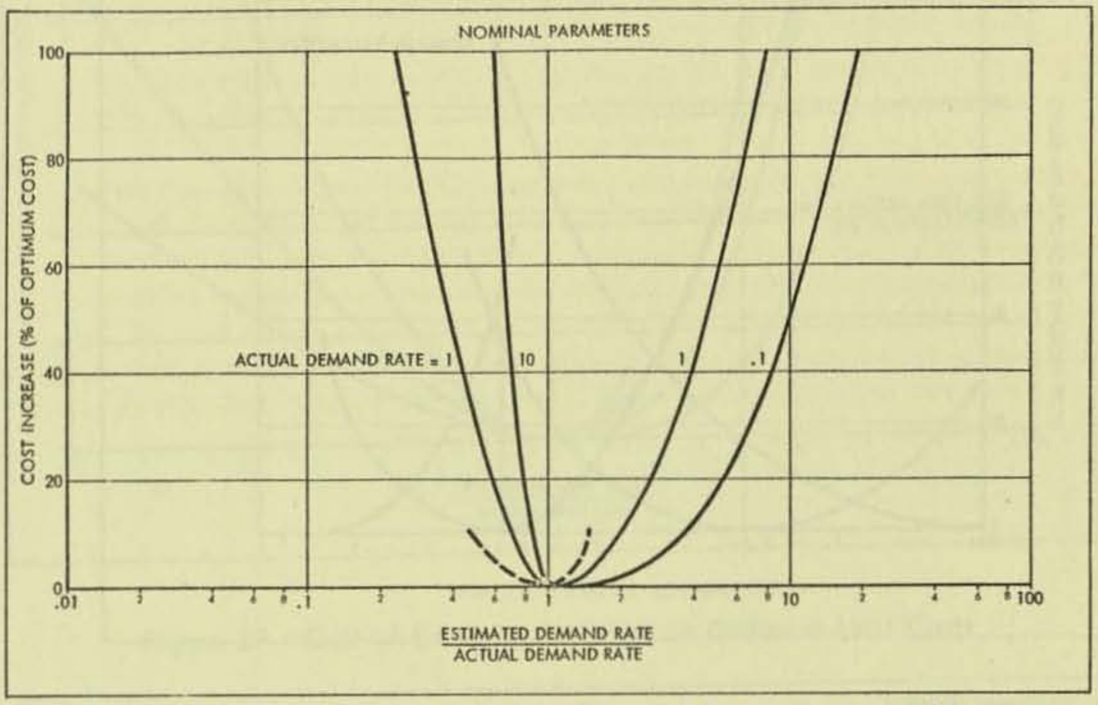

Figure 27. Cost of Errors in Demand Rate at Different Demand Rates

Unit Cost

The cost increase caused by adopting unit cost categories is shown in Figure 28. The cost for a part with a unit cost of 0.045 shortage costs is seen to be about 12 percent greater than the optimum cost for it. This increase is incurred whether the policy that is optimum for a unit cost of 0.01 or 0.1 is adopted.

The equal cost point between the 0.001 and 0.01 categories falls at a unit cost of about 0.0033. The cost increase is 5 percent. As the unit cost becomes still lower, the cost increase for a given error factor in unit cost becomes still less.

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[1] Demand rate estimation is discussed in RM 58TMP-25, Demand Forecasting Techniques for WS-107A (Mod III) Guidance Equipment Spares, T. B. Slattery, Technical Military Planning Operation, General Electric Company, Santa Barbara, California, 2 September 1958.

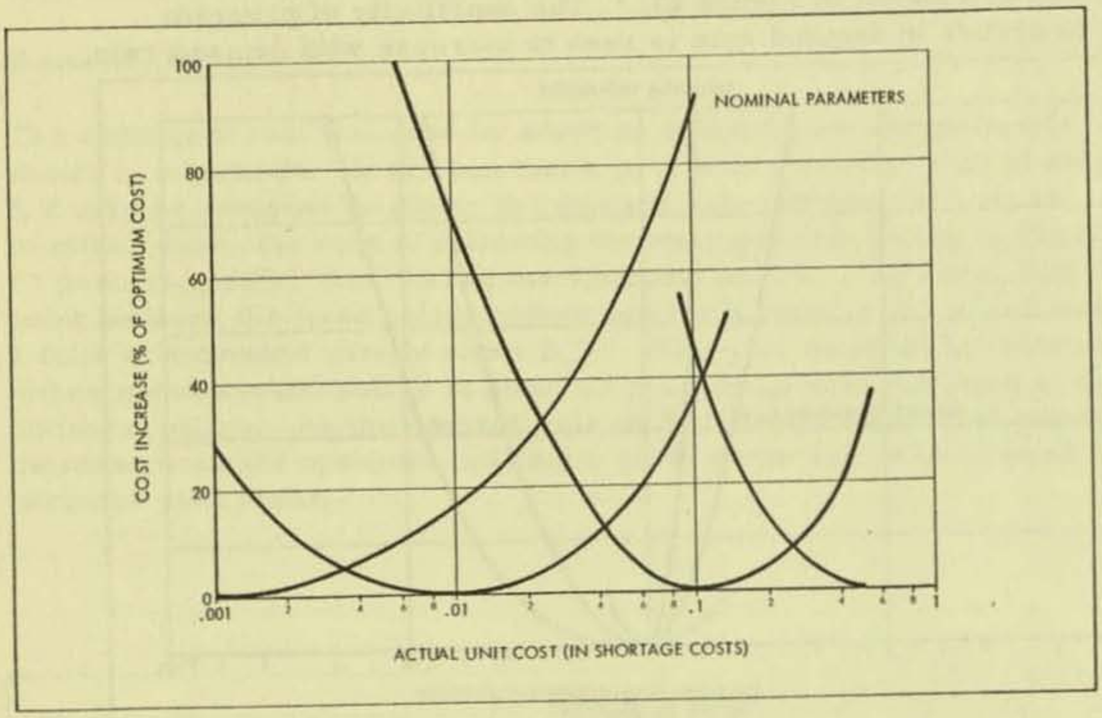


Figure 28. Cost of Following a Fixed Policy with Varying Unit Cost

The cost increase varies with errors in unit cost estimation as shown in Figure 29 for various unit costs. The error range of unit costs within which the cost increase is less than a given percentage of the optimum is seen to become smaller as unit cost increases.

Shortage Cost

The cost increase over the optimum caused by adopting shortage cost categories is shown in Figure 30. The order-of-magnitude categories shown have equal cost increase dividing points at factors of about two and one-half and at one-fourth times the shortage cost for which the fixed policy is optimum. The cost increase at these points is seen to decrease slowly as the shortage cost increases. The values are all under 10 percent.

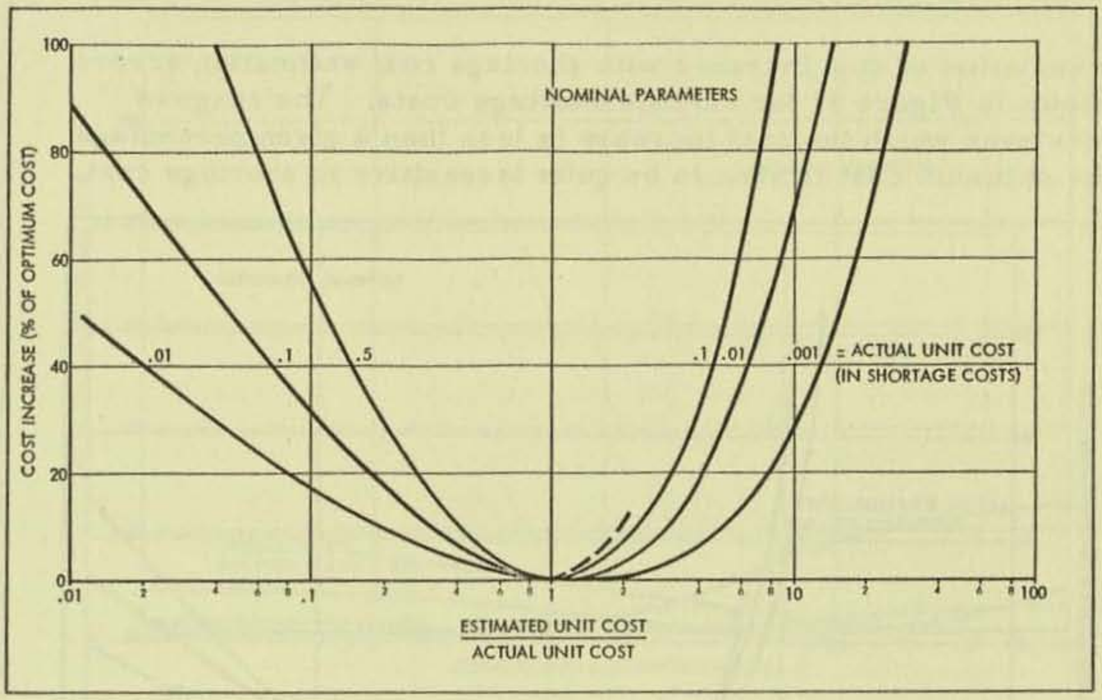


Figure 29. Cost of Errors in Unit Cost at Different Unit Costs

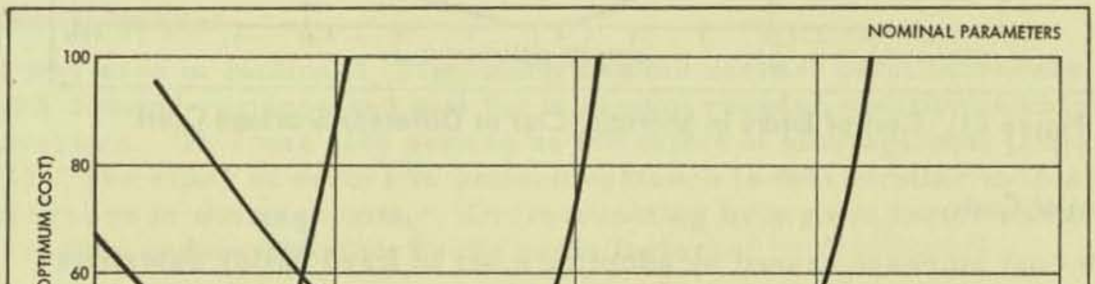


Figure 30. Cost of Following a Fixed Policy with Varying Shortage Cost

The variation of cost increase with shortage cost estimation errors is shown in Figure 31 for various shortage costs. The range of errors over which the cost increase is less than a given percentage of the optimum cost is seen to be quite insensitive to shortage cost.

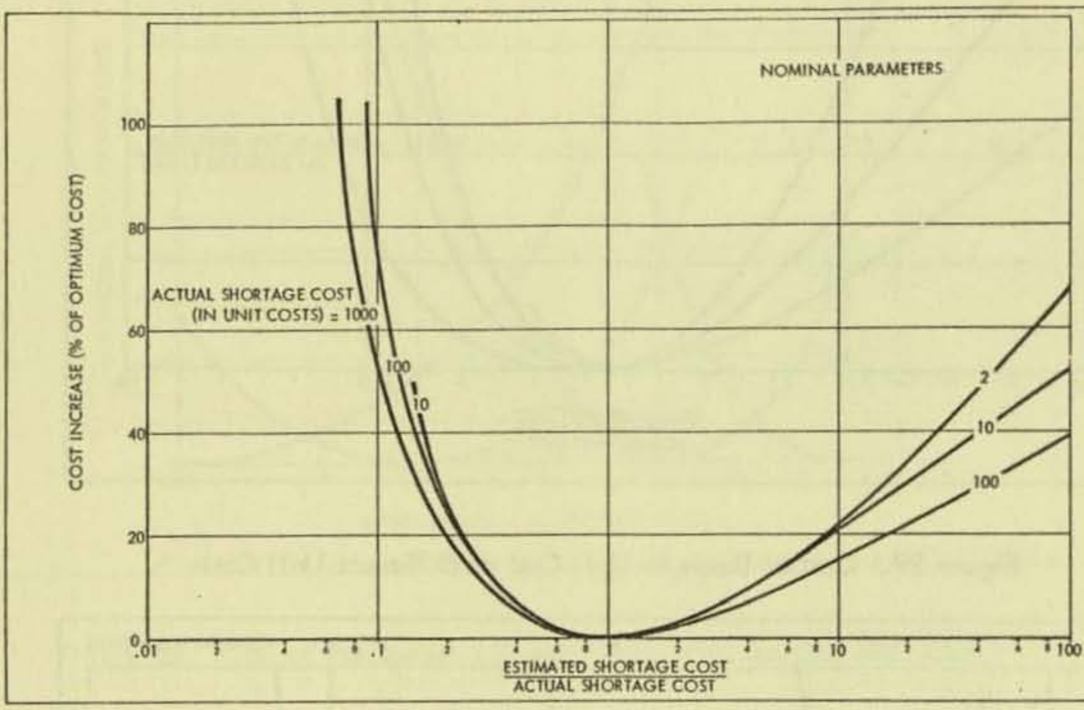


Figure 31. Cost of Errors in Shortage Cost at Different Shortage Costs

Reorder Cost

The cost increase caused by adopting a set of fixed policy categories for reorder cost is shown in Figure 32. It is seen that the fixed policy which is optimum for a reorder cost of 10 gives an increase of less than 5 percent of the optimum cost for reorder costs up to 1000 unit costs. Program costs are thus influenced but little by errors in estimating reorder cost.

Interest Rate and Holding Cost

It was seen in Section II (Figures 23 and 24) that optimum policies are relatively insensitive to interest rate and holding cost. Within the range of probable error in estimating these parameters the policies are invariant for all practical purposes. Accordingly, the cost increase caused by errors in estimating these parameters is negligible.

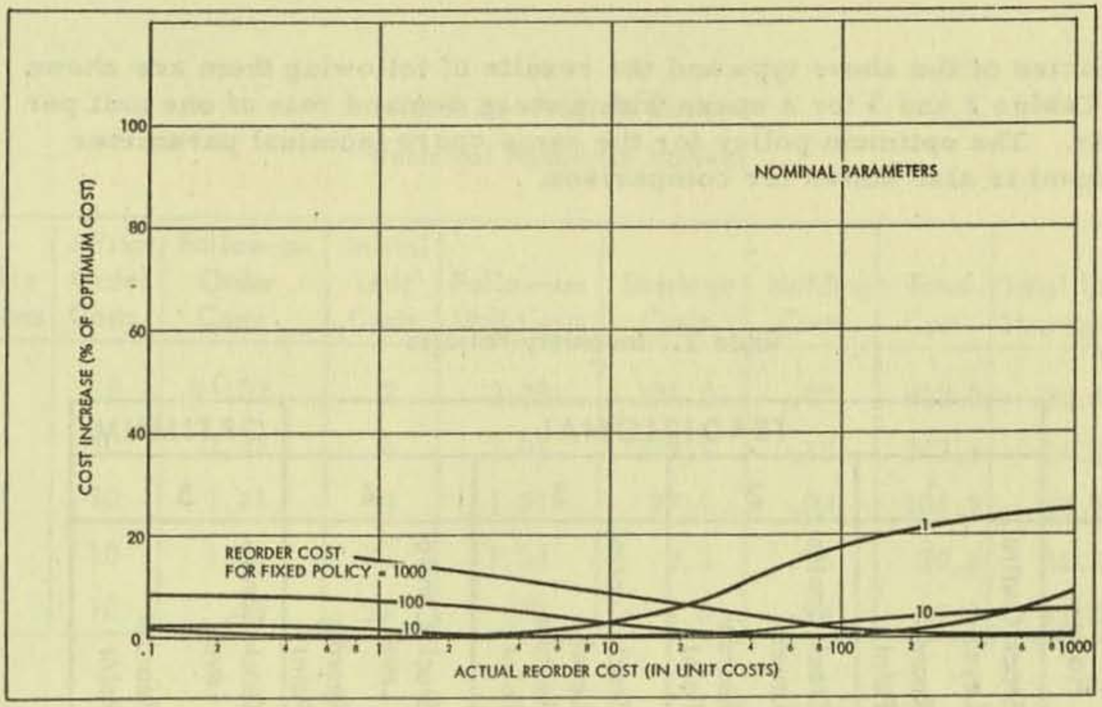


Figure 32. Cost of Following a Fixed Policy with Varying Reorder Cost

Demand Variance

It was seen in Section II (Figure 25) that the reorder point increases with demand variance and that the minimum reorder quantity remains invariant. This was also seen to be the effect of shortage cost (Figure 21). The effect of errors in demand variance is thus similar to that of errors in shortage cost. Overestimating by a given factor costs less than underestimating by the same factor.

TRADITIONAL POLICIES

Traditional inventory policies generally specify the reorder point and stock control level in terms of the expected demand during a given time interval. For example, the stock control level may be equal to two years' expected demand while the reorder point is equal to one year's expected demand. The reorder point and stock control level in such a policy is the same each year of the program unless the demand rate changes. The only parameter influencing the policy is the demand rate, although account is sometimes taken of unit cost by extending the time interval for low cost items.

Policies of the above type and the results of following them are shown in Tables 2 and 3 for a spare with a mean demand rate of one unit per year. The optimum policy for the same spare (nominal parameter values) is also shown for comparison.

Table 2. Inventory Policies

	TRADITIONAL								OPTIMUM	
	1		2		3		4		5	
Year of Program	Stock Control Level	Reorder Point	Stock Control Level	Reorder Point	Stock Control Level	Reorder Point	Stock Control Level	Reorder Point	Stock Control Level	Reorder Point
1	2	1	4	2	10	2	23	13	23	13
2	2	1	4	2	9	2	23	13	22	13
3	2	1	4	2	8	2	23	13	21	12
4	2	1	4	2	7	2	23	13	20	12
5	2	1	4	2	6	2	23	13	19	11
6	2	1	4	2	4	2	23	13	17	10
7	2	1	4	2	3	2	23	13	14	7
8	2	1	4	2	2	2	23	13	10	3

It is seen in Table 3 that as the initial number of units purchased increases, the costs of follow-on units, follow-on orders, and shortages decrease while holding costs increase. The increase in initial unit costs is more than offset by the decrease in other costs for the sequence of policies shown.

Table 3. Program Costs
(Nominal Parameter Values)

Policy Number	Initial Order Costs	Follow-on Order Costs	Initial Unit Costs	Follow-on Unit Costs	Shortage Costs	Holding Costs	Total Cost	Total Less Shortages
1	10	11.02	2	3.33	381.8	.07	408.2	26.4
2	10	7.27	4	3.05	219.1	.15	243.6	24.5
3	10	1.71	10	1.21	77.9	.37	101.2	23.3
4	10	1.04	23	1.34	3.3	.96	39.6	36.3
5	10	.63	23	.70	4.0	.96	39.3	35.3

The optimum policy (No. 5) differs from the traditional policy (No. 4) preceding it in Table 3 in that the stock control level and reorder point of the optimum policy decrease each year from the initial values shown to final values of 10 and 3, respectively. This results in follow-on order and follow-on unit costs being somewhat less for the optimum policy while shortage costs are a little more. The decreases in order and unit costs exceed the increase in shortage costs, hence program costs are reduced somewhat by going from traditional policy No. 4 to the policy which is optimum for the nominal parameter values. A policy such as No. 4 might arise if a poor data processing system made it impossible to adjust stock control levels and reorder points as a program was phasing out.

SECTION IV

COST-EFFECTIVENESS RELATIONSHIP

This Section presents a discussion of the effectiveness of inventory policies in terms of the number of shortages during the program. The point of view adopted is that there are different ways to allocate a given program cost over the relevant cost categories. A particular cost allocation is achieved by following the appropriate inventory policy. The effectiveness of the policy or cost allocation is measured by the number of shortages which result during the program.

The nature of optimum policies is first discussed. Traditional policies are then considered in relation to optimum policies.

OPTIMUM POLICIES

The optimum policies described in this Research Memorandum are determined so as to minimize total program costs. This is made possible by assigning a dollar value to a shortage and considering shortage costs as one category of program costs. The resulting minimum cost policies are such that as the shortage cost increases, program costs increase and the number of shortages decreases. This is shown in Figure 33 for a spare having nominal parameter values. For any specified number of shortages during the program the cost-effectiveness curve of Figure 33 gives the minimum expected program cost to keep shortages down to this number. Conversely, Figure 33 also shows the least number of shortages that can result from a given program cost. Points on the curve represent the best possible performance of the supply system, and can be reached only by following "optimum policies". Conversely, any other policy for the same spare results in a cost-effectiveness point lying to the right of the curve shown in Figure 33. This may also be stated another way: The best possible allocation of a given program cost is always given by the optimum policy.

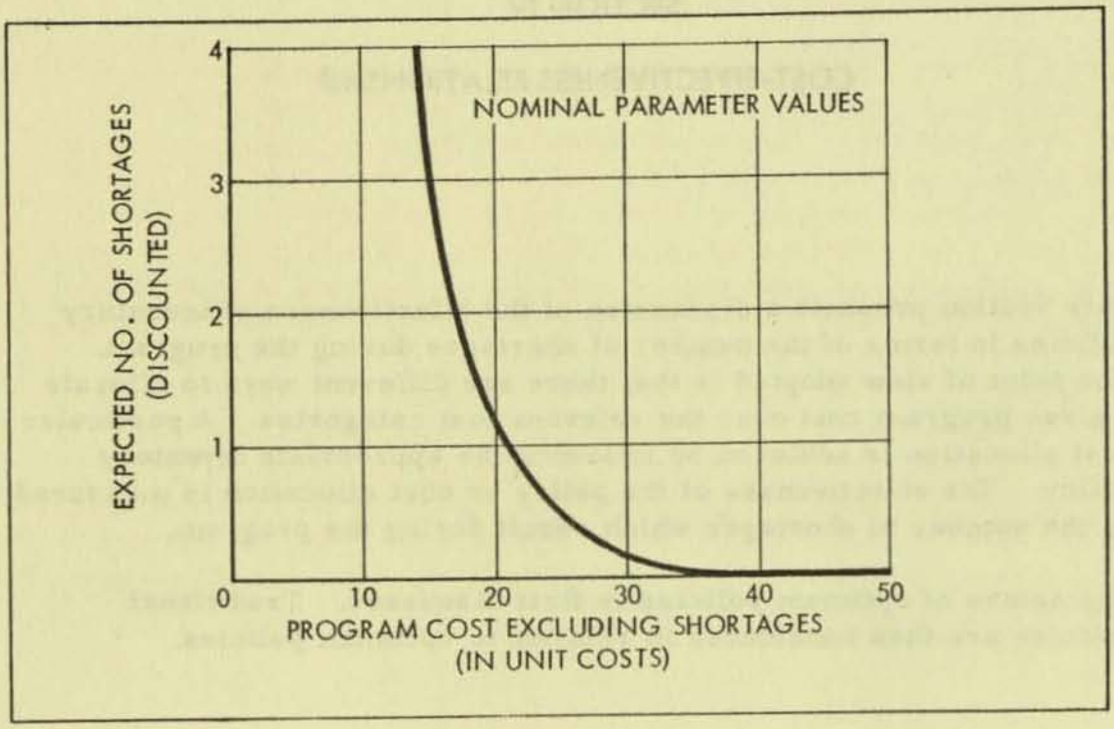


Figure 33. Cost-Effectiveness Function

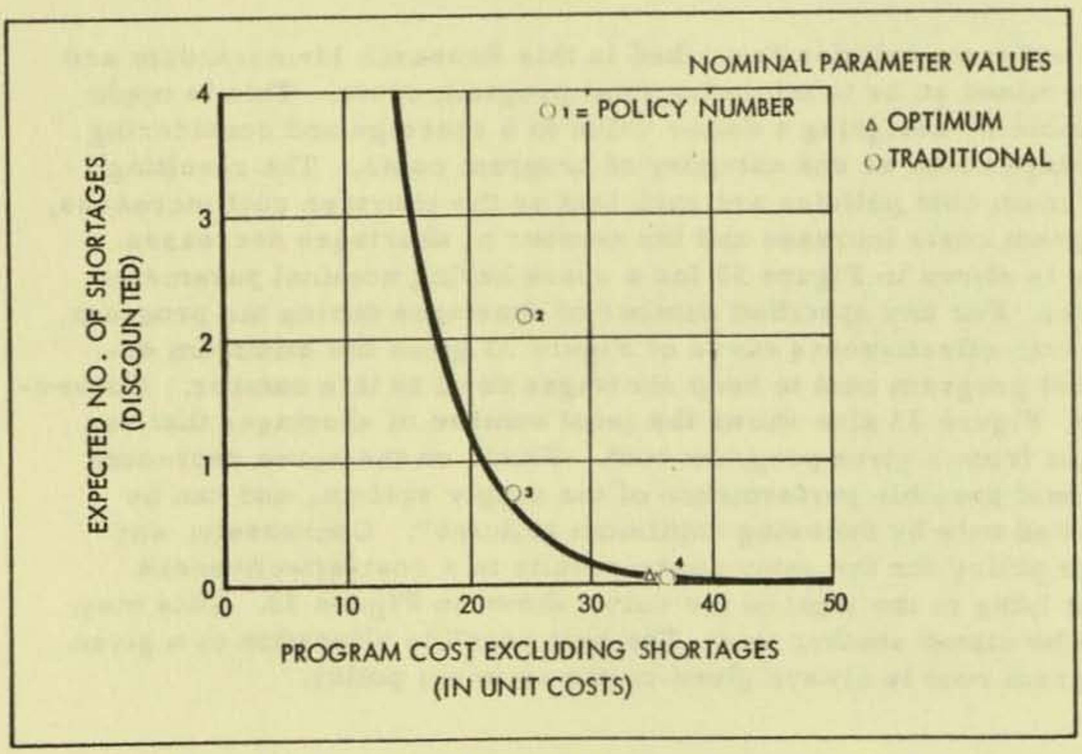


Figure 34. Traditional versus Optimum Inventory Policies

TRADITIONAL POLICIES

The traditional policies described in Section III provide an interesting illustration of the effects of non-optimum policies compared to optimum policies. These are shown in Figure 34, together with the optimum policy for a spare with nominal parameter values. It is seen that traditional policies No. 3 and No. 4 are very close to the optimum cost-effectiveness function. Policy No. 3 is seen to correspond to an optimum policy given by a lower shortage cost than the nominal value while policy No. 4 is almost the same as the optimum one. Policies No. 1 and No. 2 are the sort usually followed in practice. They are seen to result in a considerably greater number of shortages than the optimum policy corresponding to the same program cost (excluding the cost of shortages). It was seen in Section III that these traditional policies allocate less cost to the initial purchase than does the optimum policy. This results in excessive follow-on order and follow-on unit costs accompanied by greater shortage costs.

TRADITIONAL POLICIES

The traditional policies described in Section III provide an interesting illustration of the effects of using nonoptimal policies compared to optimum policies. These are shown in Figure 34, together with the optimum policy for a spare with nominal parameter values. It is seen that traditional policies No. 3 and No. 4 are very close to the optimum cost-effectiveness function. Policy No. 3 is seen to correspond to an optimum policy given by a lower shortage cost than the nominal value while policy No. 4 is almost the same as the optimum one. Policies No. 1 and No. 2 are the most widely followed in practice. They are seen to result in a considerably greater number of shortages than the optimum policy corresponding to the same program cost (excluding the cost of shortages). It was seen in Section III that these traditional policies [illegible] the initial provisioning than does the optimum policy. This results in [illegible] when [illegible] by shortage costs.

Figure 34. Traditional versus Optimum Provisioning Policies

NOTE ON THE APPENDICES

Tables 4 through 9 comprise Appendix I. They were computed according to the program described in Appendix IV. The tables show the number of events (e. g. , shortages, orders, etc.) expected to result from following the inventory policy whose identification number is listed in the tables. The number of events is discounted back to the beginning of the program so that when the number of each type of event is multiplied by its cost the result is its discounted program cost. The costs shown in the tables correspond to the parameter values which determined the inventory policy followed. If these parameter values differ from the actual values, then program costs greater than those in the tables will be incurred. The number of events may be the same, however, for they are determined solely by the inventory policy and the demand distribution. The parameter values used are listed in the tables. Where a value is not shown, it has the nominal value shown in the List of Symbols found at the end of this note.

The inventory policies used to obtain the costs shown in Appendix I are shown in Tables 10 through 14 which comprise Appendix II. They were computed according to the program described in Appendix III. The stock control level, reorder point, and minimum reorder quantity are shown for each year measured from the end of the program up to twelve years. Since the normal lead time is one year for all policies shown, no policy is shown for the start of the last year of a program.

LIST OF SYMBOLS

Symbol	Quantity	Nominal Value	Units Used
i	interest rate	0. 2	per year
α	demand variance	5	mean demands
H. C.	holding cost	0. 01	unit costs per year

Symbol	Quantity	Nominal Value	Units Used
N	program length	8	years
m	mean demand rate	1	units per year
R. C.	reorder cost	10	unit costs
S. C.	shortage cost	100	unit costs
U. C.	unit cost	1	dollars
S	stock control level	-	-
R	reorder point	-	-
Q	minimum reorder quantity	-	-

APPENDIX I

EXPECTED COSTS AND EVENTS

			DISCOUNTED NO. OF EVENTS				DISCOUNTED COSTS (IN UNIT COSTS)						
POLICY NUMBER	PARAMETER VARIED	PARAMETER VALUE	UNIT. YRS. HELD	SHORTAGES	FOLLOW-ON ORDERS	FOLLOW-ON UNITS	INITIAL UNITS	INITIAL ORDER	FOLLOW-ON ORDER	INITIAL UNITS	FOLLOW-ON UNITS	SHORTAGES	HOLDING	TOTAL
110	I	.001	546	.0810	.647	14.78	98	10	6.47	98	14.78	8.10	5.46	142.81
111		.1	391	.0634	.555	13.43	91	10	5.55	91	13.43	6.34	3.91	130.23
13		.2	270	.065	.654	18.13	75	10	6.54	75	18.13	6.5	2.70	118.87
112		.5	150	.0623	.495	12.45	62	10	4.95	62	12.45	6.23	1.50	97.13
113		1.0	95	.0629	.315	6.99	54	10	3.15	54	6.99	6.29	0.95	88.39
114	H.C.	.0001	300	.0622	.638	17.53	76	10	6.38	76	17.53	6.22	.03	116.16
13		.01	270	.065	.654	18.13	75	10	6.54	75	18.13	6.5	2.70	118.87
115		.05	260	.0757	.715	19.70	72	10	7.15	72	19.70	7.57	13.02	129.44
116		.1	253.8	.0835	.776	20.64	70	10	7.76	70	20.64	8.35	25.38	142.13
117		.2	240.1	.1065	.906	22.41	66	10	9.06	66	22.41	10.65	48.02	166.14
118	α	1	181	.0209	.693	19.29	54	10	6.93	55	19.29	2.09	1.81	95.12
119		3	228	.0429	.710	19.51	65	10	7.10	65	19.51	4.29	2.28	108.18
13		5	270	.065	.654	18.13	75	10	6.54	75	18.13	6.5	2.70	118.87
120		9	330	.0764	.613	17.87	88	10	6.13	88	17.87	7.64	3.30	132.94
118	N	2						10	0	48	0	5.73	.80	64.53
118		4						10	1.80	66	2.95	5.44	1.69	87.88

Table 4. Variations About Nominal Point

Nominal Point: $i = 0.2$, $\alpha = 5$, H.C. = 0.01, N = 8, R.C. = 10, S.C. = 100, m = 10

			DISCOUNTED NO. OF EVENTS					DISCOUNTED COSTS (IN UNIT COSTS)						
POLICY NUMBER	R.C.	S.C.	UNIT YRS. HELD	SHORTAGES	FOLLOW-ON ORDERS	FOLLOW-ON UNITS	INITIAL UNITS	INITIAL ORDER	FOLLOW-ON ORDER	INITIAL UNITS	FOLLOW-ON UNITS	SHORTAGES	HOLDING	TOTAL
1	.01	2	110.19	3.967	2.812	28.26	33	.01	.028	33	28.26	7.934	1.102	70.334
2		10	162.37	.709	3.009	29.51	44	.01	.030	44	29.51	7.09	1.624	82.264
3		100	231.78	.063	2.997	29.85	59	.01	.030	59	29.85	6.3	2.318	97.508
4	.1	2	109.92	4.013	2.649	28.17	33	.1	.265	33	28.17	8.026	1.099	70.66
5		10	165.31	.673	2.617	28.87	45	.1	.262	45	28.87	6.73	1.653	82.615
6		100	231.49	.064	2.785	29.78	59	.1	.278	59	29.78	6.4	2.315	97.873
7	1	2	115.12	4.130	1.582	25.42	36	1	1.582	36	25.42	8.26	1.152	73.414
8		10	168.73	.691	1.760	27.13	47	1	1.760	47	27.13	6.91	1.687	85.483
9		100	235.12	.065	1.903	27.99	61	1	1.903	61	27.99	6.5	2.351	100.744
10		1000	299.47	.005	1.902	28.14	75	1	1.902	75	28.14	5.-	2.995	114.037
11	10	2	143.98	4.791	.508	14.89	49	10	5.08	49	14.89	9.582	1.440	89.992
12		10	202.02	.726	.598	16.86	61	10	5.98	61	16.86	7.26	2.020	103.12
13		100	269.65	.065	.654	18.13	75	10	6.54	75	18.13	6.5	2.697	118.867
14		1000	334.59	.006	.667	18.32	89	10	6.67	89	18.32	6.-	3.346	133.336
15	100	10	298.50	.884	.041	1.54	90	100	4.1	90	1.54	8.84	2.985	207.465

Table 5. Variations in Reorder and Shortage Costs, m = 10

Fixed Parameters: $i = 0.2, \alpha = 5$, H.C. = 0.01, N = 8

			DISCOUNTED NO. OF EVENTS					DISCOUNTED COSTS (IN UNIT COSTS)						
POLICY NUMBER	R.C.	S.C.	UNIT YRS. HELD	SHORTAGES	FOLLOW-ON ORDERS	FOLLOW-ON UNITS	INITIAL UNITS	INITIAL ORDER	FOLLOW-ON ORDER	INITIAL UNITS	FOLLOW-ON UNITS	SHORTAGES	HOLDING	TOTAL
21	.01	2	18.74	1.952	.772	2.48	5	.01	.0077	5	2.48	3.904	.187	11.589
23		10	45.49	.419	.687	2.38	11	.01	.0069	11	2.38	4.190	.455	18.042
23		100	86.99	.045	.765	2.43	20	.01	.0076	20	2.43	4.5	.870	27.818
24	.1	2	18.39	2.009	.603	2.38	5	.1	.06	5	2.38	4.018	.184	11.742
25		10	45.19	.429	.539	2.29	11	.1	.054	11	2.29	4.29	.452	18.186
26		100	89.85	.041	.489	1.85	21	.1	.049	21	1.85	4.1	.899	27.998
27	1	2	21.26	1.834	.305	1.81	6	1	.305	6	1.81	3.668	.213	12.996
28		10	47.93	.401	.264	1.65	12	1	.264	12	1.65	4.01	.479	19.403
29		100	89.25	.043	.264	1.65	21	1	.264	21	1.65	4.3	.892	29.106
30		1000	134.73	.004	.218	1.45	31	1	.218	31	1.45	4.	1.347	39.015
31	10	2	27.88	1.765	.075	.77	8	10	.75	8	.77	3.53	.279	23.329
32		10	50.46	.432	.083	.91	13	10	.83	13	.91	4.32	.505	29.565
33		100	95.81	.040	.063	.70	23	10	.63	23	.70	4.	.958	39.288
34		1000	137.38	.004	.065	.75	32	10	.65	32	.75	4.	1.374	48.774
35	100	10	62.53	.389	.012	.21	16	100	1.2	16	.21	3.89	.625	121.925
36		100	103.67	.041	.009	.17	25	100	.9	25	.17	4.1	1.037	131.207
37		1000	145.10	.005	.009	.17	34	100	.9	34	.17	5.	1.451	141.521
38		10,000	191.12	.000	.008	.16	44	100	.8	44	.16	--	1.911	146.871
39	1000	100	112.56	.044	.001	.03	27	1000	1.	27	.03	4.4	1.126	1033.536
40		1000	153.97	.005	.001	.03	36	1000	1.	36	.03	5.-	1.540	1043.57

Table 6. Variations in Reorder and Shortage Costs, m = 1

Fixed Parameters: i = 0.2, α = 5, H.C. = 0.01, N = 8

			DISCOUNTED NO. OF EVENTS					DISCOUNTED COSTS (IN UNIT COSTS)						
POLICY NUMBER	R.C.	S.C.	UNIT YRS. HELD	SHORTAGES	FOLLOW-ON ORDERS	FOLLOW-ON UNITS	INITIAL UNITS	INITIAL ORDER	FOLLOW-ON ORDER	INITIAL UNITS	FOLLOW-ON UNITS	SHORTAGES	HOLDING	TOTAL
90	.01	2	4.32	.455	0	.28	0	.01	0.00	0	.33	1.21	.043	1.593
91		10												
92		100	36.40	.0436	0	.26	8	.01	0.00	8	.26	4.36	.364	12.994
93	.1	2												
94		10												
95		100												
96	1	2												
97		10												
98		100	40.67	.0338	.03	.14	9	1	.03	9	.14	3.38	.407	13.957
99		1000	77.49	.00377	.03	.14	17	1	.03	17	.14	3.77	.775	22.715
100	10	2												
101		10												
102		100	40.49	.0351	.009	.08	9	10	.09	9	.08	3.51	.405	23.085
103		1000	77.28	.0039	.007	.07	17	10	.07	17	.07	3.90	.773	31.813
104	100	10	17.53	.229	.57	.06	4	100	.57	4	.06	2.29	.175	107.095
105		100	40.39	.0368	.0039	.05	9	100	.39	9	.05	3.68	.404	113.524
106		1000	77.15	.00412	.0013	.02	17	100	.13	17	.02	4.12	.772	122.042
107		10,000	118.58	.000390	.0009	.02	26	100	.09	26	.02	3.90	1.186	131.19
108	1000	100	49.52	.0283	.00082	.01	11	1000	.82	11	.01	2.83	.495	1015.155
109		1000	81.71	.00336	.00033	.01	18	1000	.33	18	.01	3.36	.817	1022.517

Table 7. Variations in Reorder and Shortage Costs, $m = 0.1$

Fixed Parameters: $i = 0.2$, $\alpha = 5$, H.C. $= 0.01$, $N = 8$

			DISCOUNTED NO. OF EVENTS					DISCOUNTED COSTS (IN UNIT COSTS)						
POLICY NUMBER	R.C.	S.C.	UNIT YRS. HELD	SHORTAGES	FOLLOW-ON ORDERS	FOLLOW-ON UNITS	INITIAL UNITS	INITIAL ORDER	FOLLOW-ON ORDER	INITIAL UNITS	FOLLOW-ON UNITS	SHORTAGES	HOLDING	TOTAL
65	10	1000	27.58	.004	.001	0.000	6	10	.010	6.000	0.000	4.000	.276	20.286
68	100	1000	27.58	.004	0.000	0.000	6	100	0.000	6.000	0.000	4.000	.276	110.276
69		10,000	64.41	.000	0.000	.000	14	100	0.000	14.000	0.000	0.000	.644	114.644
70	1000	100	4.58	.033	.001	.01	1	1000	1.000	1.000	0.010	3.300	.046	1005.356
71		1000	27.58	.004	.000	.00	6	1000	1.000	6.000	0.000	4.000	.276	1010.276

Table 8. Variations in Reorder and Shortage Costs, $m = 0.005$

Fixed Parameters: $i = 0.2$, $\alpha = 5$, H.C. $= 0.01$, $N = 8$

			DISCOUNTED NO. OF EVENTS					DISCOUNTED COSTS (IN UNIT COSTS)						
POLICY NUMBER	R.C.	S.C.	UNIT YRS. HELD	SHORTAGES	FOLLOW-ON ORDERS	FOLLOW-ON UNITS	INITIAL UNITS	INITIAL ORDERS	FOLLOW-ON ORDERS	INITIAL UNITS	FOLLOW-ON UNITS	SHORTAGES	HOLDING	TOTAL
1	.01	2	130	3.826	3.802	39.39	33	.01	.038	33	39.39	7.652	1.30	81.390
2		10	189	.715	3.986	40.01	44	.01	.040	44	40.01	7.150	1.89	93.100
3		100	269	.063	3.981	40.17	59	.01	.040	59	40.17	6.300	2.69	108.21
4	.1	2	129	3.872	3.604	39.31	33	.1	.360	33	39.31	7.744	1.29	81.804
5		10	193	.659	3.453	39.57	45	.1	.345	45	39.57	6.59	1.93	93.535
6		100	268	.064	3.669	40.08	59	.1	.367	59	40.08	6.40	2.68	108.627
8	1	10	196	.692	2.311	37.85	47	1	2.311	47	37.85	6.92	1.96	97.041
9		100	273	.066	2.479	38.44	61	1	2.479	61	38.44	6.66	2.73	112.309
10		1000	347	.005	2.478	38.52	75	1	2.478	75	38.52	5.00	3.47	125.468
11	10	2	169	4.480	.763	27.62	49	10	7.63	49	27.62	8.960	1.69	104.900
12		10	235	.737	.831	28.66	61	10	8.31	61	28.66	7.370	2.35	117.690
13		100	313	.065	.896	29.75	75	10	8.96	75	29.75	6.50	3.13	133.34
14		1000	388	.006	.902	28.83	89	10	9.02	89	28.83	6.00	3.88	146.73

Table 9. Variation in Reorder and Shortage Costs, N = 12

Fixed Parameters: $i = 0.2$, $\alpha = 5$, H.C. = 0.01, $m = 10$

APPENDIX II

OPTIMUM POLICIES

POLICY NUMBER	PARAMETER VARIED	PARAMETER VALUE	POLICY	YEARS TO END OF PROGRAM 2	3	4	5	6	7	8	9	10	11	12
110	i.	.001	S	49	61	71	80	89	98	107				
			R	40	48	53	56	57	58					
			Q	9	13	18	24	32	40					
111		.1	S	49	60	68	76	83	88	91				
			R	39	47	50	50	49	49					
			Q	10	13	18	26	34	39					
13		.2	S	48	58	66	72	75	76	75				
			R	39	45	47	46	46	46	46				
			Q	9	13	19	26	29	30	29				
112		.5	S	47	55	60	62	62	62					
			R	37	42	42	42	42	42					
			Q	10	13	18	20	20	20					
113		1.0	S	45	51	54	54	54						
			R	35	39	38	38	38						
			Q	10	12	16	16	16						
114	H.C.	.0001	S	48	58	66	72	76	77	76				
			R	39	45	47	47	46	46					
			Q	9	13	19	25	30	31					
13		.01	S	48	58	66	72	75	76	75				
			R	39	45	47	46	46	46	46				
			Q	9	13	19	29	29	30	29				
115		.05	S	48	58	65	70	73	73	72				
			R	38	45	46	46	45	45					
			Q	10	13	19	24	28	28					
116		.1	S	48	57	64	69	70	70	70				
			R	38	45	46	45	45	45					
			Q	10	12	18	24	25	25					
		.2	S	47	56	62	65	66	65	66				
			R	38	44	45	44	44	44					
			Q	9	12	17	21	22	21					
118	α	1	S	31	41	50	57	52	54	55	54	54	54	
			R	26	28	28	27	27	28	28	28	28	28	
			Q	5	13	22	30	25	26	27	26	26	26	
119	α	3	S	41	50	59	65	67	65	65	66	66	66	
			R	33	38	38	38	38	38	38	38	38	38	
			Q	8	12	21	27	29	27	27	28	28	28	
13	α	5	S	48	58	66	72	75	76	75	75	75		
			R	39	45	47	46	46	46	46	46			
			Q	9	13	19	26	29	30	29	29			
120	α	9	S	59	70	77	82	86	87	88	88	87	87	87
			R	48	56	59	59	59	58	58	58	58	58	
			Q	11	14	18	23	27	29	30	30	29	29	

Table 10. Variations About Nominal Point

Nominal Point: $i = 0.2$, $\alpha = 5$, H.C. = 0.01, N = 8, R.C. = 10, S.C. = 100, m = 10

POLICY NUMBER	R.C.	S.C.		YEARS TO END OF PROGRAM 2	3	4	5	6	7	8	9	10	11	12
P81	.01	1.35	S	9	23	28	30	30	30					
			R	7	22	27	28	28	28					
			Q	2	1	1	2	2	2					
72		1.5	S	11	24	29	31	31	31					
			R	10	23	28	29	29	29					
			Q	1	1	1	2	2	2					
1		2.	S	16	27	32	33	33	33	33	33	33	33	33
			R	15	26	30	31	31	31	31	31	31	31	
			Q	1	1	2	2	2	2	2	2	2	2	
2		10.	S	32	40	44	44	44	44	44	44	44	44	44
			R	31	39	43	43	43	43	43	43	43	43	
			Q	1	1	1	1	1	1	1	1	1	1	
3		100.	S	48	56	59	59	59						
			R	47	54	57	58	58						
			Q	1	2	2	1	1						
82	.1	1.35	S	9	23	28	30	30	30					
			R	6	21	26	26	26	26					
			Q	3	2	2	4	4	4					
73		1.5	S	11	24	29	31	31	31					
			R	9	22	27	27	27	27					
			Q	2	2	2	4	4	4					
4		2.0	S	16	27	32	33	33	33					
			R	14	25	29	30	30	30					
			Q	2	2	3	3	3	3					
5		10.	S	32	40	44	45	45	45					
			R	30	38	41	41	41	41					
			Q	2	2	3	4	4	4					
6		100	S	48	56	59	59	59						
			R	46	54	56	56	56						
			Q	2	2	3	3	3						
83	1	1.35	S	9	23	29	32	33	33	33				
			R	2	17	22	22	22	22	22				
			Q	11	6	7	10	11	11	11				
74	1	1.5	S	11	24	30	33	33	33					
			R	4	19	23	23	23	23					
			Q	7	5	7	10	10	10					
7	1	2.	S	16	27	33	35	36	36	36				
			R	10	22	25	25	25	25	25				
			Q	6	5	8	10	11	11	11				
8	1	10	S	32	41	45	47	47	47					
			R	28	35	38	37	37	37					
			Q	4	6	7	10	10	10					

Table 11. Variations in Reorder and Shortage Costs, m = 10
Fixed Parameters: i = 0.2, α = 5, H.C. = 0.01, N = 8

POLICY NUMBER	R.C.	S.C.		YEARS TO END OF PROGRAM 2	3	4	5	6	7	8	9	10	11	12
9	1	100	S	48	56	60	61	61	61					
			R	44	51	53	52	52	52					
			Q	4	5	7	9	9	9					
10	1	1000	S	63	70	74	75	75	75					
			R	59	65	66	66	66	66					
			Q	4	5	8	9	9	9					
11	10	2	S	16	28	37	43	48	50	49	49	49		
			R	-5	13	18	18	17	17	17	17	17		
			Q	21	15	19	25	31	33	32	32	32		
12	10	10	S	32	43	50	57	60	61	61	60	61	61	61
			R	21	29	31	31	30	30	30	30	30	30	
			Q	11	14	19	26	30	31	31	30	31	31	
13	10	100	S	48	58	66	72	75	76	75	75	75		
			R	39	45	47	46	46	46	46	46	46		
			Q	9	13	19	26	29	30	29	29	29		
14	10	1000	S	63	72	80	86	89	90	89	89	89		
			R	53	60	61	61	60	60	60	60	60		
			Q	10	12	19	25	29	30	29	29	29		
15	100	10	S	32	44	55	65	74	82	90	96	101	105	106
			R	3	16	20	21	21	20	20	20	20	19	19
			Q	29	28	35	44	53	62	70	76	81	86	87
16	100	100	S	48	61	72	82	91	100	107	113	118	121	121
			R	27	35	38	38	38	38	37	37	37	37	37
			Q	21	26	34	44	53	62	70	76	81	84	84
17	100	1000	S	63	76	87	97	106	114	122	128	133	136	136
			R	43	50	53	53	53	52	52	52	52	52	52
			Q	20	26	34	44	53	62	70	76	81	84	84
18	100	10,000	S	76	89	101	111	120	128	135	141	146	144	144
			R	58	64	67	66	66	66	66	66	66	66	66
			Q	18	25	34	45	54	62	69	75	80	84	84
19	1000	100	S	48	63	75	88	99	110	120	130	139	148	157
			R	7	18	23	24	24	24	24	24	24	24	24
			Q	41	45	52	64	75	96	96	106	115	124	133
20		1000	S	63	78	91	104	115	126	137	147	156	165	174
			R	29	37	40	41	41	41	41	41	41	40	40
			Q	34	41	51	63	74	85	96	106	115	125	134

Table 11. Continued.

POLICY NUMBER	R.C.	S.C.		YEARS TO END OF PROGRAM 2	3	4	5	6	7	8	9	10	11	12
84	.01	1.35	S	0	1	2	3	3	4	4	4			
			R	-1	0	1	2	2	3	3	3			
			Q	1	1	1	1	1	1	1	1			
75	.01	1.	S	0	1	2	3	4	4	4				
			R	-1	0	1	2	3	3	3				
			Q	1	1	1	1	1	1	1				
21	.01	2	S	0	2	3	4	5	5	5				
			R	-1	1	2	3	4	4	4				
			Q	1	1	1	1	1	1	1				
22	.01	10	S	5	8	9	10	11	11	11				
			R	4	7	8	9	9	10	10				
			Q	1	1	1	1	2	1	1				
23	.01	100	S	14	17	18	19	20	20	20				
			R	13	16	17	18	19	19	19				
			Q	1	1	1	1	1	1	1				
85	.1	1.35	S	0	1	2	3	3	4	4	4			
			R	-1	0	1	1	2	2	2	2			
			Q	1	1	1	2	1	2	2	2			
76	.1	1.5	S	0	1	2	3	4	4	4				
			R	-1	0	1	2	2	3	3				
			Q	1	1	1	1	2	1	1				
24	.1	2	S	0	2	3	4	5	5	5				
			R	-1	1	2	3	3	3	4				
			Q	1	1	1	1	2	2	1				
25	.1	10	S	5	8	9	10	11	11	11				
			R	4	6	8	8	9	9	10				
			Q	1	2	1	2	2	2	1				
26	.1	100	S	14	17	18	19	20	20	21	21	21		
			R	12	15	17	18	18	19	19	19	19		
			Q	2	2	1	1	2	1	2	2	2		
86	1.	1.35	S	0	1	2	3	4	4	5	5	5		
			R	-9	-1	-1	0	0	1	1	1	1		
			Q	9	2	3	3	4	3	4	4	4		
77	1.	1.5	S	0	1	3	4	4	5	5	5			
			R	-5	-1	0	0	1	1	1	1			
			Q	5	2	3	4	3	4	4	4			
27	1	2	S	0	2	4	4	5	6	6	6			
			R	-2	0	1	1	2	2	2	2			
			Q	2	2	3	3	3	4	4	4			
28	1	10	S	5	8	9	10	11	12	12	12			
			R	2	5	6	7	7	8	8	8			
			Q	3	3	3	3	4	4	4	4			

Table 12. Variations in Reorder and Shortage Costs, $m = 1$

Fixed Parameters: $i = 0.2, \alpha = 5$, H.C. $= 0.01$, $N = 8$

POLICY NUMBER	R.C.	S.C.		YEARS TO END OF PROGRAM 2	3	4	5	6	7	8	9	10	11	12
29	1	100	S	14	17	18	19	20	21	21	21			
			R	11	14	15	16	16	17	17	17			
			Q	3	3	3	3	4	4	4	4			
30	1	1000	S	23	26	28	29	30	30	31	31	31		
			R	20	23	24	25	26	26	26	27	27		
			Q	3	3	4	4	4	4	5	4	4		
31	10	2	S	0	2	4	5	6	7	8	8	8		
			R	−16	−5	−3	−2	−1	−1	−1	−1	−1		
			Q	16	7	7	7	7	8	9	9	9		
32	10	10	S	5	8	10	11	12	13	13	14	14	15	15
			R	−1	1	2	3	4	4	4	4	4	4	
			Q	6	7	8	8	8	9	9	10	10	11	
33	10	100	S	14	17	19	20	21	22	23	23	24	24	24
			R	7	10	11	12	12	13	13	13	13	13	
			Q	7	7	8	8	9	9	10	10	11	11	
34	10	1000	S	23	26	28	30	31	31	32	33	33	33	
			R	16	19	20	21	22	22	22	23	23	23	
			Q	7	7	8	9	9	9	10	10	10	10	
35	100	10	S	5	8	11	12	14	15	16	17	18	18	19
			R	−13	−6	−3	−2	−2	−1	−1	−1	−1	−1	−1
			Q	18	14	14	14	16	16	17	18	19	19	20
36	100	100	S	14	17	20	21	23	24	25	26	27	27	28
			R	1	3	4	5	6	6	6	7	7	7	7
			Q	13	14	16	16	17	18	19	19	20	20	21
37	100	1000	S	23	27	29	31	32	33	34	35	36	37	37
			R	9	12	13	14	15	15	16	16	16	16	16
			Q	14	15	16	17	17	18	18	19	20	21	21
38	100	10,000	S	33	36	39	40	42	43	44	45	46	46	47
			R	18	21	23	24	24	25	25	25	25	26	26
			Q	15	15	16	16	18	18	19	20	21	20	21
39	1000	100	S	14	18	21	23	25	26	27	29	30	31	32
			R	−11	−5	−3	−2	−1	−1	−1	−1	−1	−1	−1
			Q	25	23	24	25	26	27	28	30	31	32	33
40		1000	S	23	27	30	32	34	35	36	38	39	40	41
			R	1	4	5	6	6	7	7	7	7	7	8
			Q	22	23	25	26	28	28	29	31	32	33	33

Table 12. Continued.

POLICY NUMBER	R.C.	S.C.		2	3	4	5	6	7	8	9	10	11	12
				YEARS TO END OF PROGRAM										
90	.01	2	S	-0	-0	-0	-0							
			R	-1	-1	-1								
			Q	1	1	1								
92		100	S	4	6	7	8	8	8					
			R	3	5	6	7	7	7					
			Q	1	1	1	1	1	1					
95		100	S	4	6	7	8	8	8					
			R	3	5	6	6	7	7					
			Q	1	1	1	2	1	1					
98	1	100	S	4	6	7	8	8	9	9	9			
			R	2	3	4	5	5	6	6	6			
			Q	2	3	3	3	3	3	3	3			
99	1	1000	S	12	14	15	16	16	17	17	17			
			R	9	11	12	13	13	14	14	14			
			Q	3	3	3	3	3	3	3	3			
100	10	2	S	-0	-0	-0								
			R	-16	-5	-4								
			Q	16	5	4								
102	10	100	S	4	6	7	8	8	9	9	9			
			R	-1	1	1	2	2	3	3	3			
			Q	5	5	6	6	6	6	6	6			
103	10	1000	S	12	14	15	16	16	17	17	17			
			R	5	7	8	9	10	10	10	10			
			Q	7	7	7	7	6	7	7	7			
104	100	10	S	-0	1	2	2	3	3	4	4	4		
			R	-14	-7	-5	-4	-4	-4	-3	-3	-3		
			Q	14	8	7	6	7	7	7	7	7		
105	100	100	S	4	6	7	8	9	9	9				
			R	-2	-1	-1	-1	-1	-1	-1				
			Q	6	7	8	9	10	10	10				
106	100	1000	S	12	14	15	16	16	17	17	18	18	18	
			R	0	2	3	3	4	4	4	5	5	5	
			Q	12	12	12	13	12	13	13	13	13	13	
107	100	10,000	S	20	22	24	25	25	26	26	26			
			R	7	9	10	11	12	12	12	12			
			Q	13	13	14	14	13	14	14	14			
108	1000	100	S	4	7	8	9	10	10	11	11	12	12	
			R	-13	-7	-5	-4	-4	-3	-3	-3	-3	-3	
			Q	17	14	13	13	14	13	14	14	15	15	
109	1000	1000	S	12	14	15	16	17	17	18	18	18		
			R	-2	-1	-1	-1	-1	-1	-1	-1	-1		
			Q	14	15	16	17	18	18	19	19	19		

Table 13. Variations in Reorder and Shortage Costs, $m = 0.1$

Fixed Parameters: $i = 0.2, \alpha = 5$, H.C. $= 0.01$, $N = 8$

POLICY NUMBER	R.C.	S.C.		YEARS TO END OF PROGRAM 2	3	4	5	6	7	8	9	10	11	12
61	.01	100	S	0	0	0								
			R	-1	-1	-1								
			Q	1	1	1								
65	10	1000	S	3	4	5	5	6	6	6				
			R	-1	-1	0	0	0	1	1				
			Q	4	5	5	5	6	5	5				
68	100	1000	S	3	4	5	5	6	6	6				
			R	-1	-1	-1	-1	-1	-1	-1				
			Q	4	5	6	6	7	7	7				
69	100	10,000	S	9	11	12	13	13	14	14	14			
			R	-1	0	1	1	2	2	2	2			
			Q	10	11	11	12	11	12	12	12			
70	1000	100	S	0	0	1	1	1						
			R	-13	-7	-5	-4	-4						
			Q	13	7	6	5	5						
71		1000	S	3	4	5	6	6	6					
			R	-2	-1	-1	-1	-1	-1					
			Q	5	5	6	7	7	7					

Table 14. Variations in Reorder and Shortage Costs, $m = 0.005$

Fixed Parameters: $i = 0.2, \alpha = 5$, H.C. $= 0.01$, $N = 8$

APPENDIX III

COMPUTATION OF OPTIMUM POLICIES

The following is a list of symbols of the parameters and random variables used to describe an inventory model:

j = subscript denoting j^{th} time period of an N period program

$u_j(x)$ = quantity-dependent price per unit such that $u_j(x)$ is a monotone non-increasing function of x, and $x\,u_j(x)$ is monotone non-decreasing. (The cost of buying a lot of size x items in the j^{th} time period is $x\,u_j(x)$.)

h_j = unit holding cost in the j^{th} time period

r_j = reorder cost in the j^{th} time period

d_j = unit shortage penalty in the j^{th} time period

z_j = demand in the j^{th} period, a random variable, and z_j's are statistically independent

$p_j(z_j)$ = probability that the demand is z_j in the j^{th} period

x_j = the stock-on-hand at the beginning of the j^{th} period

y_j = stock control level, i. e., if an order is made, the amount is $y_j - x_j$

q_j = $y_j - x_j$ = reorder quantity

$$\alpha = \frac{1}{1+(\text{rate of interest per period})} = \text{discount factor,}$$

(Assume the periods are all of equal length, and the interest ratio is constant)

$$I(x, y) = \begin{cases} 1, & x < y \\ 0, & \text{otherwise} \end{cases}$$

m_j = expected demand in the j^{th} period

$C_j(x_j)$ = minimum conditional expected cost, given x_j, from the j^{th} period on (not discounted back to beginning).

The inventory model is as follows: At the beginning of the j^{th} period, x_j is stock-on-hand. We order $y_j - x_j \geq 0$ stock, which we receive at the beginning of the next period. We then have $x_{j+1} = y_j - z_j$ stock-on-hand. The most general type of inventory policy will be N functions $y_1, \ldots, y_N$ such that $y_j(x_j) \geq x_j$. The criterion in selecting the y_j's is to minimize the total expected cost of the program where the expected cost of the j^{th} year is discounted j-1 times by "α" to obtain its present value.

We find the optimal solution by a backward induction on the time periods.

Given x_N units of stock-on-hand, and we order $y_N - x_N$ units, the conditional expected cost is the sum of the holding cost, expected shortage cost, and the <u>not-discounted</u> value of the reorder and purchase cost, i. e.,

$$C_N(x_N, y_N) = h_N x_N + d_N \sum_{z_N > x_N} (z_N - x_N) p_N(z_N) +$$

$$\left[r_N I(x_N, y_N) + q_N u_N(q_N) \right] .$$

The last term in brackets is always zero when $y_N(x_N) = x_N$. At the beginning of the last period it never pays to order since the order is not filled before the program terminates. Hence $y_N(x_N) = x_N$ is optimal.

Suppose we have found the optimal policies $y^*_{j+1}, \ldots, y^*_N$ from the $(j + 1)$st year on to the end of the program. Then the optimal y^*_j for the j^{th} year is that y_j which minimizes the conditional expected cost to the end of the program, given x_j, i. e.,

$$C_j(x_j,\, y^*_j) = \min_{y_j \geq x_j} C_j(x_j,\, y_j) = \min_{y_j \geq x_j} \Bigg\{ h_j x_j + d_j \sum_{z_j > x_j} (z_j - x_j) p_j(z_j) +$$

$$r_j I(x_j,\, y_j) + q_j u_j(q_j) + \alpha \sum_{z_j \geq 0} C_{j+1}(y_j - z_j,\, y^*_{j+1}) p_j(z_j) \Bigg\} \;,$$

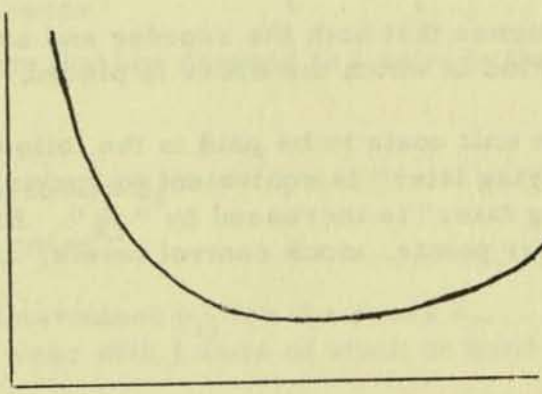
where $C_{j+1}(x_{j+1},\, y^*_{j+1})$ = minimum conditional expected cost to the end of the program using the optimal policies $y^*_{j+1}, \ldots, y^*_N$, given stock-in-hand x_{j+1}.

Define $C_j(x_j) = C_j(x_j,\, y^*_j)$ when we do not wish to explicitly emphasize the optimal policy y^*_j.

In general, the stock control level is a function of the stock on hand. S. Karlin has shown that if the purchase cost is convex, the stock control level increases as the stock-on-hand increases, but reorder smaller amounts; if purchase cost is concave, the stock control level decreases as stock-on-hand increases.

Let us suppose that our conditions are sufficiently regular such that at each period the inventory policy may be described by a reorder point and a stock control level which is not a function of the stock-on-hand. In other words, in the j^{th} period, if the stock-on-hand x_j is less than or equal to s_j, order up to S_j; otherwise do not order. In particular, if unit price is constant, i. e. , $u_j(q_j) = u_j(1)$, we find S_j by examining the marginal cost of ordering one more unit. When $\sum_{z_j} C_{j+1}(y-z_j)$ $p_j(z_j)$ is convex,

$$\sum_{z_j} C_{j+1}(y-z_j)p_j(z_j)$$



y,

we want S_j to be the biggest y such that if we order, the reduction in future costs by ordering up to y rather than y - 1 is greater than or equal to the unit cost. Therefore

$$S_j = \max_y \left\{ y: \alpha \sum_{z_j} \left[C_{j+1}(y-1-z_j) - C_{j+1}(y-z_j) \right] p_j(z_j) \geq u_{j+1}(1) \right\}.$$

The reorder point is found in a similar way; it is the largest integer such that the total cost of ordering up to the optimal amount S_j is less than the total cost of not ordering. Expressed mathematically

$$S_j = \max_{x \leq S_j} \left\{ \text{x: for x units of stock on hand, the total cost of ordering up to } S_j \text{ is less than the total costs of not ordering} \right\}$$

$$= \max_{x \leq S_j} \left\{ x: r_j + (S_j - x) u_{j+1}(1) + \alpha \sum_{z_j \geq 0} C_{j+1}(S_j - z_j) p_j(z_j) \right.$$

$$\left. < \alpha \sum_{z_j \geq 0} C_{j+1}(x - z_j) p_j(z_j) \right\}$$

or

$$S_j = \max_{x \leq S_j} \left\{ x: r_j + (S_j - x) u_{j+1}(1) < \alpha \sum_{z_j \geq 0} \left[C_{j+1}(x - z_j) \right. \right.$$

$$\left. \left. - C_{j+1}(S_j - z_j) \right] p_j(z_j) \right\}.$$

This model assumes that both the reorder and units costs are incurred in the same period in which the order is placed.

If one wants the unit costs to be paid in the following period, then the problem of "paying later" is equivalent to "paying now" if the unit price in "paying later" is increased by "$\frac{1}{\alpha}$". Either problem has the same reorder points, stock control levels, and total costs.

APPENDIX IV

COMPUTATION OF PROGRAM COSTS

In comparing the total expected costs of two inventory policies designed to meet the same demand distribution, it is interesting to see why the total costs differ. The total expected cost is itemized into the following components: An initial order and purchase cost, the total expected holding cost, the total expected shortage penalty, the total expected reordering costs and the total expected cost of units purchased. To obtain these totals, it is necessary to obtain these costs for each year. (If the interest rate is not zero, the yearly costs must be discounted back to the beginning of the program).

To find the above expectations, the probabilities p_{ij} of starting the i^{th} year with j units of stock on hand must be determined for each year in the program, and for all possible stock positions.

We now describe an iterative procedure for finding these proabilities p_{ij}. For the dynamic programming policy, the first year stock control level minimizes the total expected cost. Hence, we arbitrarily start both policies at their respective first year stock control levels. Suppose we can describe our inventory policy by a stock control level, S_i, and a reorder point, R_i, in every i^{th} year.

By assumption

$$P_{1j} = \begin{cases} 1, & \text{if } j = S_1 \\ 0, & \text{otherwise} \end{cases}$$

Let $q_i(j)$ be the probability that the demand is j units in the i^{th} year.

Then

$$P_{2j} = \begin{cases} q_1(S_1 - j) & \text{for } j \leq S_1 \\ 0, & \text{otherwise} \end{cases}$$

Now, suppose we have determined p_{ij} for the years $1, \ldots, i$. Now, we can start the $(i+1)$st year with j units of stock on hand in the following ways:

i) if the i^{th} year began with $k \geq \max(j, R_i + 1)$ units of stock-on-hand, and the demand in the i^{th} year was $k-j$, or

ii) the i^{th} year began with $k \leq R_i$ units of stock-on-hand, and the demand was $S_i - j$.

Therefore

$$p_{i+1,j} = \sum_{k \geq \max(j, R_i + 1)} p_{ik} q_i(k-j) + q_i(S_i - j) \sum_{k \leq R_i} p_{ik} .$$

For the i^{th} year, the different expected costs are as follows:

The expected holding cost is

$$h_i \sum_{j \geq 1} j p_{ij} ;$$

The expected shortage cost is

$$d_i \left[\sum_{j \geq 0} \sum_{k \geq j} (k-j) q_i(k) p_{ij} + \sum_{j < 0} \left(j + \sum_{k \geq 0} k q_i(k) \right) p_{ij} \right]$$

where $\sum_{k \geq 0} k q_i(k)$ is just the average demand in the i^{th} year;

The expected reorder cost is

$$r_i \sum_{j \leq R_i} p_{ij} ; \text{ and}$$

The expected cost of units purchased is

$$\sum_{j \leq R_i} (S_i - j) \, u_i \, (S_i - j) \, p_{ij} .$$

The total itemized costs are obtained by discounting and summing the special yearly costs.

The computation of the expected shortage cost may be simplified by noticing that for $n \geq 0$,

$$\sum_{k \geq n+1} (k-n-1) q_i(k) = \sum_{k \geq n} (k-n) q_i(k) - \sum_{k \geq n+1} q_i(k),$$

i. e. , the conditional expected shortage in the i^{th} year, given a starting stock position $n+1$, is just the conditional expected shortage, given n, minus the tail of the demand distribution from $n+1$ to the end.

Sometimes the inventory policy cannot be described by a stock control level, S_i, and a reorder point, R_i, in every i^{th} year. If the demand distribution is bimodal, or the unit price varies with the size of the order, the stock control level may vary with the amount of stock-on-hand. A stock control level function $S_i(x_i)$, $S_i(x_i) \geq x_i$, now describes the inventory policy for every i^{th} year, where x_i is the stock-on-hand at the beginning of the i^{th} year.

If we know the probabilities of stock positions for the i^{th} year, then the probabilities of stock positions in the $(i+1)$st year are:

$$P_{i+1,j} = \sum_k P_{ik} q_i(S_i(k)-j),$$ where the summation is taken over all possible stock positions, k, and

$$q_i(z) = \begin{cases} \text{probability of demand of } z \text{ in } i^{th} \text{ year, if } z \geq 0 \\ 0, \text{ otherwise.} \end{cases}$$

Similar changes would have to be made in the costing procedure for this more general type of inventory policy.

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