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A Presentation

ESPECIALLY PREPARED

FOR

MARCHANT Calculators, Inc.

Home Office: OAKLAND 8, CALIFORNIA, U. S. A.

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INDEX OF MARCHANT METHODS AND TABLES

relating to

BASIC AND STATISTICAL MATHEMATICS

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Useful in paper industry, plywood, etc.

MM-435 G Diameter of Forming Tools with Top Rake and Gash.
The geometry of tool design for screw-machines, turret lathes, and so forth.

TRIGONOMETRIC TABLES:

T-99 Table: Seven-Place Natural Sines, Cosines, Tangents and Cotangents with Increments to Seconds, by Charles E. Sharp, Jr. (Price 25 cents)

Based upon Benson's Tables (after correction). Tangents above 45° (and Cotangents below 45°) are obtained by computing Reciprocal of Tangent of Complementary Angles (and similarly for Cotangent).

T-192 Table: Seven-Place Table of Natural Cosines with the Argument in Natural Sines, by R. A. Davis.

Argument Interval (.001) - 0 to 1.000

Also useful for problems in the form of $y = 1 - x^2$

T-193 Table: Six-Place Table of Radians with the Argument in Natural Sines, by R. A. Davis.

Argument Interval (.001) - 0 to 1.000

ALGEBRAIC EQUATIONS:

MM-434 B 1 A Short Method of Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients, by Prescott D. Crout, Ph.D.

Reprint of paper presented at A.I.E.E. Summer Convention June 16 - 20, 1941.

A distinct advance in this field. Also shows short-cut when equations are symmetrical, as in certain applications of Statistical Method.

MM-434 B 2 Notes on Marchant Calculator Application to the Crout Method of Solving Simultaneous Equations (see MM-434 B1.)

Gives schematic outline and calculating pattern.

MM-434 C Evaluation of a Polynomial.

Special adaptation of the method of synthetic substitution to the Marchant so that no copying of intermediate results is required.

MM-434 D Birge-Vieta Method of Finding a Real Root of Rational Integral Function.

An exceedingly useful method which greatly simplifies the usual processes to accomplish this result and also provides superior accuracy control. This method was developed as a result of investigation by Dr. Raymond T. Birge, Ph.D., Professor of Physics and Chairman of Department, University of California.

MM-434 F Setting up an Approximating Polynomial of Degree "n" from equidistant Tabulated Values of a Function.

A basic application for use when it is desired to express experimental or approximate values in algebraic form. Solving such an equation by the Birge-Vieta Method (see MM-434D) provides means of inverse interpolation. The equation resulting from application of this method is also helpful in identifying the natural laws which govern experimentally determined values, etc.

MM-434 H How Many Figures for the Answer?

Computing procedure for obtaining the Probable Error or Standard Deviation of a function when the Probable Errors or Standard Deviations of its elements are known.

MM-434 J The Nogrady Method of Solving Cubic Equations.

Shows application of Marchant to procedure described in the monograph, "A New Method for the Solution of Cubic Equations," by Henry A. Nogrady.

INTERPOLATION:

MM-439 E1 Direct Interpolation - Straight-Line and Curvilinear.
A complete explanation, with short-cut method, showing use of Bessel or Everett Central Difference Formulas, the Comrie "Throw-Back," etc. Includes curve of Bessel and Everett 2nd Difference Coefficients. Method takes into account 4th Differences up to 1000.

MM-439 E 2 Appendix to MM-439 E 1, giving elementary mathematical basis of the method.

MM-439 E 3 Appendix to MM-439 E 1, giving elementary mathematical basis of the method.

MM-439 D Direct Curvilinear Interpolation with LaGrange Coefficients.
With Table of Rutledge-Crout Exact 5-Point Coefficients.

T-228 Table: 7-Place LaGrange 5-Point Interpolation Coefficients for values of p from 0 to 2, with Argument to 0.001. (Price 15 Cents.)

MM-439 G Direct Curvilinear Interpolation, Assuming Constant Second Differences.
A build-up method from increments of sub-divisions with adjustment at pivotal points. A rapid method under conditions that warrant its use.

MM-439 H Inverse Curvilinear Interpolation -- Short-Cut Method Includes Effect of 2nd Differences Only.
Using Bessel's Central Difference Formula.

MM-439 J 1 Inverse Curvilinear Interpolation and Finding of Roots of Tabulated Function.
The Comrie "Two-Calculator" Method, Using Bessel's Central Difference Formula.
NOTE: See also MM-439J2, MM-434D, MM-434F, and MM-439K for alternate methods of Inverse Curvilinear Interpolation.

MM-439 J 2 Curvilinear Interpolation with Unequal Intervals of Argument.
"Divided Difference" method using LaGrange-Newton formula.

MM-439 K The A. C. Aitken Method of Curvilinear Interpolation.
For equal or unequal intervals of the argument.

NUMERICAL INTEGRATION AND SOLUTION OF DIFFERENTIAL EQUATIONS:

MM-437 B 1 Area Below Curve for Fractional Portion of Distance between Equidistant Ordinates.
Original contribution for simplifying approximate integration of continuous function with limits not an integer. Assumes constant third differences. Includes curve of coefficients. (Useful in ship and tank design, etc.)

MM-437 B 2 Area Below Curve when End Section has Different Spacing from that of Balance of Sections.
Similar to MM-437 B1, except that it relates to unequal spacing of ordinates in end section, as compared with spacing in adjacent sections. Includes curve of coefficients. (Useful in ship and tank design, etc.)

MM-437 D Moment of Inertia of Sections Composed of Rectangular Areas.
A systematic work-sheet for computing the constants of structural shapes.

MM-437 E 3 Milne Method of Integration of Ordinary Differential Equations.
Complete explanation and systematic work-sheets for this popular method. Includes much heretofore unpublished information.

MM-437 E 4 Appendix to above relating to 2nd-order equations without any term of first order.

MM-437 E 5 Supplement to MM-437 E 4, showing coefficients of 7-9-11 point formulas.

MM-437 E 1 Milne-Method Starting Values for applying MM-437 E 3.

MM-437 E 2 Taylor-Series Method for obtaining starting values for applying MM-437 E3 or MM-437 E4.

STATISTICAL METHOD, AND LEAST SQUARES.

MM-441 A Summations in Statistical Method.

A full explanation of short-cut self-proving methods for obtaining the various kinds of summations used in Statistical Method. (MM-441B should also be used in connection with this.)

MM-441 B Summations of X^2 and/or (UX^2) or X^3.

This also provides means of accumulating products of three-factor multiplication when number of digits in all three factors does not exceed 10.

MM-441 C Summations for Linear Multiple Correlation.

Short-cut method for cases of two or more independent variables. (To be used in connection with MM-441A.)

MM-441 D Summations of X, XY and XY^2.

A special simplified method.

MM-442 Summation of Factors of the Type of $\frac{AB}{K}$ when A, B, and K are Variable.

MM-444 A Pearson Correlation Coefficient.

With formula especially adapted to calculator computation.

Also see MM-444 A1 and MM-444 A2.

MM-445 A Standard Deviations -- Data Grouped by Class Intervals.

A rapid method of obtaining this value when scores are grouped by tallies.

MM-446 Linear "Least Squares" Line of Regression and Coefficient of Regression.

A short-cut method for solving the most usual form of Least Squares problem. (Should also be considered in the light of MM-441 A3.)

MM-447 Hansen-Ahlberg Method of Extending Parabolic Curves.

A method based upon constant 2nd differences for rapidly extrapolating any second-degree function.

MM-448 Precision of Measurements of Equal Importance and Weight.

Most probable value, probable error of mean, and probable error of a single measurement.

MM-SQC2 Statistical Quality Control

Methods covering specific examples from Marchant booklet, Form No. SQC-1.

1 026 6436 8

CONTINUOUS MULTIPLICATION AND DIVISION WITHOUT MANUAL TRANSFERRING OF INTERMEDIATE AMOUNTS TO WORK SHEET OR TO KEYBOARD DIAL

REMARKS: *Whereas Marchant's Keyboard Dial directly adjacent to and in perfect alignment with the Middle Dial offers no difficulty in transferring intermediate products to the Keyboard Dial, especially when it can be double checked by reverse multiplying by "1" to clear Middle Dial to ciphers, there still arise special conditions where it is better to transfer the intermediate amounts to the Upper Dial instead of to the Keyboard Dial. This is particularly true when the product of a series of multiplications is to be divided by the product of another series of multiplications.*

EXAMPLE:

$$\frac{525.2 \times 623.2 \times 8162.9 \times .3762 \times .87432}{189.3 \times 422.5} = 10987.8$$

OPERATIONS: Decimals: Upper Dial 4, Middle Dial 9, Keyboard Dial 5. Use 10 Column Marchant.

(1) Set up 525.2 in Keyboard Dial and multiply by 623.2.

(2) Clear Keyboard Dial, set up in Keyboard Dial the first divisor, 189.3, and divide.

(3) Clear Keyboard Dial, set up in Keyboard Dial 8162.9, and with Manual Counter Control toward the operator, multiply by the amount that appears in the Upper Dial, reducing it to ciphers.

(4) With Manual Counter Control away from the operator, change Keyboard Dial to read 2nd divisor (422.5) and divide.

(5) With Manual Counter Control toward the operator, change Keyboard Dial to read .3762, and multiply by the amount that appears in the Upper Dial, reducing it to ciphers.

(6) With Manual Counter Control away from the operator, clear Keyboard Dial, change Keyboard Dial to read "1." and divide.

(7) With Manual Counter Control toward the operator, change Keyboard Dial to read .87432 and multiply by the amount that appears in the Upper Dial, reducing it to ciphers.

10987.8 appears in Middle Dial.

NOTE: *Permitting remainders of the divisions to remain in the Middle Dial produces a more nearly correct result than if they are cleared.*

This method is of greatest benefit when there are two less division factors than multiplication factors. If there are no division factors it is better to transfer the intermediate results to the Keyboard Dial, and if it is thought necessary, double check by reverse multiplying by "1."

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PERCENTAGE PRORATION - - with LOCKED DIVISOR

Dept.	Expense	Per cent to Total
A	126	7.29%
B	309	17.88
C	362	20.95
D	553	32.00
E	378	21.88
	1728	100.00%

Formula:

$$\frac{\text{Each Amount}}{\text{Total}} = \% \text{ to Total}$$

Division Clear-Return away
(Carriage in Position 1)

1 LOCK [LOCK OR CLEAR] Total in right of Keyboard Dials: [ ] [ ] [ ] [ ] [1] [7] [2] [8]

Set Keyboard Dials DECIMAL left of this amount

Touch TAB KEY directly below left-hand figure of divisor 5 5 4 3 2 1

To let the decimal point "find itself" divide 1728 by 1728 -

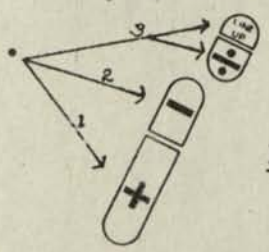
make Keyboard Dials read: [1] [7] [2] [8] [1] [7] [2] [8]

and 1) Add; 2) Subtract; 3) Line-up and Divide

Upper Dials [ ] [ ] [1] [0] [0] % [0] [0]
7 6 5 4 3 2 1

2 ÷ Enter each amount in Keyboard Dials left of decimal point....then 1) add, 2) subtract, 3) line-up and divide:

Touch . . . .



Answer appears in Upper Dials correctly pointed off: [ ] [ ] [ ] [ ] [7] . [2] [9]
7 6 5 4 3 2 1

Let remainders accumulate [ ] [ ] [ ] [ ] [2] [8] [8]

(Upon completion, Middle Dials show all zeros . . thus proving each amount was correctly entered in calculator.)

3 + Percentages = 100.00% . . thus proving the copied answers are correct.

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(over)

Note: When individual amounts are in Dollars and Cents, drop the cents and round up the dollars when cents equal 50¢ or more -(do not round up total figure).*

To carry out percentages to more decimal places in Upper Dials, move Keyboard Dials decimal point marker one place left for each additional decimal place desired in the Upper Dials answer.

* Model EFA, 8 column capacity, will easily accomodate a proration with a 4 digit total; Model 10 EFA, a 5 digit total. Therefore, when the dollars are 3 or 4 figure amounts, it is safe to drop the cents and use only the dollar figures. If, however, the amounts are small (for example, $1.23 or $20.25) enter the entire amount, and set the decimals as described in this method.

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Model 10 EFA

CONSTANT DIVIDEND
TO BE DIVIDED BY A SERIES OF VARIABLE DIVISORS

This problem appears in various forms. The principles used, as shown here, may be adapted to other similar problems.

Of course, if a table of Reciprocals is available, it is possible to solve such problems by multiplying the constant dividend by the reciprocal of each of the variable divisors.

EXAMPLE:

389.7 ÷ 2.648 = 147.2
389.7 ÷ 4.13 = 94.4
389.7 ÷ 11.402 = 34.2
389.7 ÷ 7.6 = 51.3

OPERATION:

Set decimals: Keyboard Dial 8 and 1; Upper Dial 2; Middle Dial 10. Set Tab Key No. 5.

(1) *To lock Constant Dividend* (389.7) in Keyboard Dial, around 1st decimal, hold down LOCK OR CLEAR Key while entering the figures.

Touch Tab Key No. 10 to shift Carriage into Position 10.

With the Carriage in Position 10, touch Add Bar. Dividend now appears in Middle Dial around decimal.

(2) Lightly touch Tab Key No. 1 to shift Carriage to Position 5.

Enter first *Variable Divisor* (2.648) in Keyboard Dial around 8th decimal; touch Division Key.

Quotient appears in Upper Dial (147.16; copy as *147.2*)*

(3) Now shift Carriage into Position 10; touch Add Bar to enter dividend; touch Tab Key No. 1 to re-position Carriage.

Enter next *Variable Divisor* (4.13) in Keyboard Dial around 8th decimal; touch Division Key. Quotient appears in Upper Dial (94.4).

(4) *Continue*, as above, for remaining problems in the example.

* The slight increase in the divisor because the dividend remains at right of Keyboard Dial while division is taking place does not affect 5 digit accuracy of the quotients. In the application of this method, however, the possibility of this must be considered when preparing the exact form of its use in specific cases.

Consult your Marchant Representative for other MARCHANT-METHODS which may be applied to your figurework.

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DIVISION BY AMOUNTS THAT CAN BE PUT INTO THE FORM OF (1 - X) OR (1 + X)

REMARKS: *It is sometimes desirable to divide an amount by (1 - X) or (1 + X) using the method of Summation of Series, instead of subtracting or adding X from or to "1", and resetting the amount as a divisor. This occurs first when it is difficult to obtain a suitable decimal setting on the calculator for using the ordinary process, second, when (1 - X) is a constant divisor; and third, when X is the principal known factor and it is inconvenient to use its complement. This latter consideration applies only for the case of dividing by (1 - X). An example is in Marchant Method MM-157, Problem C.*

OUTLINE: The principle of this method is the relationship:

(a) $$\frac{A}{1 - X} = A(1 + X + X^2 + X^3 + \ldots X^{n-1})$$

(b) $$\frac{A}{1 + X} = A(1 - X + X^2 - X^3 + \ldots \pm X^{n-1})$$

CASE A: DIVIDING BY (1 - X)

EXAMPLE: $$\frac{129.25}{1 - .0312} = 133.41247 -$$

OPERATIONS: Decimals; Upper Dial 5, Middle Dial 10, Keyboard Dial 5. Use any Marchant model.

(1) Set up in Keyboard Dial 1.0312 and multiply by 129.25.

(2) Clear the "1" at left of decimal in Keyboard Dial so it reads .0312, and multiply from left to right by such amount as will cause Upper Dial to read same as Middle Dial when multiplication has been completed.

In this case, shift to 6th position and increase Upper Dial figure from 9 to 3 by multiplying by 4; shift to 5th position and increase it from 2 to 4 by multiplying by 2; shift to 4th position and reduce it from 5 to 1 by reverse multiplying by 4; shift to 3rd position and multiply by 2; shift to 2nd position and multiply by 4; shift to 1st position and multiply by 7.

NOTE: This operation is extremely simple! Once in awhile multiplication will cause a Middle Dial figure to increase at left of the one being copied. In such cases, shift to right and build up corresponding Upper Dial by "1"

133.41247 appears in both Upper and Middle Dials.

(over)

CASE B: DIVIDING BY (1 + X)

EXAMPLE: $\frac{129.25}{1+.0312} = 125.33941$

OPERATIONS: Same decimals as Case A, Manual Counter Control toward operator.

(1) Set up in Keyboard Dial 129.25 and add into Middle Dial.

(2) Clear Upper Dial, set up in Keyboard Dial .0312 and reverse multiply from left to right by the amount that appears in Middle Dial so that upon completion, the Upper and Middle Dials read the same.

The procedure is similar to that described in step 2 of Case A.

125.33941 appears in Upper and Middle Dials.

1 026 64368

MULTIPLICATION OF LARGE FACTORS

It is not often that problems beyond the capacity of the calculator must be solved. However, when it is necessary, apply the following method.

EXAMPLE:

Assume only an 8-column Marchant is available and the problem is:

98,765,432,155 × 87,687 = *8,660,444,449,375,485*

Note that the product above is also the accumulated sum of the following individual products, which are both within the capacity of the calculator:

(a) 98,765,432,000 × 87,687 *or* 98,765,432 × 87,687,000 = 8,660,444,435,784,000

(b) 155 × 87,687 = 13,591,485

8,660,444,449,375,485

OPERATION:

Set Decimals at "O" in all dials.
Enter 98,765,432 in Keyboard Dial, and multiply by 87,687,000
Clear Upper and Keyboard Dials only.
Enter 87,687 in Keyboard Dial, and multiply by 155.
Answer appears in Middle Dial 8,660,444,449,375,485

Refer to MM-422 A2 for other examples containing larger factors.

Refer to MM-421 B1 for discussion of division factors which exceed the capacity of the calculator.

Consult your Marchant Representative for other MARCHANT-METHODS which may be applied to your figurework.

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DIVISION—WHEN FACTORS EXCEED CAPACITY OF CALCULATOR

If the *dividend* has more digits than the number of columns of the Marchant, as in the following example, proceed as follows:

EXAMPLE:

$$\frac{1345678912345}{234567} = 5736863$$

(1) With the Carriage in Position 1, *use a 10-coulmn Marchant.* Enter the nine digits at the right of the dividend in the Keyboard Dial (678912345), and touch the Add Bar.

Shift Carriage to Position 10, and enter the digits remaining at the left (1345) in the right of Keyboard Dial; touch Add Bar.

(Middle Dial now shows 1345678912345)

(2) Enter divisor (234567) in Keyboard Dial. Touch Division Key.

Answer appears in Upper Dial (5736863).

Rule for decimal points is Keyboard Dial + Upper Dial = Middle Dial. Therefore, in the case of large factors, which cannot both be placed around the same pre-set decimal, determine the answer decimal point as follows: Middle Dial - Keyboard Dial = Upper Dial

Note: If the *dividend* has more digits than number of columns of the Marchant, division can also be accomplished by *splitting the dividend* into two parts, dividing each by the divisor and adding the respective quotients.

$$\frac{12345678912345}{123456} = \frac{12345678910000}{123456} + \frac{2345}{123456}$$

Refer to MM-421A1 for discussion of multiplication when factors exceed capacity of the calculator.

10 EFA

DIVISION WITH A USEFUL FRACTIONAL REMAINDER

EXAMPLE I:

There are occasions when it is desired to have the fractional part of a quotient remain in true fractional form rather than in decimal form.

For example, let's say a wholesale grocery ships a total of 1813 cans in assorted case and broken case lots, and wants to determine the number of full cases shipped as well as the number of cans left over from the last full case. Assume 36 cans to a case.

Move the "Division Clear-Return" lever away from you before dividing.

Divide in the usual manner, with the dividend in the Middle Dial, enter the Divisior in the Keyboard Dial, and touch the Division Key (or "Line-up" and Division Keys, depending upon the particular problem).

EXAMPLE II:

It is sometimes desirable to obtain remainder in units, as above, and also in decimal form as it may be needed for multiplying by another factor.

For example, determine how many bushels and pounds, and bushels and pounds expressed as a decimal fraction, there are in 13,849 pounds of Shelled Corn, each bushel containing 56 pounds.

In this case, set decimals to correspond to the number of decimals required for the quotient, and after the quotient has been obtained as far as the decimal point, touch *STOP* Key while division is proceeding in the dial immediately left of Middle Dial decimal.

13,849 lbs. Shelled Corn = *247 bu., 17 lbs.* or *247.303 bu.*

Set decimals: Keyboard Dial 2, Upper Dial 3, Middle Dial 5.
Division Clear-Return lever *away* from operator.
Carriage in position 4.

Operation:

(1) Enter number of pounds (13849.) in *Middle Dial* around decimal.
Enter "lbs. per bu." (56.) in Keyboard Dial.

Touch Line-up and Division Keys simultaneously:

(2) When Active Dial Indicator *(orange arrow)* points to the Dial *left* of Upper Dial decimal (in this example, Position 4), touch Stop Key *once*.

Number of bushels (247.) appears in Upper Dial.
Number of pounds (17.) appears in Middle Dial.

(3) Now continue the division by touching Stop and Division Keys so that Upper Dial does not clear:

Hold down the Stop Key while touching Division Key by a rolling motion of two fingers (touch STOP Key; touch DIVISION Key; then release STOP; release DIVISION).

Number of bushels with remainder expressed as a decimal fraction (247.303) appears in Upper Dial.

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SUM OF A SEQUENCE OF NUMBERS

Remarks: A much used control-figure in accounting and tabulating-machine practice is the sum of a series of numbers that are in sequence. For example, a batch of cards will have serial numbers punched in sequence. The sum of these serial numbers is entered into the accounting machine and as the cards are run, the amount of each serial is subtracted from the control-figure. If no cards are missing, the control-figure reduces to ciphers, regardless of the arrangement of the cards.

In terms of the highest serial number (a) and lowest serial number (b), the number of cards (n) is $a - b + 1$. The sum of all serial numbers of the entire sequence of cards is, then

$$\frac{(a+b)n}{2}; \text{ or in equivalent form: } \frac{(a+1)a - (b-i)b}{2}$$

Example: Find the sum of all serial numbers from 51207 to 51468, inclusive.

GENERAL METHOD FOR ALL CONDITIONS

Operations: Decimals; Upper Dial 0, Middle Dial 2, Keyboard Dial 2.

(1) Set up in Keyboard Dial a number that is one higher than Highest Serial (51469) and multiply by Highest Serial (51468).

(2) Change Keyboard Dial to read a number that is one lower than the Lowest Serial (51206) and reverse-multiply by Lowest Serial (51207). Number of items less 1 (261) appears in Upper Dial.

(3) Change Keyboard Dial to "2" and divide.
Sum of all Serials (13,450,425) appears in Upper Dial.

SPECIAL METHOD FOR SERIALS NOT EXCEEDING 100,000, WITH LESS THAN 1000 ITEMS

Operations: Decimals; Upper Dial 6 and 1, Middle Dial 10 and 1, Keyboard Dial 9, 4, and 0. Division-Clear Lever away from operator. No. 7 Tab Key and Non-Shift Key down on any 10-column Marchant.

(1) With carriage in 7th position, set up Lowest Serial Number (51207) at 4th Keyboard-Dial decimal and subtract.

(2) Similarly, set up Highest Serial Number (51468) and multiply by 1.
Number of Items less 1 (261) appears in Middle Dial. Copy to work as 262.

(3) Reverse-multiply by 2, which converts Middle-Dial amount to $-(a+b)$ ..9999897325.

(4) Change Keyboard Dial to read "2" at 9th decimal and Number of Items (262) at "0" decimal and multiply by 1.* Then depress Division Key.
Sum of all Serials (13,450,425) appears at 1st Middle-Dial decimal.

(*) If row of 9's at left of Middle Dial is not eliminated with multiplication by 1, continue to multiply by 1 until they clear.

SUM OF A SEQUENCE OF NUMBERS WHEN GIVEN NUMBER OF ITEMS IN THE SEQUENCE

REMARKS: *Marchant Method. MM-418A showed means of obtaining the sum of a sequence of numbers when the number of items in the sequence is unknown. When the number of items is known, the process may be shortened by applying the method below. Sums of sequences of numbers are required for control purposes in certain systems of accounting.*

EXAMPLE: Find the sum of the 262 serial numbers from 51207 to 51468, and check the correctness of the number of items.

OPERATIONS: Decimals: Upper Dial 4 and 1, Middle Dial 6 and 3, Keyboard Dial 5 and 2. No. 9 Tab Key down on 10 column Marchant.

(1) With carriage in 9th position, set up Number of Items (262) at 2nd Keyboard-Dial decimal, and multiply by Highest Serial (51468).

(2) Move Manual Counter Control toward the operator, tap Clear-Tab Key only, which shifts carriage to 9th position without dial clearance, and multiply by Lowest Serial (51207).

One Less than Number of Items (261) appears in Upper Dial

(3) Move Manual Counter Control away from the operator, Clear-Tab Keyboard Dial only, which shifts carriage to 9th position, set up "2" at 5th Keyboard-Dial decimal, and divide.

Sum of all Serials (13,450,425) appears at 1st Upper-Dial decimal

THREE OR MORE FACTOR MULTIPLICATION BY DIAL-TRANSFER

EXAMPLE: Steel figuring: 16 I-Beams each 15'9" long @ 7.7 lbs. per ft. = *1940.4 lbs.*
Calculation is: 16 × 15.75 × 7.7 = 1940.4

SET DECIMALS: Keyboard Dial 5, Upper Dial 4, Middle Dial 9.
Set Tab Key No. 5.

OPERATION:

(1) Carriage in Position 6.

Enter first factor (16.) in keyboard Dial around decimal, and enter second factor (15.75) in *Automatic* Multiplier Keyboard around Upper Dial decimal.

Middle Dial shows product of above (252.).

(2) Clear Keyboard Dial and copy Middle Dial amount (252.) into Keyboard Dial around pre-set decimal.

Clear-TAB Upper and Middle Dials only:

Carriage has returned to Position 5, so enter third factor (7.7) in Multiplier Keyboard.

Answer appears in Middle Dial (1940.4).

(3) *MODIFICATION* of Step 2 above:

If desired, instead of clearing the Middle Dial amount, multiply by *"one whole number less"* than the actual multiplier. (In this example, by 6.7 instead of 7.7)

However, because the first method described in Step 2 above offers *complete dial proof* of multiplier entry, it is the preferred method.

NOTE: If using Figuremaster Model AB10FA, see MM–422B for application of this type problem.

Consult your Marchant Representative for other MARCHANT-METHODS which may be applied to your figurework.

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THREE-FACTOR MULTIPLICATION BY ADJUSTMENT OF MULTIPLIER

When the sum of the number of digits in all three factors does not exceed the number of keyboard columns, three-factor multiplication may be performed by a *modified form of dial-transfer,* from Middle Dial to Upper Dial.

EXAMPLE: 16 × 15.75 × 7.7 = 1940.4
The sum of the figures in the three factors is less than *ten.*
Therefore, this method can apply to a 10-column model Marchant.

OPERATION:

(1) Pre-set decimals as follows:

Keyboard Dial decimals are set to accomodate the first factor at left, and the second factor at right of Keyboard Dial. (For above example, at 8 and 2.)

Set *Upper Dial* decimal for the *sum* of the decimal places in second and third factors. (For above example, at 3.)

Middle Dial decimals are set at total of Keyboard and Upper Dials. (For above example, at 11 and 5: 3 + 8 = 11 and 3 + 2 = 5.)

(2) Enter first factor (16.) at left, and second factor (15.75) at right of Keyboard Dial around pre-set decimals.

Enter factor (7.7) in Automatic Multiplier Keyboard around Upper Dial decimal. (Middle Dial now shows 121.275 around *5th* decimal, as the product of 15.75 × 7.7)

(3) Clear second factor (15.75) from Keyboard Dial, without disturbing first factor (16.).

Use Multiplier Keys to *change* Upper Dial reading from 7.7 to the amount at right of Middle Dial, 121.275, using "Neg. X" Key when necessary.

When Upper Dial shows 121.275, Middle Dial shows product of the three factors (1940.4 at *11th* decimal).

NOTE: This method is useful in case one of the three factors is a *Constant.* In such case, the Constant is considered the *first factor,* and is *locked* in the Keyboard Dial where it remains as successive multiplications are performed.

Also see Marchant Method MM-422 A3 for application of another similar problem.

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MULTIPLICATION of FOUR-FACTORS BY DOUBLE SETUPS

This method offers the advantage of entering all four factors in succession. One additional multiplication then completes the work. *Use a 10-column Marchant.*

EXAMPLE: Steel figuring: 18 pcs. Angles 4" × 4" × ½" , 19' 9" long @ 11.1 lbs. per ft.
@ $4.37 Cwt. = $172.44
Calculation is: (18 × 19.75) × (11.1 × .0437) = 172.4424

SET DECIMALS: Keyboard Dial 8, 6, 2; Upper Dial 8, 4; Middle Dial 16, 10, 6.

OPERATION:

(1) Enter *first factor* (18.) in Keyboard Dial around 8th decimal. Enter *second factor* (19.75) in Automatic Multiplier Keyboard around 8th Upper Dial decimal.

Product of above appears in Middle Dial (355.5) around 16th decimal.

(2) Clear Keyboard Dial only. Enter *third factor* (11.1) in Keyboard Dial around 2nd decimal. Enter *fourth factor* (4.37 Cwt. = .0437) in Multiplier Keyboard around 4th Upper Dial decimal.

Product of above appears in Middle Dial (.485070) around 6th decimal.
Note the amount in left of Middle Dial (355.5) still remains in view.

(3) Now multiply these two amounts together (355.5 × .485070):

Clear Upper and Keyboard Dials. Copy amount from right of Middle Dial into Keyboard Dial (.485070) around 6th decimal.

Note amount at left of Middle Dial (355.5) on work sheet. Clear Middle Dial.

Now enter this amount (355.5) in Multiplier Keyboard, around 4th Upper Dial decimal.

Answer (172.4424) appears in Middle Dial around 10th decimal.

RULE for determining the decimal setting for other problems similar to above example:

(a) The *left-hand* Keyboard, Upper and Middle Dials decimals should be set as far to the *left* as possible, but allowing enough room for the multiplication of the first and second factors.

(b) The *right-hand* Keyboard, Upper and Middle Dials decimals should be set as far to the *right* as possible, but allowing enough room for the multiplication of the third and fourth factors.

(c) Set the decimals for multiplying the two Middle Dial amounts *after* they appear. Always try to use the decimals already provided for this last multiplication.

Note: Also see MM-422 A1, A2 for application of other similar problems.

Consult your Marchant Representative for other MARCHANT-METHODS which may be applied to your figurework.

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Model 10 EFA

ACCUMULATION OF PRODUCTS OF THREE-FACTOR MULTIPLICATIONS

There are two common ways of accumulating products of three-factor multiplications. However, they are limited in that the *sum* of the digits in the three amounts cannot exceed ten, and the figures must be *related decimally*.

For example: The products of 3.02 × 14.9 × 12.32 may be accumulated to the products of 1.80 × 5.7 × 5.46. However, it could not be accumulated to the product of .0056 × 1.2 × 450., even though the latter problem has only six significant figures.

EXAMPLE: A shoe manufacturer credits his inventory with the amount resulting from the accumulation of three-factor products applying to each item of the invoices. A typical case follows:

STANDARD COST	NUMBER OF PAIRS	ADJUSTMENT FACTOR
$ 1.39	26	1.014
1.97	14	.893
.87	15	.728

ADJUSTED TOTAL COST OF GOODS SOLD = $70.78

Calculation is: (1.39 × 26 × 1.014) + (1.97 × 14 × .893) + (.87 × 15 × .728) = 70.78

METHOD A:

Set decimals: Keyboard Dial 2, Upper Dial 9 and 0, Middle Dial 11 and 2.

(1) Enter first factor (1.39) in Keyboard Dial around 2nd decimal. Enter second factor (26.) in Automatic Multiplier Keyboard so it appears around "O" Upper Dial decimal.

(2) Change Keyboard Dial to read same as Middle Dial (36.14). Touch Neg. X Key to *negatively multiply* by "1", thus clearing Middle Dial at right.

(3) Shift Carriage to Position 10. Enter third factor (1.014) in Multiplier Keyboard around 9th Upper Dial decimal.

First three-factor product (36.65) appears in Middle Dial around 11th decimal.

Clear Upper and Keyboard Dial. Shift Carriage to Position 2.

(4) Repeating Steps 1, 2, 3 above for each of the next items. The final total (70.78) appears in Middle Dial around 11th decimal.

METHOD B: (see other side)

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METHOD B:

Set Decimals: Keyboard Dial 9 and 2, Upper Dial 2 and 0, Middle Dial 11 and 2.

(1) Enter first factor (1.39) in Keyboard Dial around 2nd decimal. Enter second factor (26.) in Multiplier Keyboard, around "O" Upper Dial decimal.

Product (36.14) appears in Middle Dial around 2nd decimal.

(2) Clear Upper and Keyboard Dials. Enter third factor (1.014) with *right-hand figure reduced by "1"* in Keyboard Dial around 9th decimal, and *fill in with* 9's the Keyboard Dial to the right. (Keyboard Dial now shows 1.0139999.99) *

With Carriage in Position 4, enter amount appearing in Middle Dial (36.14) in Multiplier Keyboard.

First three-factor product (36.65) appears in Middle Dial around 11th decimal.

Clear Keyboard and Upper Dials.

(3) Repeat Steps 1 and 2 above for each of the next items. The final total (70.78) will appear in Middle Dial around 11th decimal.

**BRIDGE OF 9's:*

To understand the "bridge of 9's" theory – note the left-hand part of the Keyboard Dial entry in Step 2, above example – 1.0139.

This is quite close to 1.014. The more 9's that are affixed, the closer it will be. With the six 9's in this case, it is so close to 1.014 that it may be considered the same as 1.014, insofar as the left-hand Middle Dial amount is concerned.

What happens at the right end of the Middle Dial ?

Consider the row of 9's by itself, and assume that the Keyboard Dial left of the decimal is filled with 9's (..999999.99). If this amount is multiplied by 36.14, the answer is 99999963.86, the complement of 36.14.

What happened is this: An accumulative multiplication was performed in Step 2 at the right end of Middle Dial. The 36.14 that was already in the Middle Dial had its complement 99999963.86 accumulated to it.

Thus, the sum shows all zeros, because *36.14 added to 99999963.86* (which equals a minus 36.14), is zero.

This then, saves the operation of negatively multiplying by "1" – step 2, *Method A.*

Consult your Marchant Representative for other MARCHANT-METHODS which may be applied to your figurework.

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SUMMATION OF THREE PRODUCTS
WITH EXAMPLE FROM THE SHIPBUILDING INDUSTRY

Obtaining the sum of two products by means of a single Keyboard and Upper Dial entry is a well-known Marchant Application when the number of figures in the factors does not exceed three (see MM-441 A). The method described herein extends this process to finding the *sum of three products*, but the number of figures in their factors cannot exceed two and the sum of the three products cannot exceed 9999. The figures of the factors of any product also must bear the same relation to decimal point as do corresponding figures of the factors of the other products.

EXAMPLE:

A Progress-Report Summary from the shipbuilding industry shows that the budgeted per cent that each section of the work when fully completed bears to the total when fully completed is in accordance with Column A of the table shown below. At the date of the report, the per cent of each section that is completed is shown in Column B. Find the weighted per cent of completion of the ship as a whole.

	(A) Budget	(B) To Date
Hull	41%	84%
Machinery	36	64
Outfitting	23	9
	100 %	60%

OPERATION:

Set Decimals: Keyboard Dial 10, 6, and 2; Upper Dial 10, 6, and 2; Middle Dial 12.
Enter Budget Percentages (as ratios) from left to right in Keyboard Dial (.4100.3600.23), and multiply by the "To Date" Percentages in opposite order (.0900.6400.84).
Weighted Per cent Complete to Date (.5955, written as 60%) appears at *right* of Middle Dial decimal.

Consult your Marchant Representative for other MARCHANT-METHODS which may be applied to your figurework.

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MULTIPLICATION BY AMOUNTS NOT DECIMALLY RELATED —EXEMPLIFIED BY STREET-RAILWAY PAY-ROLL—

Remarks: When amounts expressed as feet and inches, hours and minutes, pounds and ounces, etc., are used as a factor of a multiplication, it is customary to convert them to terms of one unit or the other. This often requires consulting a table of decimal equivalents and entering the amount as a whole number with a lengthy decimal fraction. However, in cases where one factor is a constant, time often may be saved by multiplying by the unconverted amounts in the manner outlined below.

Example: Street-Railway time cards show hours and minutes worked daily at two rates; 75 and 83 cents per hour. Semi-monthly hours and minutes for each are obtained by adding hours and minutes and converting to nearest total hours and minutes by use of Marchant Method MM863B. The cards are then sorted according to rate. One such case in the 83 cts.-per-hr. group shows 95 hrs., 55 minutes. What is Total Pay?

Operations: Upper Dial 6 and 0, Middle Dial 8,* Keyboard Dial 8 and 2.

Preliminary: For group with pay at .83 per hr., set up Earnings per Min. (.0138) at 8th Keyboard-Dial decimal, and Earnings per Hr. (.83) at 2nd Keyboard-Dial decimal, and lock as constant.

(1) With carriage in 8th position, multiply by hours (95).

(2) Shift to 2nd position and multiply by minutes (55).
Total Pay (79.61) appears in Middle Dial.
The figures at extreme left and right of Middle Dial may be ignored.*

(3) Clear-tab Upper and Middle Dials and repeat for next case.
After finishing all cases for .83 per hr., change Keyboard-Dial constant for next rate and repeat for that group.

* Decimal markers placed at 5 and 11 of Middle Dial will be helpful in setting-off the answer for quick recognition.

SIMULTANEOUS DIVISION AND DIFFERENTIAL MULTIPLICATION

SPECIAL METHOD

The Product of Two Factors (1st and 2nd) Divided by a 3rd Factor

REMARKS *This method is particularly suited to cases where the difference between 2nd and 3rd factors is easily known or is a part of the problem. The 2nd factor is not set on the Keyboard. Only the difference between it and the 3rd factor is so set. The method is reasonably well adapted to large figures*

The limitations of this method are: (1) The number of digits of the first factor cannot exceed the full keyboard capacity less the number of digits in the 3rd factor; (2) The quotient of the first factor divided by 3rd factor cannot be carried to more digits than the capacity of the keyboard less the sum of the number of digits in 3rd factor and the number of digits in the difference between 2nd and 3rd factor; that is to say, the quotient cannot be longer than the intermediate 0's in the Keyboard of Operation No. 2.

EXAMPLE A dealer in metals inventories the finished product at cost or market whichever is lower. This requires adjustment of "inventory at market" at each statement date.

Lot 46 Zinc $68512.95 based on Market 4.95 per Cwt. What is inventory at Market 4.73 per Cwt?

OPERATIONS: Decimals: Upper Dial 7 and 0, Middle Dial 9 and 2, Keyboard Dial 9 and 2. Use 10 Column Marchant.

(1) Set up in Keyboard Dial the Prior Market Value (68512.95) at 2nd decimal and with carriage in 1st position multiply by 1 " Shift to 8th position and add.

This places the Prior Market Value in Middle Dial written around 9th and 2nd decimals.

(2) Set up in Keyboard Dial at 9th decimal the Prior Market Price (4.95) and at 2nd decimal the Prior Market Price minus Current Market Price (4.95 - 4.73 equals .22). See Note B if there is an increase instead of a decrease.

(3) Line up and Divide.

Current Market Price (65467.03) appears in Middle Dial at 2nd decimal. If it is desired to know the number of Cwts the Upper Dial shows it to be 13841.

NOTE A *Applying the above to the general problem see Remarks, it is noted that 1st factor equals 68512.95. 2nd factor equals 4.73. 3rd factor equals 4.95 It will be recognized that the Middle Dial at right develops:*

over.

$$A - \left(\frac{A}{C} \times (C - B)\right) \text{ ; which equals } \frac{A \times B}{C}$$

NOTE B: *If in Operation No. 2 the Current Market Price is an increase, such as say 5 14 per Cwt. the amount set up at 2nd decimal is (4.95 — 5.14 equals 99981 and in order that the 9's will not affect the correctness of the division the right hand digit of the Prior Market Price is reduced by "1" Keyboard then reads 4 9499999 81.*

NOTE C *When it is necessary to find the difference in value for a fraction of a Cwt., or other unit. the division may be carried to the number of decimal places required for accuracy to ½¢.*

In the example above if prior value were $68514.27. pre set decimals at Upper Dial 2 and 9 Middle Dial 4 and 11, Keyboard Dial 2 and 9.

Proceed as above and in Operation No. 3 depress the Stop Key when recording of the 5th significant digit of the quotient starts. Then proceed as follows

(4) Copy the value in dollars (65469) to Work Sheet and re start the division.

(5) Then a second glance at the Middle Dial shows .1928 and the record is completed by writing .19 to the right of the dollars and if desired also noting from the Upper Dial the number of pounds, 1384126, or 13841.26 Cwt. worth $65469.19.

NOTE D: *Short first factors may often be set up at left and right of Keyboard Dial and a single addition enters them both into Middle Dial.*

NOTE E *If the first factor is not entered in Middle Dial at right hand decimal, the actual gain or loss will be shown instead of the new total provided losses are entered in complementary form the same as gains.*

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FINDING NET WEIGHT PER UNIT VOLUME OF LIQUID (SPECIFIC GRAVITY) WHEN GIVEN GROSS WEIGHT, TARE AND VOLUME

Remarks: Work of this type is frequently done in laboratories and in commercial processing. The application shown here relates to the wine industry.

Pycnometer used for determination of Sp. Grav. of wines weights 23.4934 gr. and has volume of 24.9844 cubic cm. (recip. .040025). It is used for a series of tests of Specific Gravity of various wines. The gross weights of a series of samples, each of which includes the tare weight of the Pycnometer, are as follows:

Test A	47.7753	Test D	47.8729
" B	47.7843	" E	47.8437
" C	47.7351	" F	47.6985

Find Specific Gravity of each Sample. All tests at 20° C.

Operations: Decimals: Upper Dial 4, Middle Dial 10, Keyboard Dial 6.
Manual Counter Control toward the operator.

(1) Set up in Keyboard Dial the reciprocal of the volume of the Pycnometer (.040025) and reverse multiply by its weight (23.4934).

(2) Clear Upper Dial, move Manual Counter Control away from operator, lock all Clear Keys and multiply by the first of the series of Gross Weights (47.7753).
Specific Gravity of Test A (.971883) appears in Middle Dial.

(3) Without clearing any dials, adjust the Upper Dial until it reads the Gross Weight of Test B (47.7843).
Specific Gravity of Test B (.972243) appears in Middle Dial.

(4) Proceed in the same manner until all Specific Gravities have been determined.

Note: This method is a general solution of problems of the form of $\frac{X-A}{B}$ or $(X-A)B$ when A and B are constants and X is variable. It is particularly useful when successive values of X do not differ greatly from each other, which is usually the case in sampling. Step (1) computes —BA. Step (2) adds BX to it, the sum being (X-A)B.

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DIFFERENCING

REMARKS: Many users have occasion to prepare tables for temporary or permanent use. Table preparation is also a constant task in scientific institutions. The most powerful plan of checking the correctness of the usual table is by taking the difference between the successive amounts, thus producing "first differences." Any lack of uniformity in the original data, or errors of transcription, will usually appear by inspection of the "first differences." The first differences may also be subjected to the same analysis, thus producing "second differences." Errors usually show in bold relief by a lack of reasonable uniformity of trend of the second differences.

Also whenever it is desired to make exact "curvilinear interpolations" using The Comrie End Figure Process, or direct multiplication by Bessellian or Everett coefficients, an essential part of the work is obtaining the first, second, third, and higher "orders" of differences.

EXAMPLE

In the preparation of Marchant Table 52B, it is desired to check the accuracy of the final work as transcribed to the printer's copy by the method of differencing.

Sufficient values to illustrate the principle are shown below with their first differences obtained by this method and also their second differences obtained by inspection:

100 000				101 163			
	296					4	
100 296			4		282		
	292			101 445			
100 588			3		279		
	289			101 724			
100 877			3		276		
	286			102 000			

PRELIMINARY Divide the table into two convenient sections, preferably parallel vertical columns if there be such, otherwise let the sections be the upper and lower halves of a single column table.

OPERATIONS Decimals: Upper Dial 0, Middle Dial 5 and 0, Keyboard Dial 5 and 0. Use 10 column Marchant.

(1) Set up at 5th Keyboard Dial decimal a reasonable number of digits of the highest amount in first section (877) and at "0" decimal a reasonable number of digits of the highest amount in the second section (2000) and touch Add Bar.

(2) Set up in Keyboard Dial the corresponding next lesser amounts (588 and 1724) and reverse multiply by "1."
Copy differences (289) and (276) from Middle Dial.

(3) Clear Middle Dial and touch Add Bar.

(4) Set up in Keyboard Dial the next lesser amounts (296 and 1445) and reverse multiply by "1."
Copy differences (292 and 279) from Middle Dial

(5) Repeat Operations Nos. 3 and 4 with the successively lesser amounts, thus obtaining the desired difference.

Table 52B

PIECE-WORK OR PRODUCTION BONUS OVERTIME ADJUSTMENT FACTORS* BY ¼ HOURS

42 Hour Base With Time and One-Half for Overtime

HOUR	EVEN	1/4	1/2	3/4
42	1.00 000	1.00 296	1.00 588	1.00 877
43	1.01 163	1.01 445	1.01 724	1.02 000
44	1.02 273	1.02 542	1.02 809	1.03 073
45	1.03 333	1.03 591	1.03 846	1.04 098
46	1.04 348	1.04 595	1.04 839	1.05 080
47	1.05 319	1.05 556	1.05 789	1.06 021
48	1.06 250	1.06 477	1.06 701	1.06 923
49	1.07 143	1.07 360	1.07 576	1.07 789
50	1.08 000	1.08 209	1.08 416	1.08 621
51	1.08 824	1.09 024	1.09 223	1.09 420
52	1.09 615	1.09 809	1.10 000	1.10 190
53	1.10 377	1.10 563	1.10 748	1.10 930
54	1.11 111	1.11 290	1.11 468	1.11 644
55	1.11 818	1.11 991	1.12 162	1.12 332
56	1.12 500	1.12 667	1.12 832	1.12 996
57	1.13 158	1.13 319	1.13 478	1.13 636
58	1.13 793	1.13 948	1.14 102	1.14 255
59	1.14 407	1.14 557	1.14 706	1.14 854

**NOTE: These factors are also satisfactory in adjusting for overtime the Total Earnings at Regular Hourly Rate of Pay for Total Hours Worked.*

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COMPOUND ADDITION AND CONVERSION

EXAMPLE 1:

It is often necessary to reduce to *lowest terms* amounts which are not related decimally, such as feet and inches, dozens and pieces, gross and dozens, etc. These can be added simultaneously and the inches converted to feet, pieces to dozens, dozens to gross, etc.

Example:

Yards	*Feet*	*Inches*
15	5	4
17	2	11
30		9
	17	7
62 yd.,	24 ft.,	31 in—*or* 70 yd., 2 ft., 7 in.

Decimals set at:

Keyboard Dial	0,3,6	
Upper Dial	0	
Middle Dial	0,3,6	

NOTE: The decimal setting depends upon circumstances and is therefore subject to alteration depending upon the range of factors used.

1. With the Carriage in Position 1, add the Yards, Feet, and Inches simultaneously:
   a. Enter in Keyboard Dials: 1 5.0 0 5.0 0 4. and touch Add Bar.
   b. Continue for remaining items.
   c. Middle Dials show total: 6 2.0 2 4.0 3 1.
2. Enter the *complements** of yards, feet, and inches in Keyboard Dials:
   a. Enter the complement of 12 (the number of inches in one foot) in the Keyboard Dials directly below the total number of inches shown in the Middle Dials. Also enter a "9" between the "88" and the decimal point at the left—so that each subtraction of 12 inches will increase the number of feet by "1".
   b. Enter the complement of 3 (number of feet in one yard) in the Keyboard Dials directly below the total number of yards shown in the Middle Dials. Also enter "99" to join this to the decimal at the left.

   Keyboard Dials now read: 0 0.9 9 7.9 8 8
3. Add so the Keyboard Dials do not clear: With Non Shift Key down, use Multiplier Key "1."
   a. Add once. (Notice that the number of inches is reduced by 12; the number of feet is reduced by 3 and increased by 1; the number of yards is increased by 1.)
   b. Watching *only* the dials containing the number of *inches*, add until a figure appears which is *less* than 12.
4. Clear "988" from Keyboard Dials.

   Now watching only the dials containing the number of *feet*, add until a figure appears which is *less* than 3.
5. Middle Dials show 7 0.0 0 2.0 0 7.

   Record answer as a total of *70 yards, 2 feet, 7 inches.*

*The *complement* of a number is the difference between that number and the next higher power of 10; i.e., complement of 12 is 88 (100—12); of 3 is 7 (10—3).

(over)

ADDITION OF COMMON FRACTIONS

This method is useful when it is not desired to refer to decimal equivalent tables. It is based upon the fact that mental estimation of conversions of 1/2, 1/4, 1/8, etc. to 16ths, 32nds or 64ths is extremely easy.

EXAMPLE 2:

	Feet	Inches	Fractions of Inch
	1	8	31/32
	3	3	15/16
	7	2	39/64
	5	4	7/8
Total:	17	8	25/64

OPERATIONS

Decimal Setup, on 10-column model: Keyboard Dial 0, 5, 8; Middle Dial 0, 5, 8. Carriage in Position 1. *Note that the smallest fractional unit is 64ths, so all fractional parts of one inch will be entered as if they were 64ths.*

(1) Enter successively in Keyboard Dial the amounts as feet, inches and 64ths. Enter the feet around 8th decimal, inches around 5th decimal and 64ths at 0 decimal (as shown below), and touch Add Bar after each complete Keyboard Dial entry:

01.008.00062.
03.003.00060.
07.002.00039.
05.004.00056.

Upon completion of above, 16.017.00217 appears in Middle Dial.

(2) Enter *complement* of 64 (99936) in Keyboard Dial at 0 decimal. Touch Multiplier Key "1" until "fractions of an inch" section (.00217.) is reduced to a figure below 64. (After three such multiplications, the "fractions" section reads .00025.) Clear Keyboard Dial.

(3) Enter complement of 12 (988) in Keyboard Dial at 5th decimal. Touch Multiplier Key "1" until "inches" section (.020.) is reduced to a figure below 12. (After one such multiplication, the "inches" section reads 8.) Middle Dial now reads 17.008.00025; written as 17 feet, 8-25/64 inches.

NOTE: If a multiplication proceeds too far, as indicated by 9's appearing in Middle Dial, *negatively multiply* to restore proper amount.

Consult your Marchant Representative for other MARCHANT-METHODS which may be applied to your figurework.

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Model FA

ADDITION AND SUBTRACTION OF UNUSUAL FRACTIONS

EXAMPLE: 2/37 plus 7/9 minus 5/7

To find: Total .1175

OPERATIONS: Decimals: Keyboard Dials 0. Tabulator Key #6 down.

(1) With carriage in 6th position, enter numerator of first fraction (2) Keyboard Dials at 0 decimal. Touch Add Bar. Enter denominator of first fraction (37) in Keyboard Dials at 0 decimal. Touch Division Key. Decimal equivalent of first fraction (.05405) appears in Upper Dials automatically pointed off by Division Symbol.

(2) Enter numerator of second fraction (7) in Keyboard Dials at 0 decimal. Touch Add Bar and then Subtract Bar. Enter denominator of second fraction (9) in Keyboard Dials at 0 decimal. With Stop Key depressed touch Division Key, releasing Stop Key first. Decimal equivalent of the sum of the first two fractions (.83182) appears in Upper Dials.

(3) With Manual Upper Dials Control toward operator, enter numerator of third fraction (5) in Keyboard Dials at 0 decimal. Touch Add Bar and then Subtract Bar. Enter denominator of Third fraction (7) in Keyboard Dials at 0 decimal. With Stop Key depressed touch Division Key, releasing Stop Key first. Total of the three fractions (.1175) appears in Upper Dials automatically pointed off by Division Symbol.

CONVERSION OF DECIMAL EQUIVALENTS TO COMMON FRACTIONS

Remarks: Marchant Method MM-415B covers the solution of problems of this type when the common fraction has a large numerator and denominator as in machine designing. However, it is often desired to obtain the common fraction which corresponds to a decimal equivalent when it is known that the numerator and denominator are comparatively small. Such cases occur in connection with distribution of oil-well income, royalties, and the like.

Operations: Decimals: Upper Dial 0, Middle Dial 5, Keyboard Dial 5. Use any Marchant model.

Outline: Set up in Keyboard Dial the known decimal equivalent and multiply successively by "1". After each multiplication, note the reading of the Middle Dial, stopping the process whenever there is shown at right of decimal an amount which is very close to 10000, 20000, 25000, 33333, 50000, 99999, or 00000. In cases where the process is stopped with Middle Dial at right of decimal reading close to 10000, 20000, 25000, or 33333, increase Upper Dial respectively to 10, 5, 4, or 3 times its reading. The numerator of the desired fraction then appears in Middle Dial; the denominator appears in Upper Dial.

Example A: What common fraction has the decimal equivalent .53846?

(1) Set up in Keyboard Dial the given decimal equivalent (.53846) and multiply successively by "1". When Middle Dial reads "6.99998," the Upper Dial reads "13".

The desired common fraction is 7/13.

Example B: What common fraction has the decimal equivalent .5625?

(1) Set up in Keyboard Dial the given decimal equivalent (.5625) and multiply successively by "1". When Middle Dial reads "2.25000" the Upper Dial reads "4".

(2) By reference to the Outline (see above), it is noted that the Upper Dial should be increased to four times its present reading. This is done by multiplying by 12, at which point the Middle Dial reads "9.00000."

The desired common fraction is 9/16.

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MACHINE DESIGNING

SIMPLE AND COMPOUND GEAR RATIOS

REMARKS: Because of gears having teeth, it is obvious that exact gear ratios in machine designing must be common fractions. It is often desired to determine what common fraction is within the allowable error of the desired decimal speed ratio and yet have the numerator and denominator of the fraction (which represent teeth of the gear) of such amount that gears may be constructed having such number of teeth within the limits of available space. Sometimes this requires compound gearing. The method shown herein greatly simplifies the calculation as compared with the usual methods.

EXAMPLE: It is desired to convert a speed ratio of .199146 to a common fraction gear ratio so that the number of teeth of the largest gear is within .003 of the theoretical number of teeth that would produce the exact speed ratio of .199146

OPERATIONS: Decimals: Upper Dial 8 and 0, Middle Dial 16 and 8, Keyboard Dial 8. Use any 10 column Marchant.

PRELIMINARY: Obtain reciprocal of .199146, which is 5.02144156

NOTE: If the decimal portion of the reciprocal is below 1/2 or .500, proceed as in Section A, below. If it is above 1/2 or .500, proceed as in Section B, below:

In the example, the procedure to be followed is that of Section A.

SECTION A

(1) Set up "1" in Keyboard Dial and with carriage in 9th position add into Middle Dial.

(2) Set up the decimal portion of the reciprocal of the desired ratio (.02144156) and divide, stopping division after one or two digits appear after decimal.

Upper Dial shows 46.6. Copy it to report as a whole number only (46.), as "largest multiplier." Do not "round-up" the whole number.

(3) Clear Upper and Middle Dials, complete the reading in the Keyboard Dial by setting up a "5" in 9th column, so it now reads (5.02144156) and multiply at "0" decimal by the whole number (46.) copied from Step 2.

The first trial common fraction appears with numerator in Upper Dial and denominator in Middle Dial as 46/230.98631176. It will be observed that the largest gear varies about .014 from 231, which is not close enough.

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(4) Increase the Upper Dial reading by 1, making it read 47.

The second trial common fraction is 47/236.00775332, which is not close enough.

(5) Increase the Upper Dial reading by the largest multiplier (46), making it read 93.

The third trial common fraction is 93/466.99406508, which is not close enough.

(6) Increase the Upper Dial reading by 1, making it read 94.

The fourth trial common fraction is 94/472.01550664, which is not close enough.

(7) Increase the Upper Dial reading by the largest multiplier (46), making it read 140.

The fifth trial common fraction is 140/703.0018184, which is close enough.

NOTE: If the amount in the Middle Dial that is at right of decimal is slightly below "1", the Upper Dial amount is to be increased by 1. If it is slightly above "1", do not multiply by 1, but again multiply by the largest multiplier (46).

This principle is better illustrated if the above case be modified so that the reciprocal of the speed ratio is 5.21144156. In this case the largest multiplier is 4. The successive readings of the Upper Dial in this case are:

4, 5, 9, 10, 14, 15, 19, 23, 24, 28, 29, 33, 34, 38, 42, 43, 47, 48, 52, 53, 57, 61, 62, 66, 67, 71, 75, 76, 80, 81, 85, 86, 90, 94, 95, 99, 100, 104, 105, 109, 113, 114, 118, 119, 123, 127, 128, 132, 133, 137, 138, 142, 146, 147, 151, 152, 156, 157, 161, 165, 166, 170, 171, 175.

The gear ratio is thus 175/912.002273 --- which is close enough.

It will be observed that 1 and 4 do not always alternate as multipliers; thus, in the above two multiplications by 4 take place from 157 to 165 and similarly elsewhere in the succession.

This is explained by noting that the first multiplication builds up the Middle Dial to an amount that is somewhat below a whole number. The next multiplication by "1" builds it to an amount that is slightly above a whole number. After this has occurred another multiplication by the "largest multiplier" may bring the Middle Dial amount either just below a whole number or just above a whole number. If it builds it to just above a whole number it is of no avail to multiply by "1", so multiplication is again made by the largest multiplier in the hope that the next Middle Dial reading will be sufficiently close to a whole number to be acceptable.

SECTION B

An example to illustrate this case, as referred to in "Preliminary" on Page 1, is as follows:

EXAMPLE: Convert speed ratio of .21032472 into common fraction so that the number of teeth in the largest gear is within .003 of the theoretical number of teeth that would produce the exact speed ratio.

PRELIMINARY: Obtain reciprocal of .21032472, which is 4.75455286

(1) The decimal portion of the reciprocal is above 1/2 or .500 so its complement is obtained by subtracting .75455286 from 1.00000000, it being .24544714

(2) Proceed as in Steps 1 and 2 of Section A, obtaining the reciprocal of .24544717 as 4. plus (neglecting decimal remainder), copying to report as "largest multiplier."

(3) Clear all dials. Set up in Keyboard Dial the reciprocal of the speed ratio (4.75455286) and multiply at "0" decimal by the largest multiplier 4.

The first trial common fraction is 4/19.01821144 --- which is not close enough.

(4) Increase the Upper Dial reading by 1, making it read 5.

The second trial common fraction is 5/23.77276430 --- which is not close enough.

(5) Increase the Upper Dial reading by the largest multiplier (4), making it read 9.

The third trial common fraction is 9/42.79097574 --- which is not close enough.

It will now be observed that the decimal portion of the Middle Dial reading (.79097574) is far from the allowed error and that increasing the reading to 9 + 1 or 10 will not alter it. A multiplication by "1" is only effective if the right-hand portion of the Middle Dial reading is slightly above "0" such as .0045, etc. This being the case, continue to multiply by 4 until a number is reached at which the decimal portion of Middle Dial reading is just above ciphers. This is 57.

Middle Dial now reads 271.00951302 so multiply by "1", which causes Middle Dial to read 275.76406588

Continue multiplying by 4, until 110 is reached. Middle Dial reads 523.00081460, which is close enough.

The desired gear ratio is thus 110/523.

See Reverse Side for Information Relating to Compound Gearing.

(over)

It will be observed that the gear ratios found in this Method have more teeth than ordinarily could be accommodated on gears of ordinary size and pitch. This does not make the ratios impossible to use because there is ordinarily provision for compounding the gearing. To determine the ratios for compound gearing that correspond to those of any simple gear ratio, it is necessary to refer to a table of primes.

Applying this to the three examples herein, we have:

Ratio	Simple Gearing	Factored	Compound Gearing
.199146	$\frac{140}{703}$	$\frac{10 \times 14}{19 \times 37}$	$\frac{20*}{38} \times \frac{14}{37}$
.191885487	$\frac{175}{912}$	$\frac{5 \times 37}{16 \times 57}$	$\frac{15*}{48} \times \frac{37}{57}$
.21032472	$\frac{110}{523}$	In this case 523 is prime so ordinarily this ratio would not be a practical one to employ. (See below.)	

(*) The actual gear ratios are made multiples of the common fraction in these cases as 10 and 5 tooth pinions are ordinarily not entirely satisfactory.

INVERSE COMPOUND GEARING

Examination of Case 3 above shows that the ratio 110/523 provides a large gear which is well within the limit of error of .003. Actually it is within .0008146 of one tooth in being exactly correct. This being the case, we have remaining an allowable error of .003 - .0008146 equals .0021834. Before dropping 110/523 as a possibility due to the 523 being prime, it is advisable to explore it further to see if by compounding we can secure four gears which will produce the ratio within the limit of error. It is found that this cannot be done by direct compounding but the following inverse compound gear ratio is usable; to wit:

$$\frac{16}{77} \times \frac{83}{82} \text{ equals } \frac{1328}{6314} \text{ equals } .2103262591 \text{ (recip. } 4.75451807)$$

Multiplying this by the 110 tooth value gives as the other gear 522.996988, which is seen to be only .003012 departure from 523.

Problems of this sort are solved by taking the decimal equivalent of the ratio of the pair of practical sized gears that most nearly equals the required ratio. Then take the largest pair of gears that may be accommodated and which differ by only one tooth from each other. Multiply their ratio by the ratio of the first trial pair. Repeat for other pairs. Then take another practical ratio and apply the same process. Try the combinations which differ by 1 tooth both to increase as well as to decrease ratios. After a few trials it is usually easy to observe the trend. Works on gearing show decimal equivalents (and sometimes logarithms of such equivalents) for all common ratios likely to be encountered in practice. Such tables greatly assist the final exploration of the ratio in cases when either term of the common fraction as determined in this method is prime.

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CONVERSION TABLES AND THEIR USE

Remarks: In commercial practice, the conversion of amounts from one set of units to another is not always as simple as merely multiplying the amount by the conversion factor. Because of the custom of placing a limit on the subdivision of the amount when expressed in terms of the new unit, special computing methods are necessary which are expedited by use of a table* of the type described herein.

Example: It is desired to convert metric weights into avoirdupois pounds and ounces, expressing the latter to the nearest one-fourth ounce. Prepare a Conversion Table and show its use by applying it to converting 28.164 kilograms, taking the conversion factor as 2.20462234 lbs. equals 1 kg.

Table: Inasmuch as 1/4 oz. equals 1/64 lb., any decimal fraction of a pound will be expressed to the nearest 1/4 oz. if it lies within a range of less than 1/128 lb. either way from the nearest 1/4 oz. As 1/128 lb. equals 0.0078125 lb., a table in the form below is prepared, starting with 0.0078125, and advancing by intervals of twice this amount; viz., 0.015625.

DECIMAL FRACTIONS OF ONE POUND BOUNDING OUNCES TO NEAREST 1/4 OZ.
(Avoirdupois Weight)

0 OZ.	LBS.	2 OZ.	LBS.	4 OZ.	LBS.
1/4	0.0078125	1/4	0.1328125	1/4	0.2578125
1/2	0.0234375	1/2	0.1484375	1/2	0.2734375
3/4	0.0390625	3/4	0.1640625	3/4	0.2890625
1	0.0546875	3	0.1796875	5	0.3046875
1/4	0.0703125	1/4	0.1953125	1/4	0.3203125
1/2	0.0859375	1/2	0.2109375	1/2	0.3359375
3/4	0.1015625	3/4	0.2265625	3/4	0.3515625
2	0.1171875	4	0.2421875	6	0.3671875

In critical cases, descend; i.e., if decimal fraction exactly equals any tabular amount, the next lower line is taken for the corresponding ounces. (This is arbitrary, and computing regulations in any case may specify the opposite.)

Operations: Decimals; Upper Dial 5, Middle Dial 12, Keyboard Dial 7. Use any 8-column Marchant. The example is shown as applying to an 8-column model as it is usually of sufficient capacity for such conversions.

(1) Set up in Keyboard Dial the Kilogram-to-Pounds Conversion Factor to eight places (2.2046223), and multiply by weight in kg. (28.164).

Weight in Pounds (62.090982 . . .) appears in Middle Dial.

It is copied as 62 lbs. 1 - 1/2 oz., because the decimal fraction (.090982) falls between the tabular values that bound 1 - 1/2 oz.

(*) Tables of the type herein which show bounds instead of actual equivalents are sometimes designated "Tables in Critical Form."

EXACT ELAPSED TIME CALCULATIONS

Calculation of exact elapsed time as indicated from cards stamped "in-and-out" either by time clocks or time stamps is a frequent task.

EXAMPLE: Clock Card reads:

	MORNING		AFTERNOON	
	IN	OUT	IN	OUT
Mon.	7:30	12:02	12:33	3:34
Tue.	7:01	12:20	12:54	4:20
Wed.	7:00	12:00	12:45	4:15
Thu.	8:15	1:20	1:50	4:00
Fri.	7:00	12:01	12:30	5:35

Compute elapsed time.

OPERATION:

Decimal setup: Both Keyboard Dial and Middle Dial decimals at 0 and 5. Carriage in Position 1.

Enter hours around 5th Keyboard Dial decimal and minutes around the "0" decimal.

Example: Enter 7:30 in Keyboard Dial as 7.00030.

(1) MORNING:

Subtract each of the starting times shown under the column headed "IN".

Add each of the stopping times shown under the column headed "OUT".

Note: Stopping times that extend into the afternoon hours should be entered with a "12." added to the clock hour reading.

Example: 1:20 is entered as 13:20

Upper Dial shows all zeros, which is proof that as many stopping times have been entered as starting times.

Middle Dial shows 24.99997. (Do not clear.) Continue ...

(2) AFTERNOON:

Subtract each of the starting times.

Note: Regard a 12 o'clock starting time as "0" hours.

Example: 12:33 is entered as 0.33 (By this plan a starting time of 1:50 is entered as 1:50)

Add each of the stopping times.

Upper Dial shows all zeros; Middle Dial shows 43.99889.

Note: When upon completion the minutes are greater than 60, enter the complement of 60 in the Keyboard Dial as: .99940

Touch Repeat-Neg.× Keys*; then watching the Middle Dial figures at right of the 5th decimal, tap the Multiplier Key "1" until Middle Dial shows 60 or less.

(In this example, until Middle Dial shows 42.00009. or, 42 hours, 9 minutes.)

If minutes are less than 60, the reading is correct as hours and minutes.

*On Model ADX, hold down Neg.× Key.

TABLE FOR OBTAINING SQUARE ROOTS TO FIVE SIGNIFICANT DIGITS

MARCHANT

OPERATION: Clear-return control towards operator; Carriage in seventh position; Number 7 Tab Key depressed.

N = Number whose root is desired.

1. Beginning at extreme left of Keyboard Dial, enter N and touch Add Bar.
2. Beginning at extreme left of Keyboard Dial, enter Column B number and touch Add Bar.
3. Beginning at extreme left of Keyboard Dial, enter Column C number and touch Division Key.

The square root of N appears in Upper Dials, accurate to five digits.

A	B	C
1.00		
1.04	102	202 00
1.08	106	205 92
1.12	11	209 77
1.16	114	213 55
1.20	118	217 26
1.24	122	220 91
1.29	126	224 507
1.35	132	229 79
1.41	138	234 954
1.47	144	240 01
1.53	15	244 956
1.59	156	249 806
1.65	162	254 56
1.71	168	259 235
1.77	174	263 82
1.83	18	268 33
1.90	186	272 77
1.98	194	278 575
2.06	202	284 26
2.14	21	289 834
2.22	218	295 3
2.30	226	300 67
2.38	234	305 95
2.46	242	311 13
2.55	25	316 236
2.65	26	322 5
2.75	27	328 64
2.85	28	334 67
2.95	29	340 594
3.05	3	346 42
3.15	31	352 14
3.25	32	357 78
3.36	33	363 326
3.48	342	369 87
3.60	354	376 305
3.72	366	382 63
3.84	378	388 85
3.96	39	394 974
4.09	402	401 006
4.23	416	407 93
4.37	43	414 736
4.51	444	421 43
4.65	458	428 025
4.80	472	434 52

A	B	C
4.80		
4.96	488	441 82
5.12	504	449 006
5.28	52	456 08
5.44	536	463 04
5.61	552	469 9
5.79	57	477 5
5.97	588	484 98
6.15	606	492 35
6.33	624	499 606
6.51	642	506 76
6.7	66	513 817
6.9	68	521 543
7.1	7	529 16
7.3	72	536 66
7.5	74	544 065
7.7	76	551 37
7.9	78	558 575
8.1	8	565 69
8.3	82	572 72
8.5	84	579 66
8.7	86	586 52
8.9	88	593 3
9.1	9	600 00
9.3	92	606 63
9.5	94	613 19
9.7	96	619 68
9.9*	98	626 1
10.1	1	632 46
10.3	102	638 75
10.5	104	644 98
10.7	106	651 16
10.9	108	657 27
11.1	11	663 33
11.3	112	669 33
11.5	114	675 28
11.7	116	681 18
11.9	118	687 03
12.1	12	692 82
12.3	122	698 57
12.5	124	704 28
12.7	126	709 93
12.9	128	715 54
13.1	13	721 11
13.3	132	726 64

A	B	C
13.3		
13.5	134	732 12
13.7	136	737 57
13.9	138	742 97
14.1	14	748 33
14.3	142	753 66
14.5	144	758 95
14.7	146	764 2
14.9	148	769 42
15.1	15	774 6
15.3	152	779 75
15.5	154	784 86
15.7	156	789 94
15.9	158	794 99
16.2	16	800 01
16.6	164	809 946
17.0	168	819 763
17.4	172	829 465
17.8	176	839 054
18.2	18	848 535
18.6	184	857 91
19.0	188	867 186
19.4	192	876 36
19.8	196	885 44
20.2	2	894 43
20.6	204	903 33
21.0	208	912 15
21.4	212	920 87
21.8	216	929 52
22.2	22	938 09
22.6	224	946 58
23.0	228	954 99
23.4	232	963 33
23.8	236	971 6
24.2	24	979 8
24.6	244	987 93
25.0	248	996
25.4	252	100 4
25.8	256	101 193
26.2	26	101 981
26.6	264	102 762
27.0	268	103 538
27.4	272	104 308
27.8	276	105 072
28.3	28	105 831

A	B	C
28.3		
28.9	286	106 9587
29.5	292	108 075
30.1	298	109 1795
30.7	304	110 273
31.3	31	111 356
31.9	316	112 4284
32.5	322	113 491
33.1	328	114 543
33.7	334	115 586
34.3	34	116 62
34.9	346	117 644
35.5	352	118 66
36.1	358	119 667
36.7	364	120 665
37.3	37	121 656
37.9	376	122 638
38.5	382	123 613
39.1	388	124 58
39.7	394	125 539
40.4	4	126 492
41.2	408	127 7505
42.0	416	128 997
42.8	424	130 2313
43.6	432	131 454
44.4	44	132 666
45.2	448	133 866
46.0	456	135 056
46.8	464	136 236
47.6	472	137 405
48.4	48	138 565
49.2	488	139 7146
50.0	496	140 855
50.8	504	141 9865
51.6	512	143 109
52.4	52	144 223
53.2	528	145 328
54.0	536	146 425
54.8	544	147 513
55.6	552	148 594
56.5	56	149 667
57.5	57	150 9974
58.5	58	152 316
59.5	59	153 6236
60.5	6	154 92

A	B	C
60.5		
61.5	61	156 2056
62.5	62	157 481
63.5	63	158 746
64.5	64	160 001
65.5	65	161 246
66.5	66	162 481
67.5	67	163 708
68.5	68	164 925
69.5	69	166 133
70.5	7	167 333
71.5	71	168 5235
72.5	72	169 706
73.5	73	170 881
74.5	74	172 047
75.5	75	173 206
76.5	76	174 356
77.5	77	175 5
78.5	78	176 636
79.5	79	177 764
80.5	8	178 886
81.5	81	180 000
82.5	82	181 108
83.5	83	182 209
84.5	84	183 303
85.5	85	184 391
86.5	86	185 473
87.5	87	186 548
88.5	88	187 617
89.5	89	188 68
90.5	9	189 737
91.5	91	190 788
92.5	92	191 834
93.5	93	192 873
94.5	94	193 908
95.5	95	194 936
96.5	96	195 96
97.5	97	196 977
98.7	98	197 9905
	994	199 4
100.0	(not inclusive)	

Complete instructions for use of this table on reverse side.

*FOR SQUARE ROOTS OF NUMBERS FROM 9.9 TO, BUT NOT INCLUDING, 10 ENTER N IN STEP 1 BEGINNING IN NEXT TO LEFTMOST KEYBOARD DIAL INSTEAD OF IN LEFTMOST DIAL.

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TABLE 82A

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8 9 E 4 9 9 7 0 1

MARCHANT

TABLE FOR OBTAINING SQUARE ROOT TO FIVE SIGNIFICANT DIGITS USING 8 OR 10 COLUMN MARCHANT

If N (the number whose root is desired) is greater than 100 (Example A) or less than 1 (Example B), move the decimal point by steps of two digits to the left or to the right any even number of places required to arrive at a number between 1 and 100. In Column A of the table, note between which numbers N then lies, and use the values in Columns B and C listed in the line between those numbers.

EXAMPLES: A. Find Square Root of 5467.8 (N). Move decimal point to left so N reads 54.678. This lies between the Column A values 54.0 and 54.8 and therefore we use Column B value 544 and Column C value 147513.

B. Find Square Root of .006492 (N). Move decimal point to right so N reads 64.92. This lies between the Column A values 64.5 and 65.5 and therefore we use Column B value 65 and Column C value 161246.

OPERATION: Clear-return control towards operator; Carriage in seventh position (Active Dial Indicator pointing to Upper Dial No. 7); Number 7 Tab Key depressed.

Using Example A above,

1. Beginning at extreme left of Keyboard Dial, enter N (54.678) and touch Add Bar.
2. Beginning at extreme left of Keyboard Dial, enter Column B number (544) and touch Add Bar.
3. Beginning at extreme left of Keyboard Dial, enter Column C number (147513) and touch Division Key.

The square root of N (73.94466) appears in Upper Dials, accurate to five digits and with an error of less than 5 in the sixth digit.

If N is greater than 1, point off the root to give one digit to the left of the decimal in the root for each pair of digits to the left of the decimal in N, counting any odd digit at the left of N as a pair. (For instance, if N has five digits before the decimal, this is two pairs and one odd digit and is to be counted as three pairs, thus giving three digits before the decimal in the root.)

If N is less than 1, point off the root so that the first significant digit in the root is preceded by one zero to the right of the decimal for each pair of zeros immediately following the decimal in N, not counting odd zeros. (For instance, if there are five zeros immediately following the decimal in N, this is two pairs and an odd zero and is to be counted as two pairs, thus giving two zeros following the decimal in the root.)

RAPID METHOD FOR 6-PLACE and 8-PLACE SQUARE ROOTS

When six place or eight place accuracy is required, your Marchant Representative will supply Tables for obtaining these square roots by means of a simple Marchant operation.

MARCHANT

Table for Obtaining

Square Roots
to
Six Significant Digits

TABLE No. 81

MARCHANT CALCULATORS, INC.

Home Office: OAKLAND 8, CALIFORNIA, U. S. A.

Sales Offices and Manufacturer's Service Stations Give Service Everywhere

A	B	C
17.65	177	841 428
17.75	178	843 801
17.85	179	846 168
17.95	18	848 529
18.05	181	850 882
18.15	182	853 23
18.25	183	855 57
18.35	184	857 905
18.45	185	860 233
18.55	186	862 555
18.65	187	864 87
18.75	188	867 18
18.85	189	869 483
18.95	19	871 78
19.05	191	874 071
19.15	192	876 356
19.25	193	878 636
19.35	194	880 909
19.45	195	883 176
19.55	196	885 438
19.65	197	887 694
19.75	198	889 944
19.85	199	892 189
19.95	2	894 428
20.05	201	896 661
20.15	202	898 889
20.25	203	901 111
20.35	204	903 328
20.45	205	905 539
20.55	206	907 745
20.65	207	909 945
20.75	208	912 141
20.85	209	914 331
20.95	21	916 516
21.05	211	918 695
21.15	212	920 87
21.25	213	923 039
21.35	214	925 203
21.45	215	927 362
21.55	216	929 516
21.65	217	931 666
21.75	218	933 81
21.85	219	935 949
21.95	22	938 084
22.05	221	940 213
22.15	222	942 338
22.25	223	944 458
22.35	224	946 573
22.45	225	948 683
22.55	226	950 789
22.65	227	952 891
22.75	228	954 987
22.85	229	957 079
22.95	23	959 167
23.05	231	961 249
23.15	232	963 328
23.25	233	965 402
23.35	234	967 471
23.45	235	969 536
23.55	236	971 597
23.65	237	973 653
23.75	238	975 705
23.85	239	977 753
23.95	24	979 796
24.05	241	981 835
24.15	242	983 87
24.25	243	985 901
24.35	244	987 927
24.45		

A	B	C
24.45	245	989 95
24.55	246	991 968
24.65	247	993 982
24.75	248	995 992
24.85	249	997 998
24.95	25	100
25.05	251	100 1998
25.15	252	100 3992
25.25	253	100 5982
25.35	254	100 7968
25.45	255	100 9951
25.55	256	101 1929
25.65	257	101 3904
25.75	258	101 5874
25.85	259	101 7841
25.95	26	101 9804
26.05	261	102 1763
26.15	262	102 3719
26.25	263	102 5671
26.35	264	102 7619
26.45	265	102 9563
26.55	266	103 1504
26.65	267	103 3441
26.75	268	103 5375
26.85	269	103 7304
26.95	27	103 92313
27.10	272	104 30733
27.30	274	104 6901
27.50	276	105 0715
27.7	278	105 4515
27.9	28	105 83015
28.1	282	106 20742
28.3	284	106 5834
28.5	286	106 95802
28.7	288	107 33134
28.9	29	107 70338
29.1	292	108 07413
29.3	294	108 4436
29.5	296	108 81184
29.7	298	109 1788
29.9	3	109 5446
30.1	302	109 90914
30.3	304	110 27248
30.5	306	110 6346
30.7	308	110 99557
30.9	31	111 35535
31.1	312	111 714
31.3	314	112 0715
31.5	316	112 42783
31.7	318	112 78305
31.9	32	113 13715
32.1	322	113 49014
32.3	324	113 84207
32.5	326	114 1929
32.7	328	114 5426
32.9	33	114 8913
33.1	332	115 23895
33.3	334	115 5855
33.5	336	115 9311
33.7	338	116 2756
33.9	34	116 6191
34.1	342	116 9616
34.3	344	117 3031
34.5	346	117 6436
34.7	348	117 9831
34.9	35	118 32166
35.1	352	118 65924
35.3	354	118 99586
35.5		

A	B	C
35.5	356	119 3315
35.7	358	119 6663
35.9	36	120 0001
36.1	362	120 3329
36.3	364	120 6649
36.5	366	120 9959
36.7	368	121 3261
36.9	37	121 6553
37.1	372	121 9837
37.3	374	122 3111
37.5	376	122 6377
37.7	378	122 9635
37.9	38	123 2883
38.1	382	123 6124
38.3	384	123 9355
38.5	386	124 2579
38.7	388	124 5794
38.9	39	124 9
39.1	392	125 2199
39.3	394	125 5389
39.5	396	125 8571
39.7	398	126 1745
39.9	4	126 4911
40.1	402	126 807
40.3	404	127 122
40.5	406	127 4363
40.7	408	127 7498
40.9	41	128 0625
41.1	412	128 3745
41.3	414	128 6857
41.5	416	128 9962
41.7	418	129 3059
41.9	42	129 6149
42.1	422	129 9231
42.3	424	130 2306
42.5	426	130 5374
42.7	428	130 8435
42.9	43	131 1488
43.1	432	131 4535
43.3	434	131 7574
43.5	436	132 0606
43.7	438	132 3632
43.9	44	132 6651
44.1	442	132 9662
44.3	444	133 2667
44.5	446	133 5665
44.7	448	133 8656
44.9	45	134 1641
45.1	452	134 4619
45.3	454	134 7591
45.5	456	135 0556
45.7	458	135 3514
45.9	46	135 6466
46.1	462	135 9412
46.3	464	136 2351
46.5	466	136 5284
46.70	468	136 8211
46.90	47	137 1132
47.15	473	137 5501
47.45	476	137 9856
47.75	479	138 41975
48.05	482	138 85253
48.35	485	139 28396
48.65	488	139 71408
48.95	491	140 14286
49.25	494	140 57035
49.55	497	140 99655
49.85	5	141 42144
50.15		

A	B	C
50.15	503	141 84507
50.45	506	142 26743
50.75	509	142 68855
51.05	512	143 10844
51.35	515	143 5271
51.65	518	143 9445
51.95	521	144 36074
52.25	524	144 77576
52.55	527	145 1896
52.85	53	145 60227
53.15	533	146 01377
53.45	536	146 4241
53.75	539	146 8333
54.05	542	147 24137
54.35	545	147 6483
54.65	548	148 0541
54.95	551	148 4588
55.25	554	148 8624
55.55	557	149 2649
55.85	56	149 6664
56.15	563	150 06673
56.45	566	150 466
56.75	569	150 86425
57.05	572	151 2614
57.35	575	151 6576
57.65	578	152 0527
57.95	581	152 4468
58.25	584	152 83985
58.55	587	153 2319
58.85	59	153 623
59.15	593	154 01305
59.45	596	154 40214
59.75	599	154 79024
60.05	602	155 1774
60.35	605	155 5636
60.65	608	155 9488
60.95	611	156 333
61.25	614	156 7164
61.55	617	157 0988
61.85	62	157 4802
62.15	623	157 8608
62.45	626	158 2404
62.75	629	158 6191
63.05	632	158 9969
63.35	635	159 3738
63.65	638	159 7499
63.95	641	160 125
64.25	644	160 4993
64.55	647	160 8727
64.85	65	161 2452
65.15	653	161 6169
65.45	656	161 9877
65.75	659	162 3577
66.05	662	162 7268
66.35	665	163 0951
66.65	668	163 4626
66.95	671	163 8292
67.25	674	164 1951
67.55	677	164 5601
67.85	68	164 9243
68.2	684	165 40867
68.6	688	165 89162
69.0	692	166 37317
69.4	696	166 8533
69.8	7	167 3321
70.2	704	167 8095
70.6	708	168 28556
71.0	712	168 76027
71.4		

A	B	C
71.4	716	169 23365
71.8	72	169 7057
72.2	724	170 17646
72.6	728	170 64592
73.0	732	171 11408
73.4	736	171 58096
73.8	74	172 0466
74.2	744	172 51095
74.6	748	172 97406
75.0	752	173 43595
75.4	756	173 8966
75.8	76	174 35604
76.2	764	174 81426
76.6	768	175 2713
77.0	772	175 72714
77.4	776	176 1818
77.8	78	176 6353
78.2	784	177 0876
78.6	788	177 5388
79.0	792	177 98884
79.4	796	178 43774
79.8	8	178 8855
80.2	804	179 33216
80.6	808	179 7777
81.0	812	180 22215
81.4	816	180 6655
81.8	82	181 1078
82.2	824	181 54896
82.6	828	181 9891
83.0	832	182 4281
83.4	836	182 86614
83.8	84	183 3031
84.2	844	183 739
84.6	848	184 1739
85.0	852	184 60776
85.4	856	185 0406
85.8	86	185 4724
86.2	864	185 9033
86.6	868	186 3331
87.0	872	186 76194
87.4	876	187 1898
87.8	88	187 6167
88.2	884	188 0426
88.6	888	188 4676
89.0	892	188 8916
89.4	896	189 3146
89.8	9	189 7367
90.2	904	190 1579
90.60	908	190 5781
91.00	912	190 9974
91.40	916	191 4158
91.80	92	191 83335
92.25	925	192 35393
92.75	93	192 8731
93.25	935	193 39088
93.75	94	193 90729
94.25	945	194 4223
94.75	95	194 93596
95.25	955	195 4483
95.75	96	195 95926
96.25	965	196 4689
96.75	97	196 97724
97.25	975	197 48426
97.75	98	197 98998
98.25	985	198 4944
98.75	99	198 99757
99.20	994	199 3991
99.60	998	199 7999
100.0	(not inclusive)	

ABBREVIATED INSTRUCTIONS

With carriage in 8th position and beginning all entries at extreme left of keyboard:

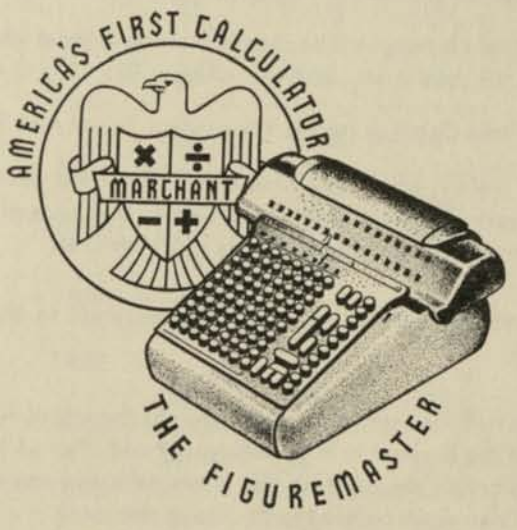
1. Enter N and touch Add Bar.
2. Enter Column B value and touch Add Bar.
3. Enter Column C value and touch Division Key.

TABLE 81

1026 64368

MARCHANT

Table of Factors for Square Roots to Eight Significant Digits



MARCHANT CALCULATING MACHINE COMPANY

Home Office: OAKLAND 8, CALIFORNIA, U. S. A.

Sales Offices and Manufacturer's Service Stations Give Service Everywhere

RAPID METHOD FOR

5-PLACE and 6-PLACE

SQUARE ROOTS

When five or six place accuracy is sufficient, your Marchant Representative will supply simple Tables for obtaining square roots by means of a single automatic Marchant operation.

1 026 6436 8

See reverse side hereof for method of calculating 10 Figure Roots.

SQUARE ROOT DIVISORS

For Calculating Square Root to 5 Significant Figures

MARCHANT

For Speed, Accuracy and Ease of Operation

See reverse side hereof for detailed application to Marchant Calculator

100 – 202

A	Col. 1	Col. 2
100	2 000 000	6 324 555
102	2 019 901	6 387 488
104	2 039 608	6 449 806
106	2 059 126	6 511 528
108	2 078 461	6 572 671
110	2 097 618	6 633 250
112	2 116 601	6 693 280
114	2 135 416	6 752 777
116	2 154 066	6 811 755
118	2 172 556	6 870 226
120	2 190 890	6 928 203
122	2 209 072	6 985 700
124	2 227 104	7 042 727
126	2 244 994	7 099 296
128	2 262 742	7 155 418
130	2 280 351	7 211 103
132	2 297 825	7 266 361
134	2 315 167	7 321 202
136	2 332 381	7 375 636
138	2 349 468	7 429 670
140	2 366 432	7 483 315
142	2 383 277	7 536 577
144	2 400 000	7 589 466
146	2 416 609	7 641 989
148	2 433 105	7 694 154
150	2 449 490	7 745 969
152	2 465 766	7 797 435
154	2 481 935	7 848 567
156	2 497 999	7 899 367
158	2 513 961	7 949 843
160	2 529 822	8 000 000
163	2 553 429	8 074 652
166	2 576 820	8 148 620
169	2 600 000	8 221 922
172	2 622 975	8 294 577
175	2 645 751	8 366 600
178	2 668 333	8 438 009
181	2 690 725	8 508 819
184	2 712 932	8 579 044
187	2 734 959	8 648 699
190	2 756 810	8 717 798
193	2 778 489	8 786 353
196	2 800 000	8 854 377
199	2 821 347	8 921 883
202	2 842 534	8 988 882

205 – 384

A	Col. 1	Col. 2
205	2 863 564	9 055 385
208	2 884 441	9 121 403
211	2 905 168	9 186 947
214	2 925 748	9 252 027
217	2 946 184	9 316 652
220	2 966 479	9 380 832
223	2 986 637	9 444 575
226	3 006 659	9 507 891
229	3 026 549	9 570 789
232	3 046 309	9 633 276
235	3 065 942	9 695 360
239	3 091 925	9 777 525
243	3 117 691	9 859 006
247	3 143 247	9 939 819
251	3 168 596	1 001 998
255	3 193 744	1 009 951
259	3 218 695	1 017 841
263	3 243 455	1 025 671
267	3 268 027	1 033 441
271	3 292 416	1 041 153
275	3 316 625	1 048 809
279	3 340 659	1 056 409
283	3 364 521	1 063 955
287	3 388 215	1 071 448
291	3 411 744	1 078 888
295	3 435 112	1 086 278
299	3 458 323	1 093 618
303	3 481 379	1 100 909
307	3 504 283	1 108 152
311	3 527 038	1 115 348
315	3 549 648	1 122 497
319	3 572 115	1 129 602
324	3 600 000	1 138 420
329	3 627 671	1 147 170
334	3 655 133	1 155 855
339	3 682 391	1 164 474
344	3 709 447	1 173 030
349	3 736 308	1 181 524
354	3 762 978	1 189 958
359	3 789 459	1 198 332
364	3 815 757	1 206 648
369	3 841 875	1 214 907
374	3 867 816	1 223 111
379	3 893 584	1 231 260
384	3 919 184	1 239 355

389 – 692

A	Col. 1	Col. 2
389	3 944 617	1 247 397
394	3 969 887	1 255 388
399	3 994 997	1 263 329
405	4 024 922	1 272 792
411	4 054 627	1 282 186
417	4 084 116	1 291 511
423	4 113 393	1 300 769
429	4 142 463	1 309 962
435	4 171 331	1 319 091
441	4 200 000	1 328 157
447	4 228 475	1 337 161
453	4 256 759	1 346 106
459	4 284 857	1 354 991
465	4 312 772	1 363 818
471	4 340 507	1 372 589
477	4 368 066	1 381 304
483	4 395 452	1 389 964
489	4 422 668	1 398 571
495	4 449 719	1 407 125
502	4 481 071	1 417 039
509	4 512 206	1 426 885
516	4 543 127	1 436 663
523	4 573 839	1 446 375
530	4 604 346	1 456 022
537	4 634 652	1 465 606
544	4 664 762	1 475 127
551	4 694 678	1 484 588
558	4 724 405	1 493 988
565	4 753 946	1 503 330
572	4 783 304	1 512 614
579	4 812 484	1 521 841
587	4 845 617	1 532 319
595	4 878 524	1 542 725
603	4 911 212	1 553 062
611	4 943 683	1 563 330
619	4 975 942	1 573 531
627	5 007 994	1 583 667
635	5 039 841	1 593 738
643	5 071 489	1 603 746
651	5 102 940	1 613 691
659	5 134 199	1 623 576
667	5 165 269	1 633 401
675	5 196 152	1 643 168
683	5 226 854	1 652 876
692	5 261 179	1 663 731

701 – 999

A	Col. 1	Col. 2
701	5 295 281	1 674 515
710	5 329 165	1 685 230
719	5 362 835	1 695 877
728	5 396 295	1 706 458
737	5 429 549	1 716 974
746	5 462 600	1 727 426
755	5 495 453	1 737 815
764	5 528 110	1 748 142
773	5 560 576	1 758 408
782	5 592 853	1 768 615
791	5 624 944	1 778 764
801	5 660 389	1 789 972
811	5 695 612	1 801 111
821	5 730 620	1 812 181
831	5 765 414	1 823 184
841	5 800 000	1 834 121
851	5 834 381	1 844 993
861	5 868 560	1 855 802
871	5 902 542	1 866 548
881	5 936 329	1 877 232
891	5 969 925	1 887 856
901	6 003 332	1 898 420
912	6 039 868	1 909 974
923	6 076 183	1 921 458
934	6 112 283	1 932 874
945	6 148 170	1 944 222
956	6 183 850	1 955 505
967	6 219 325	1 966 723
978	6 254 598	1 977 878
989	6 289 674	1 988 970
999	6 321 392	1 999 000

Table intervals are so arranged that the root calculated by the method below differs from the true root by less than 5 in the sixth significant figure. Rounding up of the fifth figure of true and calculated root may lead to occasional errors of "1" in the fifth figure of the calculated rounded-up five-figure root as compared with that of the true rounded-up five-figure root.

MARCHANT METHOD

OUTLINE: To find the square root of any number N, select from Col. A the number nearest to the three left significant digits of N. Add this number to N, lining up first significant digits. Divide this sum by the factor in Col. 1 or Col. 2 opposite the number selected from Col. A, using Col. 1 if there are an odd number of digits before the decimal point in N (or an odd number of zeros immediately following the decimal), and using Col. 2 if an even number.

The root is shown as the quotient of the preceding division, there being one digit of root, each way from decimal, for each pair of digits in the number whose root is desired.

Example:

Find the square root of 5331.172

Point off into pairs, thus: '53'31.17'20, so divisor is selected from Col. 2.

The root is 5331172
530
1456022/10631172
equals 73015 . . .
which is pointed off as 73.015

Example:

Find the square root of .0005331172

Point off into pairs, thus: .00'05'33'11'72, so divisor is selected from Col. 1.

The root is 5331172
530
4604346/10631172
equals 23089 . . .
which is pointed off as .023089

TABLE 56

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(SEE OVER)

1026 64368

MARCHANT METHOD

This method of extracting square root by use of a table of divisors (or of multipliers, if reciprocals of divisor factors are used) is a concise, practical method of extracting square roots accurate to any desired number of significant figures developed from the well-known principle according to which an approximate root may be converted to one that is of a greater degree of accuracy. (See Barlow's Tables, 1935 Edition, Page XI, Introduction, by L. H. Comrie, M.A., Ph.D.)

The table on the reverse side (and its corresponding table of reciprocals for use as multipliers when the multiplication process is more desirable) is so far as known the first and only published table of its type, with the exception of the similar Table of Cube Root divisors published and copyrighted by Marchant Calculating Machine Company, April, 1940. The development of these tables is distinctly a new and useful adjunct to the calculating art because of the extreme ease with which divisions (or multiplications) may be made by modern calculating machines. The method and tables represent original work of the staff of Marchant Calculating Machine Company.

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DETAILED MARCHANT OPERATIONS—FIVE-FIGURE ROOT

EXAMPLE: Find the square root of 5331.172.

Use any model of "Silent-Speed" Marchant Calculator.

(1) With carriage in 6th position, set up at extreme left of Keyboard Dial the number whose root is desired (5331.172), setting it up without regard to decimal (5331172), and add.

(2) Set up at extreme left of Keyboard Dial the number in Col. A (530) which is nearest to the left-hand three digits of the number whose root is desired, and add.

The Middle Dial shows a number that is approximately twice the number whose root is desired.

(3) Point off into pairs the number whose root is desired, beginning from decimal point (53'31.17'20), from which it is noted that the left-hand group (53) contains two* figures. Set up at extreme left of Keyboard Dial the number that appears in Col. 2 at right of 530 (1 456 022), clear Upper Dial** and divide.

As there is one figure of the root for each group of two, each way from decimal, Upper Dial shows root of 73.015 to five significant figures.

DETAILED MARCHANT OPERATIONS—CONVERTING FIVE-FIGURE ROOT INTO ONE OF TEN FIGURES

(Based upon Method given in Barlow's Tables, 1935 Edition)

OPERATIONS: On any 10-Column Marchant, set decimals at Upper Dial 8, Middle Dial 14, Keyboard Dial 6.

(1) Assuming that the five figure root of the above example is to be expanded to one of ten figures, set up in Keyboard Dial the five-figure root 73.015 and with carriage in 9th position multiply by 2.

Twice the Five-Figure Root appears in Middle Dial (146.030). Copy to report.

(2) Clear Upper and Middle Dials and multiply by the Five-Figure Root (73.015).

(3) Shift to 9th position, change Keyboard Dial to read the number whose root is desired (5331.172) and add.

(4) Set up in Keyboard Dial the divisor copied from Step 1 (146.030), clear Upper Dial** and divide.

The Ten-Figure Root appears in Upper Dial as 73.01487519 +100/146, which is written as 73.01487520.

*If the left-hand group contained one digit, such as if the number were 533.1172, the divisor would have been taken from Col. 1 (4 604 346), the root being 23.089.

**Clearance of Upper Dial is not necessary on Marchants which are equipped with automatic clearance of Upper Dial in division.

NOTE: The decimal setting for other roots may vary, and sometimes the Marchant Upper Dial shows only 9 digits, but the 10th digit may be estimated by reference to the remainder fraction. (See Marchant Instruction Book, 1940 Edition, Page 35.)

MARCHANT CALCULATORS, INC.

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SQUARE ROOT MULTIPLIERS

For Calculating Square Root to 5 Significant Figures

MARCHANT

For Speed, Accuracy and Ease of Operation

100 – 202

A	Col. 1	Col. 2
100	5 000 000	1 581 139
102	4 950 738	1 565 561
104	4 902 903	1 550 434
106	4 856 429	1 535 737
108	4 811 252	1 521 451
110	4 767 312	1 507 557
112	4 724 556	1 494 036
114	4 682 928	1 480 872
116	4 642 383	1 468 050
118	4 602 873	1 455 556
120	4 564 355	1 443 376
122	4 526 788	1 431 496
124	4 490 136	1 419 905
126	4 454 354	1 408 590
128	4 419 417	1 397 542
130	4 385 290	1 386 750
132	4 351 942	1 376 205
134	4 319 343	1 365 896
136	4 287 464	1 355 815
138	4 256 283	1 345 955
140	4 225 771	1 336 306
142	4 195 907	1 326 862
144	4 166 667	1 317 616
146	4 138 030	1 308 560
148	4 109 975	1 299 688
150	4 082 482	1 290 994
152	4 055 535	1 282 473
154	4 029 115	1 274 117
156	4 003 204	1 265 924
158	3 977 786	1 257 886
160	3 952 847	1 250 000
163	3 916 302	1 238 443
166	3 880 752	1 227 201
169	3 846 154	1 216 261
172	3 812 465	1 205 607
175	3 779 645	1 195 229
178	3 747 658	1 185 114
181	3 716 470	1 175 251
184	3 686 049	1 165 631
187	3 656 362	1 156 243
190	3 627 381	1 147 079
193	3 599 078	1 138 129
196	3 571 429	1 129 385
199	3 544 406	1 120 840
202	3 517 988	1 112 485

205 – 384

A	Col. 1	Col. 2
205	3 492 152	1 104 315
208	3 466 876	1 096 323
211	3 442 141	1 088 501
214	3 417 929	1 080 844
217	3 394 221	1 073 347
220	3 371 000	1 066 004
223	3 348 248	1 058 808
226	3 325 951	1 051 757
229	3 304 093	1 044 846
232	3 282 661	1 038 068
235	3 261 640	1 031 421
239	3 234 231	1 022 754
243	3 207 502	1 014 301
247	3 181 424	1 006 055
251	3 155 972	9 980 060
255	3 131 121	9 901 470
259	3 106 849	9 824 717
263	3 083 132	9 749 715
267	3 059 950	9 676 411
271	3 037 283	9 604 736
275	3 015 113	9 534 625
279	2 993 421	9 466 031
283	2 972 191	9 398 894
287	2 951 407	9 333 164
291	2 931 052	9 268 803
295	2 911 113	9 205 747
299	2 891 575	9 143 961
303	2 872 425	9 083 403
307	2 853 651	9 024 033
311	2 835 240	8 965 812
315	2 817 181	8 908 710
319	2 799 462	8 852 676
324	2 777 778	8 784 104
329	2 756 590	8 717 104
334	2 735 879	8 651 604
339	2 715 627	8 587 568
344	2 695 820	8 524 931
349	2 676 439	8 463 645
354	2 657 470	8 403 658
359	2 638 899	8 344 933
364	2 620 712	8 287 421
369	2 602 896	8 231 083
374	2 585 438	8 175 873
379	2 568 328	8 121 761
384	2 551 552	8 068 713

389 – 692

A	Col. 1	Col. 2
389	2 535 100	8 016 694
394	2 518 963	7 965 665
399	2 503 131	7 915 594
405	2 484 520	7 856 743
411	2 466 318	7 799 180
417	2 448 510	7 742 869
423	2 431 083	7 687 760
429	2 414 023	7 633 809
435	2 397 316	7 580 978
441	2 380 952	7 529 230
447	2 364 919	7 478 531
453	2 349 205	7 428 835
459	2 333 800	7 380 123
465	2 318 694	7 332 357
471	2 303 878	7 285 502
477	2 289 342	7 239 536
483	2 275 079	7 194 431
489	2 261 079	7 150 155
495	2 247 333	7 106 689
502	2 231 609	7 056 969
509	2 216 211	7 008 273
516	2 201 127	6 960 575
523	2 186 347	6 913 836
530	2 171 861	6 868 028
537	2 157 659	6 823 116
544	2 143 732	6 779 077
551	2 130 072	6 735 876
558	2 116 669	6 693 494
565	2 103 515	6 651 899
572	2 090 605	6 611 072
579	2 077 929	6 570 989
587	2 063 721	6 526 056
595	2 049 800	6 482 037
603	2 036 157	6 438 893
611	2 022 783	6 396 602
619	2 009 670	6 355 134
627	1 996 807	6 314 459
635	1 984 190	6 274 557
643	1 971 807	6 235 401
651	1 959 655	6 196 973
659	1 947 723	6 159 244
667	1 936 008	6 122 195
675	1 924 501	6 085 805
683	1 913 197	6 050 061
692	1 900 715	6 010 587

701 – 999

A	Col. 1	Col. 2
701	1 888 474	5 971 878
710	1 876 467	5 933 908
719	1 864 685	5 896 654
728	1 853 123	5 860 091
737	1 841 774	5 824 200
746	1 830 630	5 788 960
755	1 819 686	5 754 352
764	1 808 937	5 720 359
773	1 798 375	5 686 962
782	1 787 996	5 654 142
791	1 777 795	5 621 881
801	1 766 663	5 586 680
811	1 755 738	5 552 129
821	1 745 012	5 518 213
831	1 734 481	5 484 909
841	1 724 138	5 452 203
851	1 713 978	5 420 075
861	1 703 996	5 388 506
871	1 694 185	5 357 483
881	1 684 543	5 326 992
891	1 675 063	5 297 014
901	1 665 742	5 267 538
912	1 655 665	5 235 673
923	1 645 770	5 204 381
934	1 636 050	5 173 642
945	1 626 500	5 143 446
956	1 617 116	5 113 769
967	1 607 892	5 084 600
978	1 598 824	5 055 924
989	1 589 908	5 027 728
999	1 581 930	5 002 501

Example:

Find square root of 5331.172

Point off into pairs, thus:
'53'31.17'20 so factor is selected from Col. 2.

The root is
(6 868 028 x 5 331 172)
plus
(6 868 028 x 5 300 000)
equals 73015 . . .
which is pointed off as 73.015

Table intervals are so arranged that the root calculated by the method below differs from the true root by less than 5 in the sixth significant figure. Rounding up of the fifth figure of true and calculated root may lead to occasional errors of "1" in the fifth figure of the calculated rounded-up five-figure root as compared with that of the true rounded-up five-figure root.

MARCHANT METHOD

OUTLINE: Select from Col. 1 or Col. 2 a factor determined as follows: If the number whose root is desired, when pointed off into pairs each way from decimal, has one digit in the left-hand pair, select factor from Col. 1. If it has two digits in the left-hand pair, select factor from Col. 2. In either case select the factor that is at the right of the number in Col. A which is nearest to the three left-hand digits of the number whose root is desired.

The factor so selected is then set up in the Keyboard Dial of the calculator and multiplied by the number whose root is desired. Upper Dial only is cleared and the factor is again multiplied by the number previously selected from Col. A. In this second multiplication, the multiplier is entered in the Upper Dial in the position occupied by the three left-hand digits of the previous multiplier; that is to say, the second multiplier is regarded as being followed by ciphers so it is as long as the first multiplier.

The root is shown in the Middle Dial, there being one digit of root, each way from decimal, for each pair of digits in the number whose root is desired.

Example:

Find square root of .0005331172

Point off into pairs, thus:
.00'05'33'11 72, so factor is selected from Col. 1.

The root is
(2 171 861 x 5 331 172)
plus
(2 171 861 x 5 300 000)
equals 23089 . . .
which is pointed off as .023089

TABLE 57 JM 9-50

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SQUARE ROOT

without using a chart

Example: Find the square root of 780 to five decimal places.

(1) Move Manual Counter Control toward operator. Division Clear-Return lever away from operator.

(2) Since the root is desired to five decimal places, point off five decimal places in the Upper Dial, five in the Keyboard Dial, and ten in the Middle Dial.

(3) Line up Keyboard Dial and Middle Dial decimals (Carriage in Position 6). Immediately to the left of the Keyboard Dial decimal, enter 780. Touch Add Bar and then the Subtract Bar. Point off the 780 in periods of two figures each, beginning at the decimal point ('07'80).

(4) Since 780 contains two "periods" before the decimal place, we must start in the second place to the left of the Keyboard Dial and Upper Dial decimals; namely, Carriage in Position 7 and the first number entered in the seventh Keyboard Dial.

(5) Since the largest digit whose square may be subtracted from 07 is 2, enter 2 in the seventh Keyboard Dial. Touch the **Neg. X** Key and "2" Multiplier Key.

(6) Increase the Keyboard Dial entry by 2, which brings it to 4, and checks as twice the Upper Dial figure.

(7) Above this 4, the Middle Dial reads 38. 4 will go into 38 nine times, but annexing 9 to 4 makes 49 (almost 50) and 50 goes into 380 only seven times. Therefore, our divisor is 47 (see note). Annex 7 to the right of the 4 set up. (Keyboard Dial then shows 47.)

NOTE: If operator should make an incorrect estimate, the simplest method of correction is to add back the amount just subtracted by again touching the same Multiplier Key (but this time without the **Neg. X** Key).

(8) Touch the **Neg. X** Key and "7" Multiplier Key to multiply negatively by the 7 estimated above.

(9) Repeat Steps 6, 7, and 8 until a total of four digits are determined.

(a) Step 6 repeated. Increase the 47 Keyboard Dial entry by 7, bringing the Keyboard Dial reading to 54 which checks as twice the 27 appearing in the Upper Dial.

Step 7 repeated. Immediately below 51 in the Middle Dial, we see 5+ in the Keyboard Dial, which will evidently be contained nine times. Therefore, annex 9 to the right of the entry, making the Keyboard Dial read 549.

Step 8 repeated. Touch the **Neg. X** Key and "9" Multiplier Key to multiply negatively by the 9 thus estimated.

(b) Step 6 repeated. Increase the 549 Keyboard Dial entry by 9, making Keyboard Dial read 558, which checks as twice the 279 in Upper Dial.

Step 7 repeated. Now 558 in the Keyboard Dial is immediately below 1590 in Middle Dial. As 5½ goes into 16 a little less than three times, we estimate the next digit as 2 and annex this figure to the right of the set up, making the Keyboard Dial read 5582.

(over)

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(c) If the operator is absolutely certain that the figure just estimated is correct he may skip the following "Step 8 repeated" and proceed directly to "Step 10". However, it is the safer practice to complete Step 8.

Step 8 repeated. Touch the **Neg. X** Key and "2" Multiplier Key to multiply negatively by the 2 thus estimated.

As four digits have now been determined in the Upper Dial, proceed directly to Step 10.

(10) Restore Manual Counter Control to normal position (away from operator), and touch the Stop and Division Keys so as to prevent clearance of Upper Dial. The Marchant will automatically complete the calculation.

The Upper Dial reads 27.92848. This is the required root.

CHECK

(11) Clear the first three digits from the Keyboard Dial and copy the last three digits from Upper Dial, making the Keyboard Dial read .02848. With Carriage in Position 3, touch "Repeat-**Neg. X**" Key – and then touch in succession the 8, 4, and 8, Multiplier Keys, restoring the last three Upper Dials to zero. The remainder in the Middle Dial now reads .0000048896. (Release "Repeat-**Neg. X**" Key.) Copy the first three digits from the Upper Dial, making the Keyboard Dial read 27.92848. Clear Upper Dial and square the Keyboard Dial entry. Middle Dial reads 780, the figure we started with, thus proving the root correct.

RAPID METHOD FOR

5-PLACE, 6-PLACE and 8-PLACE

SQUARE ROOTS

Your Marchant Representative will supply special Tables for obtaining square roots by means of a simple Marchant operation.

1 026 6436 8

EXTRACTING SQUARE ROOT
"SUCCESSIVE ODD NUMBER METHOD"
USING SEMI-AUTOMATIC MARCHANT

Example: Find the square root of 780 to 5 decimal places.

(1) Since the root is desired to five decimals, point off five decimal places in Upper Dial, five in Keyboard Dial, ten in Middle Dial. Move Manual Counter Control toward operator.

(2) Line up Keyboard Dial and Middle Dial decimals (Carriage in Position 6). Immediately to the left of the Keyboard Dial decimal, enter 780. Touch Add Bar and then the Subtract Bar. Point off the 780 in periods of two figures each, beginning at the decimal point ('07'80.)

(3) Since 780 contains two "periods" before the decimal place, we must start in the second place to the left of the Keyboard Dial and Upper Dial decimals; namely, Carriage in Position 7 and the first number entered in the seventh Keyboard Dial.

(4) Enter the digit "1" in 7th Keyboard Dial and subtract it from the amount in the Middle Dial by touching Short-Cut Bar. Enter the next higher odd number (3) and touch Short-Cut Bar, continuing to subtract successively the larger odd numbers 5, 7, 9, 11, 13, etc., until no more subtractions can be made without causing 9's to appear at left of Middle Dial.

NOTE A: In this case it is evident that if the "3" in Keyboard Dial is changed to a "5" the subtraction would cause an "overdraft of 9's", so the process is stopped after subtracting the "3".

NOTE B: Be sure to touch Short-Cut Bar a sharp tap so it makes only a single subtraction. By keeping the eye on the Upper Dial while touching the Bar it can be noted whether more than one subtraction is counted.

(5) Change the Keyboard entry to the even number (4) which is just one larger than the last odd number subtracted. Shift carriage one position to the left (to 6th Position).

(6) Enter the digit "1" in the next column to the right of those previously used in the Keyboard Dial, and subtract by touching the Short-Cut Bar. Change the "1" to a "3" and subtract by touching the Short-Cut Bar, and similarly change 3 to 5 and subtract, 5 to 7 and subtract, 7 to 9 and subtract, change 49 to 51 and subtract, 51 to 53 and subtract. It is evident that at this point any further subtraction will cause an overdraft, so increase the 53 to the next higher even number (54) and shift carriage one position to the left (to 5th position).

(7) Enter the digit "1" in the next column (5th) to the right of those used previously in the Keyboard Dial and subtract by touching Short-Cut Bar, and continue subtracting after increasing the set-up to the next higher odd number, as before, until after 9 subtractions, Keyboard Dial reads 557, and one further subtraction would produce an overdraft. Increase the entry to 558 and shift to 4th position.

(8) Enter the digit "1" in the next column (4th) to the right of those used previously in the Keyboard Dial and subtract by touching the Short-Cut Bar, and continue to subtract successive odd numbers until after two subtractions it is evident that further subtraction would cause an overdraft.

NOTE: Four digits of the root have now been determined (27.92), which is more than half the required number of digits. Therefore, proceed to step 9.

(over)

(9) Increase the right hand digit of the Keyboard entry by "1" to the nearest higher even number (5584). Move Manual Counter Control away from the operator and touch the Stop and Division Keys so as to prevent clearance of Upper Dial.

Upper Dial reads 27.92848, which is the required root.

CHECK

(10) Clear Keyboard Dial, and enter the additional digits of the root that were determined by the division (.00848). Using Short-Cut Bar and multiplying "from the right", reduce the "848" in last three Upper Dials to ciphers. The remainder in the Middle Dial now reads .0000048896. Copy the four left hand digits from Upper Dial, making the Keyboard Dial read 27.92848. Clear Upper Dial and square the Keyboard Dial entry.

RAPID METHOD FOR

5-PLACE, 6-PLACE and 8-PLACE

SQUARE ROOTS

Your Marchant Representative will supply special Tables for obtaining square roots by means of a simple Marchant operation.

1 026 643 6 8

SQUARE ROOT WITHOUT USE OF TABLE

Remarks: Extraction of square root with the aid of a Marchant Table is the fastest known calculator means of obtaining an accurate root.

However, in cases where a Table may not be available, recourse is often had to the well known process of dividing the number whose root is desired by a first approximation of the root, and then taking the mean of quotient and divisor. The root so obtained will be correct to twice as many significant figures as was the first approximation of the root. The latter method is also commonly used in engineering offices in which the first approximation of the root is obtained with aid of a slide-rule. Then, applying the process once with the aid of a calculator converts that root to one of five or six figures.

However, finding mean of quotient and divisor sometimes requires an extra calculator operation. To avoid this, the following equivalent method has been found useful. It is simple and easily remembered.

Example: Find the square root of 5331.172, when by slide-rule a first approximation of the root is estimated to be 73.0.

Operations: Use any model Marchant. Decimals are not used as the root is pointed-off by the well known rule that one digit of root each way from its decimal corresponds to each pair of digits each way from decimal in the number whose root is desired; i.e., 53'31.17'20, will have as its root an amount of the nature of 73.000. . . .

(1) With carriage at extreme right, set up at extreme left of Keyboard Dial the significant figures of the number whose root is desired (5331172) and add. Clear Upper Dial.

(2) Set up in Keyboard Dial the significant figures of the first approximation of the root (73) in such location as will cause the amount, when squared, to increase the Middle-Dial reading to approximately twice the amount it contains. In this instance, the "73" would be entered at the extreme left of Keyboard Dial as "073."* Multiply by itself (73), which causes Middle Dial to read 10660172. . . .

(3) Change Keyboard Dial to read double its amount (146) and divide. The significant figures of the root, accurate to twice as many significant figures as the first approximation, appear in Upper Dial. It is pointed off by the usual rule as 73.0149, assuming that the first approximation was correct to three significant figures.

(*) Under any condition, the entry will be either at the extreme left of Keyboard Dial, or displaced one column to the right, as in this instance.

Mathematical Note: If N is the number whose root is desired and e the absolute error of the first approximation of its root, then the absolute error of the second approximation of the root is, closely, $e^2/2N^{1/2}$.

Similarly, if r is the relative error of the first approximation of the root, then the relative error of the second approximation of the root is, closely $r^2/2$.

RAPID METHOD FOR 5–PLACE, 6–PLACE and 8–PLACE SQUARE ROOTS

Your Marchant Representative will supply simple Tables for obtaining square root.

See reverse side hereof for method of calculating 10-Figure Roots.

CUBE ROOT DIVISORS

For Calculating Cube Root to 5 Significant Figures

MARCHANT

For Speed, Accuracy and Ease of Operation

See reverse side hereof for detailed application to Marchant Calculator

100 - 197

A	Col. 1	Col. 2	Col. 3
100	300 000	139 248	646 330
102	303 987	141 098	654 920
104	307 948	142 937	663 453
106	311 883	144 763	671 932
108	315 794	146 579	680 357
110	319 681	148 383	688 731
112	323 544	150 176	697 054
114	327 384	151 958	705 328
116	331 202	153 730	713 554
118	334 998	155 492	721 732
120	338 773	157 244	729 864
122	342 527	158 987	737 951
124	346 260	160 720	745 995
126	349 973	162 443	753 995
128	353 667	164 158	761 953
130	357 342	165 863	769 869
132	360 997	167 560	777 745
134	364 635	169 248	785 581
136	368 254	170 928	793 379
138	371 855	172 600	801 138
140	375 439	174 264	808 860
142	379 007	175 919	816 545
144	382 557	177 567	824 194
146	386 091	179 208	831 808
148	389 609	180 841	839 387
150	393 111	182 466	846 932
152	396 598	184 084	854 444
154	400 069	185 696	861 923
156	403 526	187 300	869 369
158	406 967	188 897	876 784
160	410 394	190 488	884 168
162	413 807	192 072	891 520
164	417 206	193 650	898 843
166	420 591	195 221	906 136
168	423 962	196 786	913 400
170	427 321	198 345	920 634
173	432 333	200 671	931 434
176	437 317	202 985	942 171
179	442 273	205 285	952 847
182	447 200	207 572	963 464
185	452 101	209 847	974 023
188	456 976	212 109	984 524
191	461 824	214 360	994 970
194	466 648	216 599	100 536
197	471 446	218 826	101 570

200 - 375

A	Col. 1	Col. 2	Col. 3
200	476 220	221 042	102 599
203	480 971	223 247	103 622
206	485 698	225 441	104 640
209	490 402	227 624	105 654
212	495 084	229 797	106 663
215	499 743	231 960	107 666
218	504 381	234 113	108 666
221	508 998	236 256	109 660
224	513 594	238 389	110 650
227	518 169	240 513	111 636
230	522 725	242 627	112 618
233	527 260	244 733	113 595
236	531 777	246 829	114 568
240	537 769	249 610	115 859
244	543 727	252 376	117 142
248	549 654	255 127	118 419
252	555 548	257 863	119 689
256	561 411	260 584	120 952
260	567 244	263 291	122 209
264	573 047	265 985	123 459
268	578 821	268 665	124 703
272	584 566	271 332	125 941
276	590 284	273 985	127 173
280	595 973	276 626	128 398
284	601 636	279 254	129 618
288	607 272	281 870	130 833
292	612 881	284 474	132 041
296	618 466	287 066	133 244
300	624 025	289 647	134 442
304	629 560	292 216	135 635
308	635 070	294 773	136 822
312	640 557	297 320	138 004
316	646 020	299 856	139 181
320	651 460	302 381	140 353
325	658 229	305 523	141 811
330	664 963	308 648	143 262
335	671 663	311 758	144 705
340	678 329	314 853	146 142
345	684 963	317 932	147 571
350	691 565	320 996	148 993
355	698 136	324 046	150 409
360	704 676	327 082	151 818
365	711 186	330 103	153 220
370	717 666	333 111	154 616
375	724 117	336 105	156 006

380 - 703

A	Col. 1	Col. 2	Col. 3
380	730 539	339 086	157 390
386	738 209	342 646	159 042
392	745 839	346 188	160 686
398	753 431	349 712	162 322
404	760 984	353 217	163 949
410	768 500	356 706	165 568
416	775 979	360 178	167 180
422	783 423	363 633	168 783
428	790 831	367 071	170 379
434	798 205	370 494	171 968
440	805 545	373 901	173 549
446	812 851	377 292	175 124
452	820 125	380 668	176 691
458	827 367	384 030	178 251
465	835 776	387 933	180 062
472	844 143	391 816	181 865
479	852 468	395 681	183 659
486	860 753	399 526	185 444
493	868 999	403 354	187 220
500	877 205	407 163	188 988
507	885 374	410 954	190 748
514	893 504	414 728	192 500
521	901 598	418 485	194 243
528	909 656	422 225	195 979
535	917 678	425 949	197 708
542	925 666	429 656	199 429
550	934 752	433 873	201 386
558	943 794	438 071	203 334
566	952 794	442 248	205 273
574	961 751	446 405	207 203
582	970 666	450 543	209 124
590	979 541	454 663	211 036
598	988 376	458 763	212 939
606	997 171	462 846	214 834
614	100 593	466 910	216 721
622	101 465	470 957	218 599
631	102 441	475 489	220 703
640	103 413	480 000	222 796
649	104 380	484 490	224 880
658	105 343	488 958	226 954
667	106 301	493 407	229 019
676	107 255	497 835	231 075
685	108 205	502 244	233 121
694	109 151	506 634	235 159
703	110 093	511 005	237 187

712 - 1000

A	Col. 1	Col. 2	Col. 3
712	111 030	515 357	239 207
721	111 964	519 690	241 219
730	112 894	524 006	243 222
740	113 922	528 781	245 438
750	114 946	533 534	247 645
760	115 966	538 266	249 841
770	116 981	542 977	252 028
780	117 992	547 668	254 205
790	118 998	552 339	256 373
800	120 000	556 991	258 532
811	121 097	562 085	260 897
822	122 190	567 156	263 250
833	123 278	572 204	265 594
844	124 361	577 231	267 927
855	125 439	582 235	270 250
866	126 512	587 219	272 563
877	127 581	592 181	274 866
888	128 646	597 122	277 160
900	129 802	602 490	279 651
912	130 954	607 833	282 131
924	132 100	613 154	284 601
936	133 241	618 451	287 059
948	134 378	623 725	289 508
961	135 603	629 415	292 148
974	136 823	635 078	294 777
987	138 038	640 717	297 394
1000	139 248	646 330	300 000

Table intervals are so arranged that the root calculated by the method below differs from the true root by less than 5 in the sixth significant figure. Rounding of the fifth figure of true and calculated root may lead to occasional errors of "1" in the fifth figure of the calculated rounded five-figure root as compared with that of the true rounded five-figure root.

See reverse side for Extension to 10-Figure Root

MARCHANT METHOD

OUTLINE: To find the cube root of any number N, select from Col. A the number nearest to the three left significant digits of N. Add twice this number to N, lining up first significant digits. Divide this sum by the factor in Col. 1, 2, or 3 according to whether there are 1, 2, or 3 digits in the left group of digits of N after it has been pointed off in groups of three digits each way from decimal.

The root is shown as the quotient of the preceding division rounded to five figures, there being one digit of root, each way from decimal, for each group of significant digits in the number whose root is desired.

See Reverse Side for Examples and Detailed Instructions.

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(SEE OVER)

1 026 6436 8

MARCHANT METHOD

This method of extracting cube root by use of a table of divisors (or of multipliers, if reciprocals of divisor factors are used) is a concise, practical method of extracting cube roots accurate to any desired number of significant figures. It is based upon a well-known principle according to which an approximate root may be converted to one that is of a greater degree of accuracy. (See Barlow's Tables, 1935 Edition, Page XI, Introduction, by L. J. Comrie, M.A., Ph.D.)

The table on the reverse side was first published in April 1940 and is so far as known the first and only published table of its type. The method and table represent original work of the staff of Marchant Calculating Machine Company.

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MARCHANT OPERATIONS—FIVE-FIGURE CUBE ROOT

EXAMPLE: Find the cube root of 65324.74.

Use any model of "Silent-Speed" Marchant Calculator.

(1) With carriage in 6th position, set up at extreme left of Keyboard Dial the significant figures of the number whose root is desired (6532474), and add.

(2) Set up at extreme left of Keyboard Dial the number in Col. A (649) which is nearest to the left-hand three digits of the number whose root is desired, and multiply by 2.

The Middle Dial shows a number that is approximately three times the number whose root is desired.

(3) Point off into groups of three digits the number whose root is desired, beginning from decimal point (65'324.740'), from which it is noted that the left-hand group (65) contains two* figures. Set up at extreme left of Keyboard Dial the number that appears in Col. 2* at right of 649 (484490) and divide.

Upper Dial shows 402742 which rounds to 40274 as five significant figures of the root. As there is one figure of the root each way from decimal for each group of three figures each way from decimal in the number whose root is desired, the cube root is 40.274 which is correct to five significant figures; i.e., the error in the 6th figure is less than 5.

Or, arithmetically

```
          6532474
          649
          649
484490/19512474
        402742554
```

which rounds to five figures as 40.274.

CONVERTING FIVE-FIGURE CUBE ROOT TO ONE OF TEN FIGURES

Rule: Divide the number whose ten-figure root is desired by the square of its five-figure cube root, noting the quotient to ten figures. Add twice the five-figure cube root to the left-hand five figures of this quotient. One-third of this sum will be the desired cube root accurate to ten significant figures; i.e., the error of the 11th digit will not exceed 5.

EXAMPLE: Extend the above five-figure cube root to one of ten figures.

The square of the five-figure root is $40.274^2 = 1621.995076$

```
65324.74/1621.995076=40.27431462
                     40.274
                     40.274
                   3/120.82231462
                     40.27410487
```

***If the left-hand group contained one digit, such as if the number whose root is desired were 6'532.48, divisor would have been taken from Col. 1, as 104380. If it contained three digits, such as if the number were 653'247.400' divisor would have been taken from Col. 3, as 224880.**

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FIFTH ROOT

REMARKS: The table and its explanation submitted herein represent the work of Mr. Charles S. Larkey who has followed the principles exemplified in Marchant Tables 56 and 68 relating to square and cube root, respectively, to extend the method to the extraction of fifth roots. Mr. Larkey's manuscript is reproduced herewith without checking of the tabular amounts by us, but knowing as we do of his skill as a mathematician, we have complete confidence that his table will produce the desired results within the limit of accuracy set forth in the explanation.

OUTLINE

The method described below enables one to obtain directly, by means of division, the fifth root of any number, correct to five significant figures.

TO OBTAIN FIFTH ROOT CORRECT TO 5 SIGNIFICANT FIGURES

Selection of Proper Number in Column of Table:

Column N contains a sequence of numbers from 10,000 to 100,000. In finding the fifth root of a number, the number in Column N nearest the given number is selected, it being understood that the decimal point may be placed in any desired position in the tabular value in order to conform to the number whose root is sought.

For example: The fifth root of 302.456 is desired. In Column N, 30,129 is the nearest number since it may be written 301.29 by moving the decimal point, and this value is nearer 302.456 than 305.10, the next greater value in the table.

Finding Column for Divisor:

First separate the number whose fifth root is to be found into periods of five digits each, beginning at the decimal point. In general, the number of digits in the first period is also the number of the Column for the divisor; but in case of a number entirely decimal, subtract the number of zeros immediately following the decimal point from five or a multiple of five such that the remainder is equal to five or less; this remainder is the Column number of the divisor.

Example: Thus if the given number is 12,453,575, dividing it into periods it becomes 124'53575, and Column 3 is to be used; whereas, if the number is 0.012453575, Column 4 is to be used.

Determining the Fifth Root to 5 Significant Figures:

Set the number whose root is to be found in the Middle Dials of the calculating machine with the carriage to the right for division. Set in Keyboard the nearest number from Column N as found above with the corresponding digits opposite those in the Middle Dial and multiply by four; then divide by the number appearing in Column 1, 2, 3, 4 or 5, as the case may be, from the selection as explained above. The quotient will be the fifth root of the number, correct to five significant figures.

(over)

EXTENSION OF CALCULATION TO NINE SIGNIFICANT FIGURES

Divide the number whose nine-figure root is desired by the fourth power of the five-figure root already obtained, carrying the quotient to ten places. Add four times the five-figure root to this quotient and divide by five. The result will be the fifth root correct to nine significant figures.

FORMULAS FOR CALCULATIONS

The operations described above may be expressed in formulas as follows:

$$R_5 = \sqrt[5]{X} = \frac{X + 4N}{D}$$

correct to five significant figures, and

$$R_9 = \sqrt[5]{X} = \frac{1}{5}\left(\frac{X}{R_5^4} + 4R_5\right)$$

correct to nine significant figures, where X is the number whose root is sought, N is the nearest basic tabular number, and D is the indicated divisor from the table.

EXAMPLES

Example 1:

Find the fifth root of 12,453,575.

Writing this number in periods of five digits, it becomes 124'53575. The nearest corresponding value under N is 12,497, and since there are three digits in the first period, the divisor 237.911 from Column 3 is to be used. After adding four times 12497000 to 12453575, the Middle Dial reads 62441575. Dividing by 237.911, the root is found to be 26.246, the number of periods to the left of the decimal in the number determining the number of digits to the left of the decimal in the root.

To determine the root to nine places, square 26.246 by multiplying it by itself, and then square the result, thus obtaining the fourth power of the five-figure root. This gives 474,517.7888 which is contained in 12,453,575, 26.24469576 times, carrying to the full capacity of a 10-column machine. Set this quotient in the Middle Dial and add four times 26.24600000, causing the Middle Dial to read, 131.22869576. Divide by five, giving 26.2457392 as the value of the root correct to nine figures.

Example 2:

Find the fifth root of 0.000125.

The nearest value of N is 12,497. Since there are three zeros after the decimal point, subtracting from five this leaves two for the number of the column for the divisor which is found to be 37.7064. Proceeding as in Example 1, the root is found to be 0.16572.

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
10 000	5.00000	31.5479	199.054	1255.94	7924.47
10 142	5.05672	31.9058	201.312	1270.19	8014.36
10 285	5.11368	32.2651	203.579	1284.50	8104.63
10 430	5.17127	32.6285	205.872	1298.96	8195.91
10 577	5.22950	32.9959	208.190	1313.59	8288.20
10 726	5.28835	33.3672	210.533	1328.37	8381.47
10 877	5.34783	33.7425	212.901	1343.31	8475.73
11 030	5.40792	34.1217	215.293	1358.41	8570.98
11 185	5.46863	34.5047	217.710	1373.66	8667.20
11 342	5.52996	34.8917	220.152	1389.06	8764.39
11 501	5.59189	35.2824	222.617	1404.62	8862.55
11 662	5.65442	35.6770	225.107	1420.33	8961.66
11 825	5.71756	36.0754	227.620	1436.19	9061.72
11 990	5.78130	36.4775	230.158	1452.20	9162.74
12 157	5.84563	36.8834	232.719	1468.36	9264.69
12 326	5.91055	37.2930	235.303	1484.66	9367.59
12 497	5.97605	37.7064	237.911	1501.12	9471.41
12 670	6.04215	38.1234	240.542	1517.72	9576.16
12 845	6.10882	38.5440	243.196	1534.47	9681.83
13 022	6.17607	38.9684	245.874	1551.36	9788.41
13 201	6.24389	39.3963	248.574	1568.39	9895.90
13 383	6.31267	39.8302	251.312	1585.67	10004.9
13 567	6.38200	40.2677	254.072	1603.09	10114.8
13 753	6.45190	40.7088	256.855	1620.65	10225.6
13 941	6.52237	41.1533	259.660	1638.34	10337.3
14 132	6.59376	41.6038	262.502	1656.28	10450.4
14 325	6.66570	42.0577	265.366	1674.35	10564.4
14 520	6.73819	42.5151	268.252	1692.56	10679.3
14 718	6.81160	42.9783	271.175	1711.00	10795.7
14 918	6.88555	43.4449	274.119	1729.57	10912.9
15.121	6.96040	43.9172	277.099	1748.37	11031.5

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
15 326	7.03579	44.3928	280.100	1767.31	11151.0
15 534	7.11208	44.8742	283.137	1786.47	11271.9
15 744	7.18889	45.3589	286.195	1805.77	11393.6
15 957	7.26659	45.8491	289.288	1825.29	11516.8
16 172	7.34482	46.3427	292.402	1844.93	11640.8
16 390	7.42392	46.8418	295.551	1864.80	11766.1
16 610	7.50353	47.3441	298.721	1884.80	11892.3
16 833	7.58402	47.8519	301.925	1905.02	12019.9
17 059	7.66537	48.3652	305.164	1925.45	12148.8
17 289	7.74722	48.8816	308.422	1946.01	12278.5
17 518	7.82993	49.4035	311.715	1966.79	12409.6
17 752	7.91349	49.9307	315.042	1987.78	12542.0
17 989	7.99789	50.4633	318.402	2008.98	12675.8
18 229	8.08314	51.0012	321.796	2030.39	12810.9
18 471	8.16888	51.5421	325.209	2051.93	12946.8
18 716	8.25544	52.0883	328.655	2073.67	13084.0
18 964	8.34284	52.6398	332.134	2095.63	13222.5
19 215	8.43106	53.1964	335.647	2117.79	13362.3
19 469	8.52010	53.7582	339.192	2140.15	13503.5
19 726	8.60996	54.3252	342.769	2162.72	13645.9
19 986	8.70063	54.8973	346.378	2185.50	13789.6
20 249	8.79211	55.4744	350.020	2208.48	13934.5
20 515	8.88438	56.0567	353.694	2231.66	14080.8
20 785	8.97780	56.6461	357.413	2255.12	14228.9
21 058	9.07201	57.2405	361.163	2278.79	14378.2
21 334	9.16701	57.8399	364.944	2302.65	14528.7
21 613	9.26280	58.4443	368.759	2326.71	14680.5
21 895	9.35935	59.0535	372.603	2350.96	14833.6
22 181	9.45703	59.6698	376.491	2375.50	14988.4
22 470	9.55548	60.2910	380.411	2400.23	15144.4
22 762	9.65469	60.9170	384.360	2425.15	15301.7

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
23 058	9.75500	61.5499	388.354	2450.35	15460.6
23 357	9.85607	62.1876	392.377	2475.73	15620.8
23 660	9.95822	62.8321	396.444	2501.39	15782.7
23 966	10.0611	63.4814	400.541	2527.24	15945.8
24 276	10.1651	64.1375	404.680	2553.36	16110.6
24 590	10.2701	64.8003	408.862	2579.75	16277.1
24 907	10.3759	65.4677	413.073	2606.32	16444.7
25 228	10.4828	66.1418	417.327	2633.15	16614.1
25 553	10.5907	66.8226	421.622	2660.26	16785.1
25 881	10.6993	67.5080	425.946	2687.54	16957.2
26 213	10.8089	68.1998	430.312	2715.08	17131.0
26 549	10.9196	68.8983	434.719	2742.89	17306.5
26 889	11.0314	69.6033	439.167	2770.96	17483.6
27 233	11.1441	70.3147	443.656	2799.28	17662.3
27 581	11.2579	71.0326	448.186	2827.86	17842.6
27 933	11.3727	71.7570	452.756	2856.70	18024.5
28 289	11.4885	72.4877	457.366	2885.79	18208.1
28 649	11.6053	73.2247	462.017	2915.13	18393.2
29 013	11.7231	73.9680	466.707	2944.72	18579.9
29 381	11.8420	74.7177	471.437	2974.56	18768.2
29 753	11.9617	75.4735	476.206	3004.65	18958.1
30 129	12.0825	76.2356	481.014	3034.99	19149.5
30 510	12.2046	77.0059	485.874	3065.66	19343.0
30 895	12.3277	77.7823	490.773	3096.57	19538.0
31 284	12.4517	78.5648	495.710	3127.72	19734.6
31 678	12.5770	79.3553	500.698	3159.19	19933.2
32 076	12.7032	80.1520	505.725	3190.91	20133.3
32 479	12.8308	80.9566	510.801	3222.94	20335.4
32 886	12.9592	81.7671	515.916	3255.21	20539.0
33 298	13.0889	82.5856	521.080	3287.79	20744.6
33 714	13.2196	83.4100	526.282	3320.61	20951.6

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
34 135	13.3515	82.2422	531.533	3353.74	21160.7
34 561	13.4846	85.0823	536.833	3387.19	21371.7
34 992	13.6190	85.9300	542.182	3420.94	21584.6
35 427	13.7543	86.7836	547.567	3454.92	21799.0
35 867	13.8908	87.6448	553.001	3489.20	22015.4
36 312	14.0285	88.5136	558.483	3523.79	22233.1
36 762	14.1674	89.3901	564.013	3558.68	22453.8
37 217	14.3075	90.2741	569.591	3593.88	22675.8
37 677	14.4488	91.1656	575.216	3629.37	22899.8
38 142	14.5913	92.0646	580.889	3665.16	23125.6
38 612	14.7349	92.9711	586.608	3701.24	23353.3
39 088	14.8801	93.8868	592.386	3737.70	23583.3
39 569	15.0264	94.8100	598.211	3774.45	23815.2
40 055	15.1738	95.7404	604.081	3811.50	24048.9
40 547	15.3228	96.6801	640.010	3848.90	24284.9
41 044	15.4728	97.6270	615.985	3886.60	24522.8
41 546	15.6240	98.5810	622.004	3924.58	24762.4
42 054	15.7767	99.5442	628.081	3962.92	25004.4
42 567	15.9305	100.514	634.203	4001.55	25248.1
43 086	16.0857	101.494	640.382	4040.54	25494.1
43 611	16.2423	102.482	646.617	4079.87	25742.3
44 141	16.4000	103.477	652.896	4119.49	25992.2
44 677	16.5591	104.481	659.230	4159.46	26244.4
45 219	16.7196	105.494	665.621	4199.78	26498.8
45 767	16.8815	106.515	672.066	4240.45	26755.4
46 321	17.0448	107.546	678.566	4281.46	27014.2
46 881	17.2095	108.584	685.121	4322.82	27275.2
47 447	17.3755	109.632	691.731	4364.53	27538.3
48 019	17.5429	110.688	698.394	4406.57	27803.6
48 597	17·7116	111.753	705.111	4448.95	28071.0
49 182	17.8820	112.828	711.893	4491.74	28341.0

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
49 773	18.0537	113.911	718.729	4534.87	28613.1
50 370	18.2267	115.003	725.617	4578.34	28887.3
50 974	18.4013	116.104	732.570	4622.20	29164.1
51 584	18.5773	117.215	739.575	4666.40	29443.0
52 201	18.7548	118.335	746.643	4711.00	29724.4
52 825	18.9340	119.465	753.775	4756.00	30008.3
53 455	19.1144	120.604	760.958	4801.32	30294.3
54 092	19.2964	121.752	768.204	4847.04	30582.7
54 736	19.4800	122.910	775.512	4893.15	30873.7
55 387	19.6651	124.078	782.882	4939.65	31167.1
56 045	19.8518	125.256	790.314	4986.54	31463.0
56 710	20.0400	126.444	797.807	5033.82	31761.3
57 382	20.2298	127.641	805.361	5081.48	32062.0
58 061	20.4210	128.848	812.976	5129.53	32365.1
58 747	20.6138	130.064	820.651	5177.96	32670.7
59 440	20.8081	131.290	828.387	5226.77	32978.7
60 141	21.0042	132.528	836.193	5276.02	33289.4
60 849	21.2018	133.774	844.059	5325.65	33602.6
61 565	21.4011	135.032	851.995	5375.73	33918.5
62 288	21.6020	136.299	859.990	5426.17	34236.8
63 019	21.8046	137.577	868.055	5477.06	34557.9
63 758	22.0089	138.867	876.189	5528.38	34881.7
64 504	22.2146	140.165	884.381	5580.07	35207.8
65 258	22.4221	141.474	892.641	5632.19	35536.7
66 020	22.6313	142.794	900.970	5684.74	35868.3
66 790	22.8423	144.125	909.367	5737.72	36202.5
67 568	23.0549	145.466	917.831	5791.12	36539.5
68 354	23.2692	146.819	926.363	5844.95	36879.2
69 149	23.4854	148.183	934.972	5899.28	37221.9
69 952	23.7034	149.558	943.648	5954.02	37567.3
70 763	23.9230	150.944	952.390	6009.18	37915.3

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
71 583	24.1445	152.341	961.209	6064.82	38266.4
72 411	24.3676	153.749	970.094	6120.88	38620.1
73 248	24.5927	155.170	979.054	6177.41	38976.8
74 094	24.8197	156.602	988.090	6234.42	39336.6
74 948	25.0483	158.044	997.190	6291.85	39698.9
75 811	25.2788	159.498	1006.37	6349.74	40064.1
76 683	25.5111	160.964	1015.62	6408.10	40432.4
77 564	25.7453	162.442	1024.94	6466.93	40803.6
78 454	25.9814	163.931	1034.34	6526.23	41177.7
79 353	26.2193	165.432	1043.81	6585.98	41554.8
80 262	26.4593	166.947	1053.36	6646.27	41935.1
81 180	26.7011	168.473	1062.99	6707.01	42318.4
82 107	26.9448	170.010	1072.69	6768.22	42704.6
83 044	27.1905	171.560	1082.47	6829.94	43094.0
83 991	27.4382	173.124	1092.34	6892.17	43486.7
84 947	27.6878	174.698	1102.27	6954.86	43882.2
85 913	27.9394	176.286	1112.29	7018.06	44281.0
86 889	28.1930	177.886	1122.38	7081.77	44682.9
87 875	28.4487	179.499	1132.56	7145.99	45088.1
88 871	28.7064	181.125	1142.82	7210.71	45496.5
89 877	28.9660	182.763	1153.16	7275.94	45908.1
90 893	29.2277	184.414	1163.57	7341.66	46322.7
91 920	29.4916	186.079	1174.08	7407.95	46741.0
92 957	29.7575	187.757	1184.67	7474.73	47162.4
94 005	30.0255	189.448	1195.34	7542.08	47587.3
95 063	30.2956	191.152	1206.09	7609.91	48015.3
96 132	30.5678	192.870	1216.93	7678.29	48446.7
97 212	30.8422	194.601	1227.85	7747.22	48881.7
98 303	31.1188	196.347	1238.86	7816.70	49320.0
99 405	31.3976	198.106	1249.96	7886.72	49761.9
100 000	31.5479	199.054	1255.94	7924.47	50000.0

RAPID METHOD FOR

5-PLACE, 6-PLACE and 8-PLACE

SQUARE ROOTS

Your Marchant Representative will supply special Tables for obtaining square roots by means of a simple Marchant operation.

1 026 64368

FIFTH ROOTS WITHOUT LOGARITHMS

REMARKS: *The Method herein is an adaptation of that proposed by L. H. Comrie, M.A., Ph.D. Managing Director, Scientific Computing Service, Ltd., London, and lately Deputy Supt H M Nautical Almanac Office. The method was disclosed to American audiences as a part of a series of lectures given by Dr. Comrie several years ago. It is not copyrighted in the United States.*

EXAMPLE: Find the fifth root of 442.87 to four significant figures at right of decimal.

PRELIMINARY: If Barlow's Tables, showing fifth powers of numbers from 1 to 99 is available, the method may be simplified because such a table obtains a first approximation of the root to two significant figures. It is assumed herein that Barlow's Tables is not available, so there are given below the fifth powers of certain numbers from 1 to 10, which is the starting point of the calculation.

1	1	4	1024	7	16807
1.5	7.6	4.5	1845	7.5	22730
2	32	5	3125	8	32768
2.5	98	5.5	5033	8.5	44371
3	243	6	7766	9	59049
3.5	525	6.5	11603	9.5	77378
				10	100000

OUTLINE: The first step is to obtain by inverse straight line interpolation from the above table a first approximation of the root. By applying Marchant Method 439A2 the first approximation of the 5th root of 443 is 3.35. (In applying MM-439A2, the Upper Dial ratio of .709, etc., is divided by 2 because the table arguments are separated by .5)

This first approximation of the root (3.35) is then applied in a formula which produces the second approximation, as follows:

$$X_2 \text{ equals } \frac{1}{5}\left(\frac{N}{X_1^4} + 4X_1\right)$$

In this formula, N equals the number whose root is desired,
X_1 equals the first approximation of the root, and
X_2 equals the second approximation of the root.

The second approximation is then used just as if it were a first approximation, applying the same formula, thus producing a third approximation, and so on to any accuracy desired.

OPERATIONS: Decimals: Upper Dial 7, Middle Dial 14, Keyboard Dial 7.
Use 10 column Model

(1) Set up in Keyboard Dial X_1 (3.35) and multiply by X_1 (3.35).

(2) Clear Upper and Keyboard Dials and transfer X_1^2 (11.2225) from Middle Dial to Keyboard Dial. Clear Middle Dial and multiply by 11.2225

X_1^4 (125.9445) appears in Middle Dial. Copy to work sheet to two more significant figure than there are in N, i.e., 125.9445

(3) Change Keyboard Dial to read N (442.87), shift to 8th position, clear Middle Dial and add.

(4) Set up in Keyboard Dial X^4 (125.9445) as taken from Step 2, clear Upper Dial* and divide.

$\frac{N}{X_1^4}$ (3.5163901) appears in Middle Dial.

(5) Clear Keyboard and Middle Dials, transfer amount in Upper Dial (3.5163901) to Keyboard Dial, shift to 8th position and add.

(6) Clear Upper Dial, set up in Keyboard Dial X_1 (3.35) and multiply by 4.

$\frac{N}{X_1^4}$ plus $4X_1$ (16.9163901) appears in Middle Dial.

(7) Change Keyboard Dial to read "5". Divide.

X_2, Second Approximation of the root (3.383278), appears in Middle Dial.

(8) Using X_2 as if it were X_1, at a value of 3.383, X_3 is found to be 3.3826+.

(9) It is probable that the root is very nearly correct at 3.3826, so in obtaining data for the next approximation X^4 is computed as in Step 2.

X_3^4 is 130.9189693 which is copied to work sheet as possibly it may be needed later.

(10) Change Keyboard Dial to read same as Middle Dial (130.918963). Clear Upper and Middle Dials and multiply by X_3 (3.3826).

Middle Dial shows X_3^5 (442.847).

(11) To determine whether X_3 is a close enough root, shift to 4th position and multiply by .0005, which approximates the effect of 5 successive multiplications of 1 x 1 in that position.

Middle Dial shows that an increase of .001 in the root causes its 5th power to become approximately 442.91, which is greater than N (442.87), from which it is concluded that 3.3826 is the root accurate to four places at right of decimal because 442.87 is nearer to 442.847 than it is to 442.912

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TO RAISE A DECIMAL FRACTION TO AN ODD POWER

REMARKS: The interesting method submitted herein was developed by Miss Emily T. Hannan. It is made available to other computers by courtesy of the company by which she is employed, which has requested that its name be withheld. Though the method is simple, it exemplifies a unique Marchant operation; viz., the entry of a multiplier, a portion of which is negative, and a portion positive. Marchant is the only calculator that permits automatic multiplication of this type.

EXAMPLE: To find $.9754^{.285714}$

OPERATIONS: Decimals; Upper Dial 7, Middle Dial 13, Keyboard Dial 6. Use any model of Marchant.

(1) Set up in Keyboard Dial the exponent (.285714) and, with carriage in 8th position, reverse multiply by the absolute value of the characteristic (-1) and then multiply by the decimal portion (mantissa) of the logarithm of .9754 (.9891828).

Mantissa of logarithm of $.9754^{.285714}$ (.9969094) appears in Middle Dial at right of decimal.

Characteristic of logarithm is indicated by the figure at left of Middle Dial decimal. In this case, it is "9", the characteristic is thus $\bar{1}$., and the entire logarithm is thus $\bar{1}.9969094$.

NOTE: If figure at left of Middle Dial decimal is 8, 7, etc., the characteristic is correspondingly $\bar{2}$, $\bar{3}$, etc.

(2) A table of logarithms shows that the amount which has logarithm of $\bar{1}.9969094$ is .99291.

OTHER SAMPLE CALCULATIONS

Function	Log of Whole No.	Log of Function	Function
$.09754^{.285714}$	$\bar{2}.9891828$	$\bar{1}.7111954$	.514275
$.00009754^{.285714}$	$\bar{5}.9891828$	$\bar{2}.8540534$	.071458

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EQUATIONS CONTAINING $\sqrt{N}$ IN NUMERATOR OR DENOMINATOR

(Particularly Useful When There Is Much Work of the Same Type)

REMARKS: *Because Marchant Tables 56 and 57 show values of $2\sqrt{A}$ and $1/(2\sqrt{A})$, respectively, when A is an amount that is close to N, it is not necessary separately to compute $\sqrt{N}$ and record it for inclusion in the numerator or denominator of equations. A frequent use of this principle is in solving formulas in Statistical Method.*

OUTLINE: (1) If $\sqrt{N}$ appears in NUMERATOR of an equation, such as $X = BC\sqrt{N}$, it may be readily solved by the following transformation:

$$X = \frac{BC\ (N + A_n)}{2\sqrt{A_n}} = BC\ (N + A_n)\ (1/(2\sqrt{A_n}))$$

in which A_n is an amount taken* from column A of Marchant Table 57 which is nearest to N, and $1/(2\sqrt{A_n})$ is the corresponding factor from columns 1 or 2 of Table 57, column 1 being selected when the number of digits of N at left of decimal is odd, and column 2 being selected when it is even.

(2) If $\sqrt{N}$ appears in DENOMINATOR of any equation; such as, $X = BC/\sqrt{N}$, it may be readily solved by the following transformation:

$$X = \frac{BC\ (2\sqrt{A_n})}{(N + A_n)}$$

in which A_n is an amount taken* from column A of Marchant Table 56 which is nearest to N, and $2\sqrt{A_n}$ is the corresponding factor from columns 1 or 2 of Table 56, column 1 being selected when the number of digits of N at left of decimal is odd, and column 2 being selected when it is even.

It will be observed that the factors also may be used in a reverse manner from the above-suggested transformations; thus

(3) If $\sqrt{N}$ appears in NUMERATOR, $BC(N + A_n)$ may be divided by $2\sqrt{A_n}$ taken from Table 56, and

(4) If $\sqrt{N}$ appears in DENOMINATOR, BC may be divided by $(N + A_n)(1/(2\sqrt{A_n}))$ in which $1/(2\sqrt{A_n})$ is taken from Table 57.

(*) *See "Pointing Off Decimals" on next page.*

NOTE A: In any case $\sqrt{N}$ is correct to within 5 in its 6th significant figure, so accuracy of the final result must be considered as subject to this limitation.

NOTE B: Which of the various transformations it is best to use in any instance depends upon the nature of the problem, principally the number of factors, their length, etc. If the problem can be adapted to dividing by only a single factor, as in examples 2 and 3, the work is facilitated considerably. In such cases the NUMERATOR is computed first. Without clearing Middle Dial, the single factor divisor is set up and Division Key depressed.

(over)

If the DENOMINATOR has more than one factor, it is advisable to compute it first, then after computation of NUMERATOR, the DENOMINATOR may be set up from the work sheet without clearing the NUMERATOR from the Middle Dial.

POINTING OFF DECIMALS

Inasmuch as it is desirable to have all multiplication and division factors entered into the Marchant with respect to pre-set decimals, it is important to point off the decimal in A_n, $2\sqrt{A_n}$ from Table 56, and $1/(2\sqrt{A_n})$ from Table 57 as the case may be. This is easily done by those having familiarity with usual computing methods. Examples are as follows:

N	A_n from Col. A of Table 56 or Table 57	$2\sqrt{A_n}$ from Table 56	$1/(2\sqrt{A_n})$ from Table 57
56325.67	56500.	475.3946	.002103515
5632.57	5650.	150.3330	.006651899
1632.57	1630.	80.74652	.01238443
.0563257	.0565	.4753946	2.103515
.0056326	.00565	.1503330	6.651899
.0016326	.00163	.08074652	12.38443

EXAMPLE

The formula for r_{xy} of MM-444A after computing the denominator prior to extracting square root, is:

$$r_{xy} = \frac{(5 \times 35751) - (433 \times 405)}{\sqrt{12054160}}$$

Use transformation No. 2 (see page 1).

OPERATIONS: Upper Dial 3, Middle Dial 5, Keyboard Dial 2. Use any 10 column Marchant.

(1) Set up in Keyboard Dial 35751 and multiply by 5.

(2) Clear Upper and Keyboard Dials, set up 433 in Keyboard Dial, and with Manual Counter Control toward the operator, reverse multiply by 405.

(3) Clear Upper and Keyboard Dials, and transfer 3390 from Middle Dial to Keyboard Dial. Clear Middle Dial and with Manual Counter Control away from the operator, multiply by $2\sqrt{12000000}$ (6928.203), taking it from Col. 2 of Table 56 at right of "120".

(4) Clear Upper and Keyboard Dials, set up in Keyboard Dial 12054160 and increase it by changing keys at left so it shows the sum of 12000000 and 12054160 (24054160), and divide.

r_{xy} (.976) appears in Upper Dial

NOTE: The remainder fraction 97/240 that appears in Middle and Keyboard Dials, being less than one-half, the right-hand digit of the above is not rounded up.

Submitted by: Lawrence Boyd, *Newark. N. J.*

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SOLUTION OF RIGHT TRIANGLES

Location of Drilling Centers

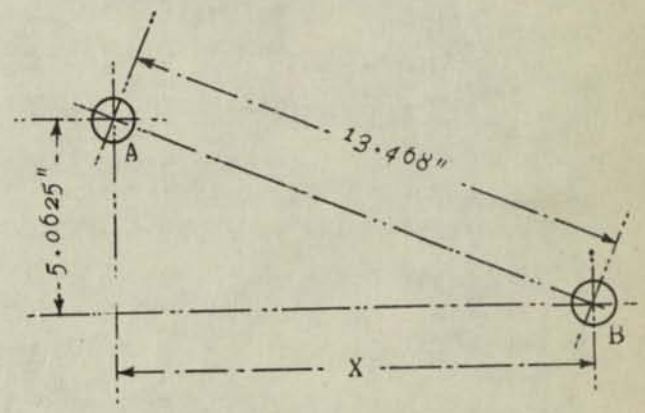
REMARKS: In machine shop layout work, particularly in connection with layout of drilling jigs, it is often necessary to locate centers of holes when they are not indicated on the drawing in the manner necessary for use with layout tools or boring machines. Obtaining the proper dimensions for such cases is easily done with the aid of a Marchant Calculator.

NOTE: In all cases as described below, the words "distance between holes" signify distance between their centers.

CASE I: Linear Location

Given all dimensions shown, to find X:

In sketch (not to scale), the location of hole A is known by dimensions that permit its center to be easily located on the work. Hole B, on the other hand, is located only by its vertical distance from A and its actual distance (on the slope) from A. For proper layout of the work with the facilities at hand, it is necessary to know the horizontal distance of hole B from hole A.



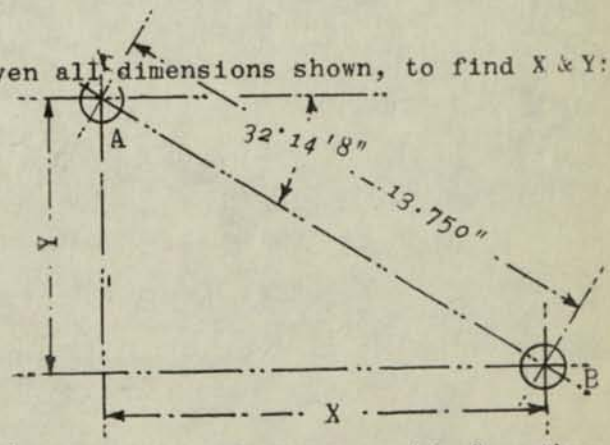
What is the horizontal distance of hole B from hole A to within ½ of 1/1000 inch?

Obviously, it is necessary to find the distance "X". This is done by solving the right triangle when given its hypotenuse (13.468) and the length of one side; thus, $X = \sqrt{13.468^2 - 5.0625^2}$

CASE II: Angular Location

Given all dimensions shown, to find X & Y:

This case is similar to Case I, except that hole B is located only by its actual distance (on the slope) from A, and by the angle that the center-line connecting both holes bears from the horizontal.



What are the horizontal and vertical distances of hole B from hole A to within ½ of 1/1000 inch?

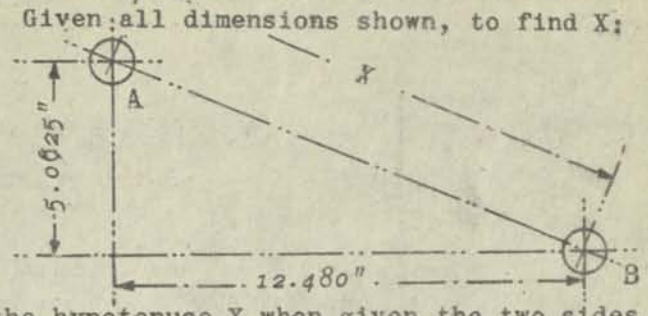
Obviously, it is necessary to find distances "X" and "Y", respectively; thus,

$$X = \sin(90 - 32°14'8") \times 13.750; \quad Y = \sin 32°14'8" \times 13.750$$

(over)

CASE III: Distance Between Holes

Given all dimensions shown, to find X:



In this case, the holes A and B are located with reference to each other by their vertical and horizontal distances.

What is the distance between the two holes to ½ of 1/1000 inch?

Obviously, it is necessary to find the hypotenuse X when given the two sides of the right triangle; thus, $X = \sqrt{12.480^2 + 5.0625^2}$

It is assumed that extracting square root by Marchant Method as outlined on Marchant Table No. 56 and obtaining angular functions from tables of Marchant Method MM-99 are understood.

MARCHANT METHOD FOR CASE I

Use any Marchant 8-column M or D model with decimals as follows: Upper Dial 6, Middle Dial 11, and Keyboard Dial 5. In all cases, set the decimals as far toward the left as will accomodate the factors in order that a 5 figure square root may be obtained without the necessity of re-setting any part of the problem.

(1) Set up 13.468 in Keyboard Dial and multiply by 13.468.

(2) Clear Upper and Keyboard Dials only, set up 5.0625 in Keyboard Dial, move Manual Counter Control toward the operator, and reverse multiply by 5.0625. Move Manual Counter Control away from operator.

(3) Clear Upper and Keyboard Dials only, shift to 7th position, set up in Keyboard Dial nearest number to 155.758, etc. that appears in Col. A of Table 56 (156.) and add.

Middle Dial now reads 311.758, etc., but it need not be separately noted.

(4) It should now be noted that the square root will contain two whole numbers (at left of its decimal), so carriage is shifted to 8th position so the root will appear properly pointed off by Upper Dial Decimal.

(5) Set up in Keyboard Dial the proper Square Root Divisor (2497999) that appears in Col. 1 adjacent to Col. A, so that left-hand figure of divisor is directly below left-hand figure of Middle Dial amount. The divisor from Col. 1 is selected because the number of which the root is desired (155.758 etc.) has one figure in the left-hand "period" when 155 is separated into periods of two figures beginning at decimal point; thus 1'55'. Depress Division Key.

Distance "X" (12.480) appears in Upper Dial. This value is correct to 5 places; that is to say, the error of 6th figure does not exceed 5, so it satisfies the conditions of the problem; viz., the amount is correct to within ½ of 1/1000 of an inch.

NOTE: Not more than 5 figures of Upper Dial amount may be used, but the 5th figure should be increased by "1" if the 6th figure is "5" or over. If a 5-figure root does not provide the desired accuracy, convert to a 9-figure root by method described on reverse side of Marchant Table 56.

MARCHANT METHOD FOR CASE II

Use any Marchant 8-column M or D model with decimals as follows: Upper Dial 4, Middle Dial 11, Keyboard Dial 7. The desired accuracy requires use of not more than 6 places of the trigonometric function, but inasmuch as the tables of Marchant Method MM-99 provide 7 places, such values will be used to avoid possibility of error in rounding off tabular values to six places.

(1) Set up in Keyboard Dial sin 32°14' from table of MM-99 (.5333685) and, with carriage in 5th position, add into Middle Dial.

(2) Set up in Keyboard Dial (at extreme right) the increment for 8 seconds corresponding to 32°15' (328), and add.

(3) Set up in Keyboard Dial the amount that appears in Middle Dial (.5334013), clear Upper and Middle Dials, and multiply by 13.750.

Distance Y (7.334...) appears in Middle Dial. Copy it as 7.334."

(4) Clear all dials, set up in Keyboard Dial 90° in terms of degrees, minutes, and seconds; thus, 8.9059060 (the decimal between 8 and 9 has no significance), shift to 1st position, and add.

(5) Similarly, set up 32°14'8" in Keyboard Dial; thus, 3.2014008 and subtract, thus producing complement of 32°14'8" (57°45'52").

(6) Clear all dials, shift to 5th position, set up in Keyboard Dial sin 57°46' (.8458830) and add.

(7) Set up in Keyboard Dial (at extreme right) the increment for 52 seconds corresponding to 57°45' (207) and subtract.

(8) Set up in Keyboard Dial the amount that appears in Middle Dial (.8458623), clear Upper and Middle Dials, and multiply by 13.750.

Distance X (11.6306...) appears in Middle Dial. Copy it as 11.631."

MARCHANT METHOD FOR CASE III

Same decimal setting as for Case I.

(1) Set up 12.480 in Keyboard Dial, and multiply by 12.480.

(2) Clear Upper and Keyboard Dials only, set up 5.0625 in Keyboard Dial, and multiply by 5.0625.

(3) Clear Upper and Keyboard Dials only, shift to 7th position, set up nearest number to 181.3... from Col. A of Table 56 (181), and add.

(4) Shift to 8th position, set up in Keyboard Dial the proper Square Root Divisor (2690725) so that left-hand figure of divisor is directly below left-hand figure of Middle Dial amount, and divide.

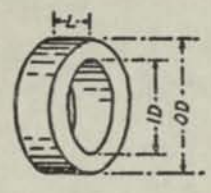
Distance "X" (13.4677...) appears in Upper Dial. Copy it as 13.468", as it is correct to five figures.

It is noted that this problem is the reverse of that of Case I.

WEIGHT OF STANDARD SHAPES

Remarks: In foundries, forge shops, and drafting rooms, it is often desired to compute the weight of castings or forgings of standard geometric shapes but with dimensions varying from case to case. If there is much work of this sort, it may be greatly simplified by using a Constant Factor which expresses weight in terms of the dimensions customarily used for measurement. Whether this Constant should be a multiplication or division factor depends upon whichever is more suitable for Marchant computation. Though the principle is exemplified herein for the case of the Annular Ring, it may similarly be applied to other standard shapes.

Example:



The Annular Ring, illustrated, is measured by its outside diam. (OD), inside diam. (ID), and axial length (L). Find suitable formula in terms of these dimensions expressed in inches when the material of which the ring is composed is steel weighing 0.28333 lbs. per cu. in. Also, find weight of ring of the following dimensions: OD — 13 13/16", ID — 11 1/16", L — 4 3/16".

Outline: Volume of ring = 3.14159 × Avg. Diam. × Radial thickness × Axial Length

" " " $= 3.14159 \times \frac{1}{2}(OD + ID) \times \frac{1}{2}(OD - ID) \times L$

Weight = 0.28333 × Volume in cu. inches. Whence, substituting and combining,

Weight $= 0.222527 \times (\overline{OD}^2 - \overline{ID}^2) \times L$, which form is not as suitable for computation as if the Constant were a Divisor factor; thus,

$$\text{Weight} = \frac{(\overline{OD}^2 - \overline{ID}^2)\ L}{4.4938}$$

NOTE: Constant used only to five figures because unit weight is expressed only to five places.

Operations: Decimals; Upper Dial 5, Middle Dial 10, Keyboard Dial 5. Use any M Model with Upper Green Shift Key down.

(1) Set up OD (13.8125) in Keyboard Dial and multiply by same amount.

(2) Change Keyboard Dial to read ID (11.0625) and reverse multiply by same amount.
Twice the radial thickness of ring (2.75) appears in Upper Dial, which is useful for checking purposes as this dimension is usually known.

(3) Clear Upper and Keyboard Dials and transfer Middle Dial amount (68.40625) to Keyboard Dial, clear Middle Dial and multiply by L (4.1875).

(4) Change Keyboard Dial to read Constant Factor (4.4938) and divide.*
Weight (63.74 lbs.) appears in Upper Dial.

(*) On certain Marchant models, it is necessary to clear Upper Dial prior to division.

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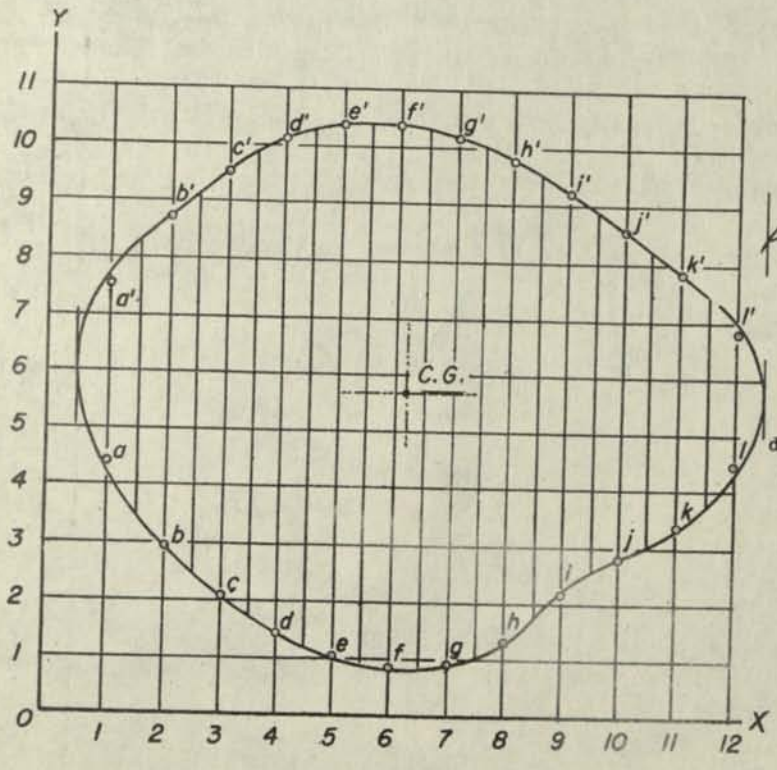
CENTER OF GRAVITY OF AN IRREGULAR PLANE FIGURE

REMARKS: *This method obtains the coordinates of the center of gravity in terms of the coordinates of selected points on the perimeter.*

OUTLINE: Assume the area outlined below. Sectionalize it into vertical strips of equal width; the more strips, the greater accuracy. Draw center-lines of each strip and extend them to a horizontal base-line (OX). Number the points of intersection of center-lines and base as 1, 2, 3, etc. These represent the scale of x-distances (departures) and numerically indicate the departure of each center-line. Draw OY perpendicular to OX and mark on it a scale of distances which need not be the same scale as that along the x-axis (though in this case it is the same).

Next, find the average mid-point of the top and bottom of each strip and tabulate their coordinates according to the respective scales of OX and OY. The small diagrams at right show how to estimate the mid-point in cases of curvature of the perimeter.



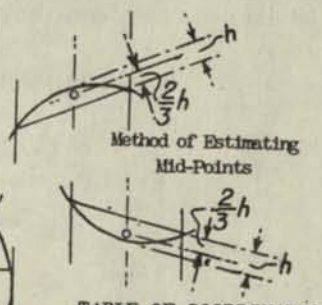


TABLE OF COORDINATES

departure	latitudes top	latitudes bottom
x	y'	y
1	7.56	4.41
2	8.70	2.95
3	9.52	2.08
4	10.13	1.44
5	10.38	1.03
6	10.37	0.84
7	10.16	0.90
8	9.74	1.30
9	9.21	2.17
10	8.55	2.80
11	7.85	3.38
12	6.80	4.45

Designating the y-ordinates of point a' as y'_a, and of point a as y_a, and so forth for the remaining lettered points, the center of gravity will lie on a horizontal line having a y-ordinate (latitude) as follows:

$$\frac{(y'_a)^2 + (y'_b)^2 + (y'_c)^2 + \ldots\ldots\ldots - (y_a)^2 - (y_b)^2 - (y_c)^2 - \ldots\ldots\ldots}{2(y'_a + y'_b + y'_c + \ldots\ldots\ldots\ldots\ldots - y_a - y_b - y_c - \ldots\ldots\ldots\ldots)}$$

The center of gravity also lies on a vertical line having an x-ordinate (departure) as follows, in which x_a, x_b, etc., are the x-ordinates of the points a'a, b'b, etc.

$$(2) \qquad \frac{x_a(y'_a - y_a) + x_b(y'_b - y_b) + x_c(y'_c - y_c) + \ldots\ldots\ldots}{y'_a + y'_b + y'_c + \ldots\ldots - y_a - y_b - y_c - \ldots\ldots}$$

The coordinates of the center of gravity are thus the latitude found by applying (1), and the departure found by applying (2).

OPERATIONS: Decimals: Upper Dial 3 and 0; Middle Dial 8 and 2; Keyboard Dial 8, 5, and 2. Upper Green Shift Key and No. 4 Tab Key down on 10-column M-series Marchant, with Division-Clear Lever away from operator.

TO FIND DIVISORS OF (1) AND (2)

(1) With carriage in 1st position, set up y'_a (7.56) and y_a (4.41) at 8th and 2nd Keyboard-Dial decimals, respectively, and add. Similarly add each of the remaining pairs of y's.

Sum of all y's (108.97) appears at 8th Middle-Dial decimal
" " " y s (27.75) " " 2nd " " "

(2) Touch Clear-Tab only (which shifts carriage to 4th position), set up right-hand Middle-Dial amount (27.75) at 5th Keyboard-Dial decimal, and subtract.

Divisor for Formula (2) (81.22) appears at 5th Middle-Dial decimal

(3) Duplicate Middle-Dial amount (81.22) in Keyboard Dial directly below and add.

Divisor for Formula (1) (162.44) appears in Middle Dial

TO SOLVE FORMULA (1)

(4) Clear all dials, set up y'_a (7.56) at 5th Keyboard-Dial decimal and multiply by same amount (7.56).

(5) Clear-Tab Upper and Keyboard Dials only and similarly multiply y'_b (8.70) by itself, and so on for all other y's of the top mid-points.

Sum of squares of y's of upper mid-points (1005.25) appears in Middle Dial

(6) Move Manual Counter Control toward operator, set up y (4.41) at 5th Keyboard-Dial decimal and reverse-multiply by same amount (4.41), and perform similar multiplications for all other y's of the lower mid-points.

Numerator of Formula (1) (922.66) appears in Middle Dial

(7) Move Manual Counter Control away from operator, change Keyboard Dial to read Divisor from Step (3) (162.44) and divide.

Y-ordinate of Center of Gravity (5.68) appears in Upper Dial

(continued on next page)

TO SOLVE FORMULA (2)

(8) Clear dials, set up y'_a (7.56) and y_a (4.41) at 8th and 2nd Keyboard Dials, respectively, and multiply by the corresponding x-ordinate (1.) at the right-hand decimal of Upper Dial.

(9) Clear Upper and Keyboard Dials, and similarly set up the next pair of values, (8.70) and (2.95), and multiply by the corresponding x-ordinate (2.), and so on for all values.

Sum of products of the y' values multiplied by the proper x-values (693.54) appears at left of Middle Dial, and similarly for the y-values (187.31) at right of Middle Dial.

(10) Clear-tab Keyboard Dial (which shifts carriage to 4th position), set up right-hand Middle-Dial amount (187.31) at 5th Keyboard-Dial decimal and subtract.

Numerator of Formula (2) (506.23) appears at left of Middle Dial

(11) Set up Divisor from Step 2 (81.22) at 5th Keyboard-Dial decimal and divide.

X-ordinate of Center of Gravity (6.23) appears in Upper Dial

CAPACITY OF CYLINDRICAL TANKS OR BINS

Remarks: A frequent calculation is that of obtaining the storage capacity of cylindrical tanks or bins. The customary method is to apply the usual formula involving the constant Pi (3.1415927). However, by combining that constant with one showing the relation between the unit of linear dimension of the tank and the unit in which its capacity is expressed, an exceedingly simple method results, which is ideally adapted to Marchant computation.

Example: Find the capacity in US gallons of a cylindrical tank 11 ft. 3 in. in diameter and 12 ft. 9 in. deep.

Operations: Decimals: Upper Dial 4, Middle Dial 10, Keyboard Dial 6.

(1) Set up Diameter in feet (11.25) in Keyboard Dial and multiply by same amount.
Diameter Squared (126.5625) appears in Middle Dial.

(2) Transfer Middle-Dial Amount (126.5625) to Keyboard Dial, clear Upper and Middle Dials, and multiply by Depth in feet (12.75).
Volume in cu. ft. of circumscribed rectangular prism (1613.67 . . .) appears in Middle Dial, but it need not be noted.

(3) Change Keyboard Dial to read Division Constant for Ft.-US gals. (0.170207) and divide.
Capacity in US gallons (9480.6) appears in Upper Dial.

Note A: The answer should be rounded to one less figure than the number of figures in the Constant of Step 3.

Note B: Constants (for Step 3) for frequently encountered conditions are listed below:

Dimensions of tank in feet:	Constant
Capacity in US gallons	0.170207
" " British Imperial gallons	0.204409
" " Barrels of 42 US gals. each	7.148710
" " US bushels	1.584491
" " British Imperial bushels	1.635276
" " Cubic Feet	1.273240
Dimensions of tank in inches:	
Capacity in US gallons	294.1183
" " British Imperial gallons	353.2196

LENGTH OF SHEETS OF KNOWN THICKNESS IN ROLLS OF KNOWN DIAMETER

REMARKS: This method is useful for finding length of plywood obtainable from log of given diameter, paper when rolled on cores, leather belting, etc. Another useful application is finding length of paper damaged when rolls are dug into by grappling hooks of transportation companies, the depth of the greatest cut into the roll being known.

EXAMPLE: A log 60 3/8" diam. is to be "peeled" into 1/7" thick plywood, the inner core (not usable) is 12" diam. How many linear feet of plywood can be obtained?

OUTLINE: The net end area of the roll is divided by the net end area of unit length of plywood. Formula is

$$L = \frac{3.14159 \ (D^2 - d^2)}{4 \times A} \text{ or, in this case } \frac{3.14159 \ (D^2 - d^2)}{4 \times 1/7 \times 12}$$

in which "D" is outside diam., "d" is inside diam., and "A" is area of unit length of sheet in units used for measuring "D" and "d". For convenience of computation, this formula, for any given thickness of sheet, is best put into the following form:

$$L = (D^2 - d^2) \text{ divided by } \frac{4 \times 1/7 \times 12}{3.14159}, \text{ as Divisor Constant } (2.18270).$$

OPERATIONS: Decimals; Upper Dial 3, Middle Dial 8, Keyboard Dial 5, use any 10 column model.

(1) Set up Outside Diameter (60.375) in Keyboard Dial and multiply by same amount.

(2) Shift to 4th position, change Keyboard Dial to square of Core Diameter, usually taken from table (144), and subtract.

(3) Set up Divisor Factor corresponding to thickness of sheet (2.18270) and divide.

Length in feet (1604.04) appears in Middle Dial.

NOTE: If core has an unusual diameter, as would be the case in problems relating to damaged outer portions of rolls of paper (in which the "core diam." is the diameter of the undamaged part of the roll), it is advisable to deduct the square of the "core diam." by means of negative multiplication in Step 2, instead of using the procedure shown.

DIAMETERS OF FORMING TOOLS HAVING TOP RAKE

REMARKS: *When a manufactured part has several diameters and it is desired to form it with a single cutting tool which has top rake (see Diagram), it is necessary to know the diameter of the forming tool in order that its cutting edges will properly turn the part to the desired diameter. In most cases the center of the work is below the center of the tool by a distance termed "gash.*

EXAMPLE: The diagram shows dimensions of such a part and the arrangement of work and forming tool. It is desired to know the diameter of forming tool sections which produce parts diameters of .150", .210", and .350" respectively, when the tool diameter is 1.750" forming the part diameter of .100"; and cutting edge in this instance is on horizontal center line of the part. Other constants are Rake Angle 10°, Gash below center .125".

OUTLINE: Before showing detailed Marchant Method, the arithmetical outline is given below: This also provides a schematic outline for recording the results of the calculation:

A) CONSTANTS FOR ANY TOOL-WORK COMBINATION:

$r = .05"$ $R_1 = .875"$ $E = 10°$ $G = .125"$ $\sin E = .173648$ $\cos E = .984808$

B) DERIVED CONSTANTS - WHICH DO NOT VARY FOR A PARTICULAR TOOL-WORK COMBINATION:

(1) $\frac{G}{R_1} = \frac{.125}{.875} = .142857$

(2) From Sin-Cos Table $\cos F = \cos \sin^{-1} \frac{G}{R_1} = \cos \sin^{-1} .142857 = .989743$ — See Marchant Method MM-192 for Sin-Cos Table.

(3) $r \cos E = .05 \times .984808 = .0492404$

(4) $R_1 \cos F = .875 \times .989743 = .866025$

(5) $r \sin E = .05 \times .173648 = .0086824$

C) CALCULATIONS WHICH INVOLVE R:

		R = .075"	.105"	.175"
(1)	$r \sin E/R = .0086824/R$	.115765	.082690	.049614
(2)	$\cos A = \cos \sin^{-1}(r \sin E/R)$ (Obtain Step 2 items from sin cos table)	.993277	.996575	.998768
(3)	$S = R \cos A - r \cos E$			
	$R \cos A - .0492404$			
(4)	$m = G + S \sin E = .125 + .173648\ S$	.0252554	.0553998	.125544
(5)	$n = R_1 \cos F - S \cos E$			
	$.866025 - .984808\ S$	.1293855	.1346201	.1468005
(6)	$m^2 + n^2$	.8411533	.8114668	.7423883
(7)	$r_1 = \sqrt{m^2 + n^2}$	.7242795	.6766601	.5726908
(8)	$d_1 = 2\ r_1$	.85105	.82259	.75676
		1.70210	1.64518	1.51352

(over)

MARCHANT OPERATIONS

DECIMALS: Upper Dial 7, Middle Dial 15, Keyboard Dial 8. Use 10-column Marchant. (If radii are less than 1, an 8 column Marchant may be used; or if fewer decimal places are required, an 8 column Marchant may be used even if radii exceed 1.)

Step B1: G/R_1 Divide in accordance with Instruction Book.

Step B2:

(a) From Sin-Cos Table it is noted that Cos of angle whose sine is .142 is .9898667. Set up this amount in Keyboard Dial, depress Add Bar, and then depress Subtract Bar.

(b) Set up in Keyboard Dial the "Difference" between the above cos and the one next below (.000144), and with Manual Counter Control toward the operator, reverse multiply by .857.

Cos F (.989743) appears in Middle Dial.

Steps B3, 4, & 5. Multiply in accordance with Instruction Book.

Step C1: Divide in accordance with Instruction Book.

Step C2: Obtain from Sin-Cos Table in a similar manner to Step B2.

Step C3: Multiply R by cos A from Step C2 and subtract .0492404.

Step C4:

(a) Set up in Keyboard Dial (.125) and add. Clear Upper Dial.

(b) Set up in Keyboard Dial (.173648) and multiply by S from Step 3 (.0252554).

m (.1293855) appears in Middle Dial.

(c) Without clearing Upper Dial, change its reading to .0553998.

m (.1346201) appears in Middle Dial.

(d) Without clearing Upper Dial, change its reading to .125544.

m (.1468005) appears in Middle Dial.

Step C5:

(a) Set up in Keyboard Dial (.866025) and add. Clear Upper Dial.

(b) Set up in Keyboard Dial (.984808) and, with Manual Counter Control toward the operator, <u>reverse</u> multiply by S (.0252554).

n (.8411533) appears in Middle Dial.

(c) Without clearing Upper Dial, and with Reverse Bar continually depressed, change its reading to S (.0553998).

n (.8114668) appears in Middle Dial.

(d) In a manner similar to above, obtain the next n (.7423883) and move Manual Counter Control away from operator.

Step C6:

(a) Set up .1293855 and multiply by .1293855.

(b) Clear Upper and Keyboard Dials only; set up .8411533, and multiply by .8411533.

m^2+n^2 (.7242795) appears in Middle Dial.

At this point it is recommended that its square root be obtained as follows:

Step C7:

(a) From Table 56, it is noted that the number in Col. A nearest to 724 is 728. Set this up in Keyboard Dial as .728 and, with carriage in 8th position, depress Add Bar. Clear Upper Dial.

(b) Take Divisor (1 706 458) from Col. 2 at right of 728 and set it up in Keyboard Dial as 1.706458 and divide.

The Five Figure Root (.85105) appears in Upper Dial.

Repeat Steps C6 and C7 for the other values of r.

Step 8: Double the results of Step C7.

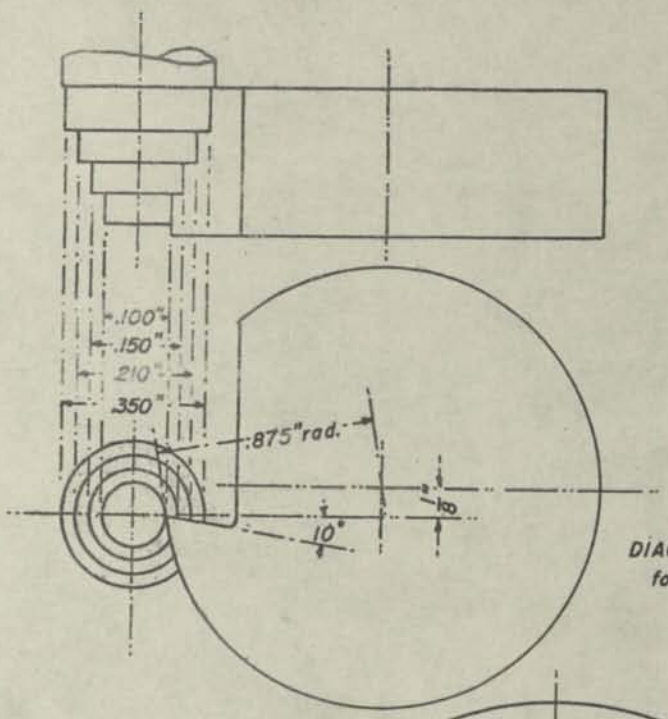


DIAGRAM of TOOL and WORK for example of this method (not to scale)

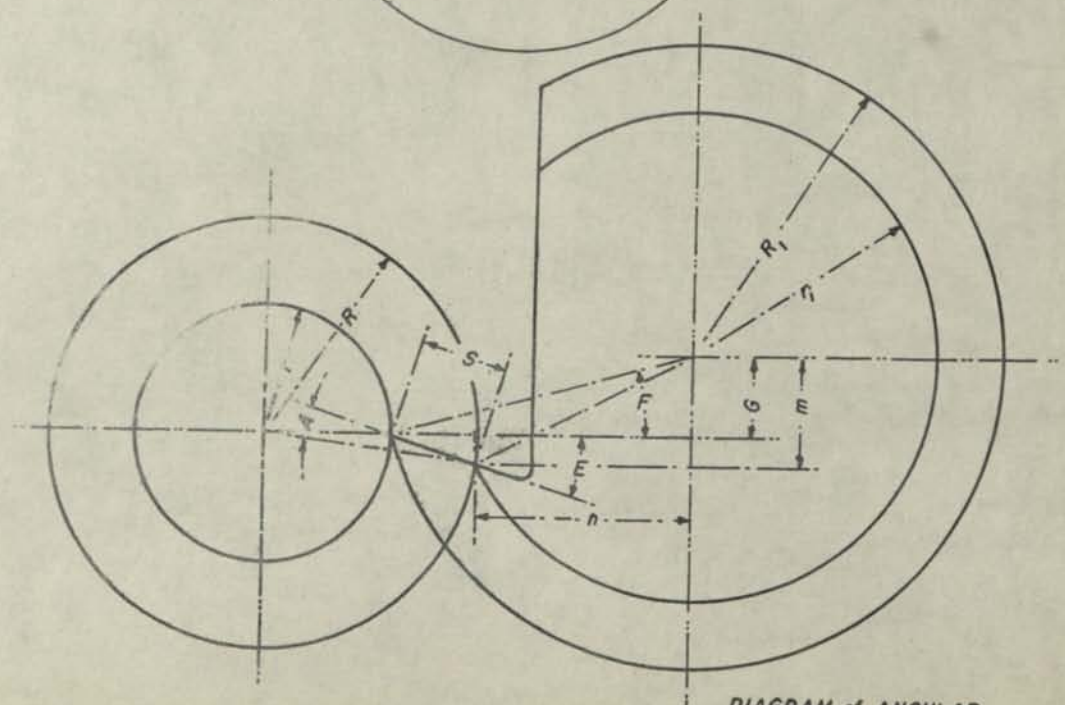


DIAGRAM of ANGULAR RELATIONS (not to scale)

TABLE 99

SEVEN-PLACE NATURAL TRIGONOMETRIC FUNCTIONS TO MINUTES WITH INCREMENTS FOR SECONDS

Copyright, 1939, by Charles E. Sharp, Jr.

PREFACE

The modern calculating machine has given rise to the use of natural trigonometric functions, thereby largely superseding the use of tables of logarithmic trigonometric functions. The purpose of the tables presented herewith is to supply this need in a form that is brief, accurate, and easy to use with modern calculating machines.

The functions for each minute were copied from "Natural Trigonometric Functions" by Robert E. Benson, who has kindly permitted their use by the author. The increments for seconds were computed and arranged for easy reference by the author. Occasionally a maximum error of 1 in the seventh place is to be expected. The author will welcome the report of any greater error.

The "seconds" increments for tangents above 45° increase out of all proportion to those for sines 0° to 90°, and tangents 0° to 45°, and become quite unwieldy. Hence the functions in these tables have been confined to these sines and tangents. Cosines may be found as the sines of their complementary angles. For tangents above 45° the reciprocal of the tangent of the complementary angle is to be used. With the calculating machine this handling is simple.

Owing to the Marchant calculator having a separate dial which shows the keyboard set-up in straight line true figures, it is usually possible to avoid separate machine addition or the noting of the values for minutes and the seconds increments. This is possible because in most cases the function is to be set up in the keyboard, and in such event it is recommended that the value for minutes be first set up so it appears "written" in the Marchant Keyboard Dial. The increment is then mentally added (or subtracted) by depression of the keyboard keys for the desired terminal total thus altering the Keyboard Dial reading; that is to say, the keyboard and the Keyboard Dial become the "scratch pad" for this adjustment.

Charles E. Sharp, Jr.

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Oakland, California

INTRODUCTION

In the following tables each page comprises 5 degrees of angle. The upper half gives the function minute by minute. In this table, the symbol A' indicates minutes from 0 to 30. Functions in this group are to be read down, in the left half of the double column for each degree. The symbol B' indicates minutes from 30 to 60. Functions in this group are to be read up, in the right half of the column.

The lower half of the page gives the increment for seconds corresponding to the given function. The symbol A" indicates seconds from 0 to 30, reading down. The symbol B" indicates seconds from 30 to 60, reading up. The zones over which the increments apply are 15 minutes in width; i.e., successive columns designate such angular intervals as 7°-15', 7°-30', 7°-45', etc. In cases where there is a difference greater than 1 in the right hand digit of adjacent horizontal increments the increment should be adjusted slightly to correspond with what it would be if there be imagined a vertical column headed with the actual minutes of the function. This can usually be done by inspection.

It is important to note that the increment for A" is the same value as that for 60-A". For example, the increment for 7" is the same as that for 53". However, the one for A" is to be added to the "minute" function, while the one for B" is to be subtracted from the function for the next higher minute. For compactness the increments are listed as whole numbers. Actually they are the right hand terminals of seven place decimal fractions as shown in the examples.

READING FUNCTIONS FROM KNOWN ANGLES

SINES

Given angle	6°-11'-19"	6°-11'-41"	76°-47'-12"	76°-47'-48"
Function from upper table	.1077102	.1079994	.9735124	.9735789
Increment from lower table	.0000916	-.0000916	.0000133	-.0000133
Function for given angle	.1078018	.1079078	.9735257	.9735656

Given angle	53°-22'-26"	53°-22'-34"	48°-37'-26"	48°-37'-34"
Function from upper table	.8024705	.8026440	.7503034	.7504957
Increment from lower table	.0000752*	-.0000752*	.0000833*	-.0000833*
Function for given angle	.8025457	.8025688	.7503867	.7504124

(*) *The increments are adjusted downwards from the value in the 30' and 45' column respectively because the minutes of the function are between those for which columns of increments are compiled.*

TANGENTS

Given angle	43°-58'-22"	43°-58'-38"	75°-37'-19"	75°-37'-41"
Complementary angle	- - -	- - -	14°-22'-41"	14°-22'-19"
Function from upper table	.9645651	.9651268	.2564463	.2561363
Increment from lower table	.0002060*	-.0002060*	-.0000982*	.0000982*
Function for complementary angle	- - -	- - -	.2563481	.2562345
Function for given angle	.9647711	.9649208	3.9009456#	3.9026751#

(*) *The increments are slightly adjusted (See note under "sines").*

(#) *The eighth significant figure is not reliable, but it is more significant than rounding off the 7th digit.*

READING ANGLES FROM KNOWN FUNCTIONS

SINES

(1)	Given function	.5909723	.1078018	.9735656
(2)	Next larger angle to (')	36°-14'	6°-12'	76°-48'
(3)	Next smaller angle to (')	36°-13'	6°-11'	76°-47'
(4)	Function for (2) to (')	.5910750	.1079994	.9735789
(5)	Function for (3) to (')	.5908404	.1077102	.9735124
(6)	(4) minus (1)	1027	1976	133
(7)	(1) minus (5)	1319	916	532
(8)	Angle less or more than 30"	more	less	more
(9)	Desired angle	36°-13'-34"-	6°-11'-19"	76°-47'-48"

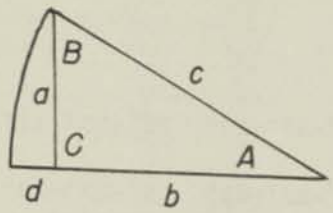
NOTE: If (6) is greater than (7), column (8) is "less," and vice versa. If (1) is clearly closer to either (4) or (5), step 7 or 6 respectively can be eliminated.

TANGENTS

If the tangent is less than 1.0000000 read in a similar manner to the above for sines.

If the tangent is greater than 1.0000000, obtain its reciprocal (divide 1.0 by it), and read angle corresponding to the reciprocal as if the latter were a tangential function. The complement of this angle is the desired angle.

TRIGONOMETRIC RELATIONSHIPS



$\text{Sin } A = a \div c$

$\text{Cos } A = b \div c$

$\text{Tan } A = a \div b$

$\text{Vers } A = d \div c$

$\text{Cosec } A = c \div a$

$\text{Sec } A = c \div b$

$\text{Cot } A = b \div a$

$\text{Sin } (A \pm B) = \text{Sin } A \text{ Cos } B \pm \text{Cos } A \text{ Sin } B$

$\text{Cos } (A \pm B) = \text{Cos } A \text{ Cos } B \mp \text{Sin } A \text{ Sin } B$

$\text{Tan } (A \pm B) = (\text{Tan } A \pm \text{Tan } B) \div (1 \mp \text{Tan } A \text{ Tan } B)$

$\text{Cot } (A \pm B) = (\text{Cot } A \text{ Cot } B \mp 1) \div (\text{Cot } B \pm \text{Cot } A)$

$\text{Sin } 2A = 2 \text{ Sin } A \text{ Cos } A$

$\text{Cos } 2A = \text{Cos}^2 A - \text{Sin}^2 A$

$\text{Tan } 2A = 2 \text{ Tan } A \div (1 - \text{Tan}^2 A)$

$\text{Cot } 2A = (\text{Cot}^2 A - 1) \div 2\text{Cot } A$

$\text{Vers } 2A = 2 \text{ Sin}^2 A$

$\text{Sin } A \pm \text{Sin } B = 2 \text{ Sin } \tfrac{1}{2}(A \pm B) \times \text{Cos } \tfrac{1}{2}(A \mp B)$

$\text{Cos } A + \text{Cos } B = 2 \text{ Cos } \tfrac{1}{2}(A + B) \times \text{Cos } \tfrac{1}{2}(A - B)$

$\text{Cos } B - \text{Cos } A = 2 \text{ Sin } \tfrac{1}{2}(A + B) \times \text{Sin } \tfrac{1}{2}(A - B)$

$\text{Tan } A \pm \text{Tan } B = \text{Sin } (A \pm B) \div \text{Cos } A \text{ Cos } B$

$\text{Cot } A \pm \text{Cot } B = \text{Sin } (B \pm A) \div \text{Sin } A \text{ Sin } B$

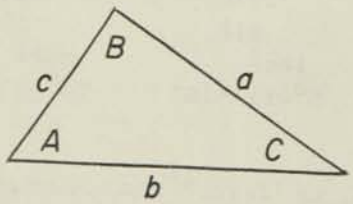
$\text{Sin } \tfrac{1}{2}A = \sqrt{(1 - \text{Cos } A) \div 2}$

$\text{Cos } \tfrac{1}{2}A = \sqrt{(1 + \text{Cos } A) \div 2}$

$\text{Tan } \tfrac{1}{2}A = \text{Sin } A \div (1 + \text{Cos } A)$

$\text{Cot } \tfrac{1}{2}A = \text{Sin } A \div (1 - \text{Cos } A)$

SOLUTION OF THE SCALENE TRIANGLE



Formulas

$$\begin{cases} \text{Cos } A = (c^2 + b^2 - a^2) \div 2bc \\ s = (a + b + c) \div 2 \\ \text{Sin } \tfrac{1}{2}A = \sqrt{(s-b)(s-c) \div bc}, \quad \text{Cos } \tfrac{1}{2}A = \sqrt{\dfrac{s(s-a)}{bc}} \\ \text{Tan } \tfrac{1}{2}A = \sqrt{(s-b)(s-c) \div s(s-a)} \end{cases}$$

Given	Sought	
a, b, c	A	
a, A, B	b, c	$b = a \text{ Sin } B \div \text{Sin } A$ $\quad c = a \text{ Sin } C \div \text{Sin } A$
c, b, A	a, C	$\text{Tan } C = (c \text{ Sin } A) \div (b - c \text{ Cos } A)$

$$a = \sqrt{b^2 + c^2 - 2bc \text{ Cos } A}$$

FUNCTION FOR EACH MINUTE

For first half (0′ to 30′) of each degree read DOWN in column marked A′
For second half (30′ to 60′) of each degree read UP in column marked B′

A′	0°		1°		2°		3°		4°		B′
	A′	B′	A′	B′	A′	B′	A′	B′	A′	B′	
0	.0000000	.0174524	.0174524	.0348995	.0348995	.0523360	.0523360	.0697565	.0697565	.0871557	60
1	.0002909	.0171616	.0177432	.0346088	.0351902	.0520455	.0526264	.0694663	.0700467	.0868660	59
2	.0005818	.0168707	.0180341	.0343181	.0354809	.0517550	.0529169	.0691761	.0703368	.0865762	58
3	.0008727	.0165799	.0183249	.0340274	.0357716	.0514645	.0532074	.0688859	.0706270	.0862864	57
4	.0011636	.0162890	.0186158	.0337366	.0360623	.0511740	.0534979	.0685957	.0709171	.0859966	56
5	.0014544	.0159982	.0189066	.0334459	.0363530	.0508835	.0537883	.0683055	.0712073	.0857067	55
6	.0017453	.0157073	.0191974	.0331552	.0366437	.0505929	.0540788	.0680153	.0714974	.0854169	54
7	.0020362	.0154165	.0194883	.0328644	.0369344	.0503024	.0543693	.0677251	.0717876	.0851271	53
8	.0023271	.0151256	.0197791	.0325737	.0372251	.0500119	.0546597	.0674349	.0720777	.0848373	52
9	.0026180	.0148348	.0200699	.0322830	.0375158	.0497214	.0549502	.0671446	.0723678	.0845474	51
10	.0029089	.0145439	.0203608	.0319922	.0378065	.0494308	.0552406	.0668544	.0726580	.0842576	50
11	.0031998	.0142530	.0206516	.0317015	.0380971	.0491403	.0555311	.0665641	.0729481	.0839677	49
12	.0034907	.0139622	.0209424	.0314108	.0383878	.0488498	.0558215	.0662739	.0732382	.0836778	48
13	.0037815	.0136713	.0212332	.0311200	.0386785	.0485592	.0561119	.0659836	.0735283	.0833880	47
14	.0040724	.0133805	.0215241	.0308293	.0389692	.0482687	.0564024	.0656934	.0738184	.0830981	46
15	.0043633	.0130896	.0218149	.0305385	.0392598	.0479781	.0566928	.0654031	.0741085	.0828082	45
16	.0046542	.0127987	.0221057	.0302478	.0395505	.0476876	.0569832	.0651129	.0743986	.0825183	44
17	.0049451	.0125079	.0223965	.0299570	.0398411	.0473970	.0572736	.0648226	.0746887	.0822284	43
18	.0052360	.0122170	.0226873	.0296662	.0401318	.0471065	.0575640	.0645323	.0749787	.0819385	42
19	.0055268	.0119261	.0229781	.0293755	.0404224	.0468159	.0578544	.0642420	.0752688	.0816486	41
20	.0058177	.0116353	.0232690	.0290847	.0407131	.0465253	.0581448	.0639517	.0755589	.0813587	40
21	.0061086	.0113444	.0235598	.0287940	.0410037	.0462347	.0584352	.0636614	.0758489	.0810687	39
22	.0063995	.0110535	.0238506	.0285032	.0412944	.0459442	.0587256	.0633711	.0761390	.0807788	38
23	.0066904	.0107627	.0241414	.0282124	.0415850	.0456536	.0590160	.0630808	.0764290	.0804889	37
24	.0069813	.0104718	.0244322	.0279216	.0418757	.0453630	.0593064	.0627905	.0767190	.0801989	36
25	.0072721	.0101809	.0247230	.0276309	.0421663	.0450724	.0595967	.0625002	.0770091	.0799090	35
26	.0075630	.0098900	.0250138	.0273401	.0424569	.0447818	.0598871	.0622099	.0772991	.0796190	34
27	.0078539	.0095992	.0253046	.0270493	.0427475	.0444912	.0601775	.0619196	.0775891	.0793290	33
28	.0081448	.0093083	.0255954	.0267585	.0430382	.0442006	.0604678	.0616292	.0778791	.0790391	32
29	.0084357	.0090174	.0258862	.0264677	.0433288	.0439100	.0607582	.0613389	.0781691	.0787491	31
30	.0087265	.0087265	.0261769	.0261769	.0436194	.0436194	.0610485	.0610485	.0784591	.0784591	30

INCREMENT FOR SECONDS

For A″ add increment to function in above table. For B″ subtract increment from function for next higher minute in above table.

A″	0°				1°				2°				3°				4°				B″
	0′	15′	30′	45′	0′	15′	30′	45′	0′	15′	30′	45′	0′	15′	30′	45′	0′	15′	30′	45′	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	49	49	49	49	49	49	49	48	48	48	48	48	48	48	48	48	48	48	48	48	59
2	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	58
3	145	145	145	145	145	145	145	145	145	145	145	145	145	145	145	145	145	145	145	145	57
4	194	194	194	194	194	194	194	194	194	194	194	194	194	194	194	194	193	193	193	193	56
5	242	242	242	242	242	242	242	242	242	242	242	242	242	242	242	242	242	242	242	242	55
6	291	291	291	291	291	291	291	291	291	291	291	291	291	290	290	290	290	290	290	290	54
7	339	339	339	339	339	339	339	339	339	339	339	339	339	339	339	339	339	338	338	338	53
8	388	388	388	388	388	388	388	388	388	388	388	387	387	387	387	387	387	387	387	386	52
9	436	436	436	436	436	436	436	436	436	436	436	436	436	436	436	435	435	435	435	435	51
10	485	485	485	485	485	485	485	485	485	484	484	484	484	484	484	484	484	483	483	483	50
11	533	533	533	533	533	533	533	533	533	533	533	533	533	532	532	532	532	532	532	531	49
12	582	582	582	582	582	582	582	582	581	581	581	581	581	581	581	581	580	580	580	580	48
13	630	630	630	630	630	630	630	630	630	630	630	629	629	629	629	629	629	628	628	628	47
14	679	679	679	679	679	679	678	678	678	678	678	678	678	678	677	677	677	677	677	676	46
15	727	727	727	727	727	727	727	727	727	727	727	726	726	726	726	726	725	725	725	725	45
16	776	776	776	776	776	775	775	775	775	775	775	775	775	774	774	774	774	774	773	773	44
17	824	824	824	824	824	824	824	824	824	824	823	823	823	823	823	822	822	822	822	821	43
18	873	873	873	873	873	872	872	872	872	872	872	872	871	871	871	871	871	870	870	870	42
19	921	921	921	921	921	921	921	921	921	920	920	920	920	920	919	919	919	919	918	918	41
20	970	970	970	970	969	969	969	969	969	969	969	968	968	968	968	968	967	967	967	966	40
21	1018	1018	1018	1018	1018	1018	1018	1018	1018	1017	1017	1017	1017	1016	1016	1016	1016	1015	1015	1015	39
22	1067	1067	1067	1067	1066	1066	1066	1066	1066	1066	1066	1065	1065	1065	1065	1064	1064	1064	1063	1063	38
23	1115	1115	1115	1115	1115	1115	1115	1114	1114	1114	1114	1114	1114	1113	1113	1113	1112	1112	1112	1111	37
24	1164	1164	1164	1163	1163	1163	1163	1163	1163	1163	1162	1162	1162	1162	1161	1161	1161	1160	1160	1160	36
25	1212	1212	1212	1212	1212	1212	1212	1211	1211	1211	1211	1211	1210	1210	1210	1209	1209	1209	1208	1208	35
26	1261	1261	1261	1260	1260	1260	1260	1260	1260	1260	1259	1259	1259	1259	1258	1258	1258	1257	1257	1256	34
27	1309	1309	1309	1309	1309	1309	1308	1308	1308	1308	1308	1307	1307	1307	1306	1306	1306	1305	1305	1305	33
28	1358	1358	1358	1358	1357	1357	1357	1357	1357	1356	1356	1356	1356	1355	1355	1355	1354	1354	1353	1353	32
29	1406	1406	1406	1406	1406	1406	1405	1405	1405	1405	1405	1404	1404	1404	1403	1403	1403	1402	1402	1401	31
30	1454	1454	1454	1454	1454	1454	1454	1454	1454	1453	1453	1453	1452	1452	1452	1451	1451	1450	1450	1449	30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	5°		6°		7°		8°		9°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.0871557	.1045285	.1045285	.1218693	.1218693	.1391731	.1391731	.1564345	.1564345	.1736482	60
1	.0874455	.1042392	.1048178	.1215806	.1221581	.1388850	.1394612	.1561472	.1567218	.1733617	59
2	.0877353	.1039499	.1051070	.1212919	.1224468	.1385970	.1397492	.1558598	.1570091	.1730752	58
3	.0880251	.1036605	.1053963	.1210031	.1227355	.1383089	.1400372	.1555725	.1572963	.1727887	57
4	.0883148	.1033712	.1056856	.1207144	.1230241	.1380208	.1403252	.1552851	.1575836	.1725022	56
5	.0886046	.1030819	.1059748	.1204256	.1233128	.1377327	.1406132	.1549978	.1578708	.1722156	55
6	.0888943	.1027925	.1062641	.1201368	.1236015	.1374445	.1409012	.1547104	.1581581	.1719291	54
7	.0891840	.1025032	.1065533	.1198481	.1238901	.1371564	.1411892	.1544230	.1584453	.1716425	53
8	.0894738	.1022138	.1068425	.1195593	.1241788	.1368683	.1414772	.1541356	.1587325	.1713560	52
9	.0897635	.1019245	.1071318	.1192704	.1244674	.1365801	.1417651	.1538482	.1590197	.1710694	51
10	.0900532	.1016351	.1074210	.1189816	.1247560	.1362919	.1420531	.1535607	.1593069	.1707828	50
11	.0903429	.1013457	.1077102	.1186928	.1250446	.1360038	.1423410	.1532733	.1595940	.1704961	49
12	.0906326	.1010563	.1079994	.1184040	.1253332	.1357156	.1426289	.1529858	.1598812	.1702095	48
13	.0909223	.1007669	.1082885	.1181151	.1256218	.1354274	.1429168	.1526984	.1601683	.1699228	47
14	.0912119	.1004775	.1085777	.1178263	.1259104	.1351392	.1432047	.1524109	.1604555	.1696362	46
15	.0915016	.1001881	.1088669	.1175374	.1261990	.1348509	.1434926	.1521234	.1607426	.1693495	45
16	.0917913	.0998986	.1091560	.1172485	.1264875	.1345627	.1437805	.1518359	.1610297	.1690628	44
17	.0920809	.0996092	.1094452	.1169596	.1267761	.1342744	.1440684	.1515484	.1613167	.1687761	43
18	.0923706	.0993197	.1097343	.1166707	.1270646	.1339862	.1443562	.1512608	.1616038	.1684894	42
19	.0926602	.0990303	.1100234	.1163818	.1273531	.1336979	.1446440	.1509733	.1618909	.1682026	41
20	.0929499	.0987408	.1103126	.1160929	.1276416	.1334096	.1449319	.1506857	.1621779	.1679159	40
21	.0932395	.0984514	.1106017	.1158040	.1279302	.1331213	.1452197	.1503981	.1624650	.1676291	39
22	.0935291	.0981619	.1108908	.1155151	.1282186	.1328330	.1455075	.1501106	.1627520	.1673423	38
23	.0938187	.0978724	.1111799	.1152261	.1285071	.1325447	.1457953	.1498230	.1630390	.1670556	37
24	.0941083	.0975829	.1114689	.1149372	.1287956	.1322564	.1460830	.1495353	.1633260	.1667687	36
25	.0943979	.0972934	.1117580	.1146482	.1290841	.1319681	.1463708	.1492477	.1636129	.1664819	35
26	.0946875	.0970039	.1120471	.1143592	.1293725	.1316797	.1466585	.1489601	.1638999	.1661951	34
27	.0949771	.0967144	.1123361	.1140702	.1296609	.1313913	.1469463	.1486724	.1641868	.1659082	33
28	.0952666	.0964248	.1126252	.1137812	.1299494	.1311030	.1472340	.1483848	.1644738	.1656214	32
29	.0955562	.0961353	.1129142	.1134922	.1302378	.1308146	.1475217	.1480971	.1647607	.1653345	31
30	.0958458	.0958458	.1132032	.1132032	.1305262	.1305262	.1478094	.1478094	.1650476	.1650476	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	5°				6°				7°				8°				9°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	59
2	97	97	97	97	97	96	96	96	96	96	96	96	96	96	96	96	96	96	96	96	58
3	145	145	145	145	145	145	145	144	144	144	144	144	144	144	144	144	144	144	143	143	57
4	193	193	193	193	193	193	193	193	192	192	192	192	192	192	192	192	192	191	191	191	56
5	241	241	241	241	241	241	241	241	241	240	240	240	240	240	240	240	239	239	239	239	55
6	290	290	290	290	289	289	289	289	289	289	288	288	288	288	288	287	287	287	287	287	54
7	338	338	338	338	338	337	337	337	337	337	337	336	336	336	336	335	335	335	335	334	53
8	386	386	386	386	386	386	385	385	385	385	385	384	384	384	384	383	383	383	383	382	52
9	435	434	434	434	434	434	434	433	433	433	433	432	432	432	432	431	431	431	430	430	51
10	483	483	483	482	482	482	482	481	481	481	481	480	480	480	479	479	479	478	478	478	50
11	531	531	531	531	530	530	530	530	529	529	529	528	528	528	527	527	527	526	526	526	49
12	580	579	579	579	579	578	578	578	577	577	577	576	576	576	575	575	575	574	574	573	48
13	628	627	627	627	627	627	626	626	626	625	625	624	624	624	623	623	622	622	622	621	47
14	676	676	676	675	675	675	674	674	674	673	673	673	672	672	671	671	670	670	669	669	46
15	724	724	724	724	723	723	723	722	722	721	721	721	720	720	719	719	718	718	717	717	45
16	773	772	772	772	771	771	770	770	770	769	769	768	768	767	767	766	766	766	765	764	44
17	821	821	820	820	819	819	818	818	818	817	817	817	817	816	816	815	815	814	813	813	43
18	869	869	869	869	868	867	867	866	866	865	865	865	864	864	863	862	862	861	861	860	42
19	918	917	917	917	916	916	915	915	914	914	913	913	912	912	911	910	910	909	908	908	41
20	966	966	965	965	964	964	963	963	962	962	961	961	960	960	959	958	958	957	956	956	40
21	1014	1014	1014	1013	1013	1012	1012	1011	1011	1010	1009	1009	1008	1008	1007	1006	1006	1005	1004	1003	39
22	1063	1062	1062	1061	1061	1060	1060	1059	1059	1058	1058	1057	1056	1056	1055	1054	1053	1053	1052	1051	38
23	1111	1110	1110	1110	1109	1108	1108	1107	1107	1106	1106	1105	1104	1104	1103	1102	1101	1101	1100	1099	37
24	1159	1159	1158	1158	1157	1157	1156	1156	1155	1154	1154	1153	1152	1152	1151	1150	1149	1148	1148	1147	36
25	1207	1207	1206	1206	1205	1205	1204	1204	1203	1202	1202	1201	1200	1200	1199	1198	1197	1196	1195	1195	35
26	1256	1255	1255	1254	1254	1253	1252	1252	1251	1250	1250	1249	1248	1247	1247	1246	1245	1244	1243	1242	34
27	1304	1303	1303	1302	1302	1301	1301	1300	1299	1299	1298	1297	1296	1295	1295	1294	1293	1292	1291	1290	33
28	1352	1352	1351	1351	1350	1349	1349	1348	1347	1347	1346	1345	1344	1343	1343	1342	1341	1340	1339	1338	32
29	1401	1400	1399	1399	1398	1398	1397	1396	1396	1395	1394	1393	1392	1391	1390	1390	1389	1388	1387	1386	31
30	1449	1448	1447	1447	1447	1446	1445	1444	1444	1443	1442	1441	1440	1439	1439	1438	1437	1436	1435	1433	30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	10°		11°		12°		13°		14°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.1763270	.1943803	.1943803	.2125566	.2125566	.2308682	.2308682	.2493280	.2493280	.2679492	60
1	.1766269	.1940784	.1946822	.2122525	.2128606	.2305618	.2311746	.2490191	.2496370	.2676374	59
2	.1769269	.1937766	.1949841	.2119486	.2131647	.2302555	.2314811	.2487102	.2499460	.2673257	58
3	.1772269	.1934748	.1952861	.2116446	.2134688	.2299492	.2317876	.2484013	.2502551	.2670141	57
4	.1775270	.1931731	.1955881	.2113407	.2137730	.2296429	.2320941	.2480925	.2505642	.2667025	56
5	.1778270	.1928713	.1958901	.2110369	.2140772	.2293367	.2324007	.2477837	.2508734	.2663909	55
6	.1781271	.1925696	.1961922	.2107331	.2143814	.2290306	.2327073	.2474750	.2511826	.2660794	54
7	.1784273	.1922680	.1964943	.2104293	.2146857	.2287244	.2330140	.2471663	.2514919	.2657680	53
8	.1787274	.1919664	.1967964	.2101255	.2149900	.2284184	.2333207	.2468577	.2518012	.2654566	52
9	.1790276	.1916648	.1970986	.2098218	.2152944	.2281123	.2336274	.2465491	.2521106	.2651452	51
10	.1793279	.1913632	.1974008	.2095181	.2155988	.2278063	.2339342	.2462405	.2524200	.2648339	50
11	.1796281	.1910617	.1977031	.2092145	.2159032	.2275003	.2342411	.2459320	.2527294	.2645226	49
12	.1799284	.1907602	.1980053	.2089109	.2162077	.2271944	.2345479	.2456236	.2530389	.2642114	48
13	.1802287	.1904587	.1983076	.2086073	.2165122	.2268885	.2348548	.2453151	.2533484	.2639002	47
14	.1805291	.1901573	.1986100	.2083038	.2168167	.2265827	.2351617	.2450068	.2536580	.2635891	46
15	.1808295	.1898559	.1989124	.2080003	.2171213	.2262769	.2354687	.2446984	.2539676	.2632780	45
16	.1811299	.1895546	.1992148	.2076968	.2174259	.2259711	.2357758	.2443902	.2542773	.2629670	44
17	.1814303	.1892533	.1995172	.2073934	.2177306	.2256654	.2360829	.2440819	.2545870	.2626560	43
18	.1817308	.1889520	.1998197	.2070900	.2180353	.2253597	.2363900	.2437737	.2548968	.2623451	42
19	.1820313	.1886507	.2001222	.2067867	.2183400	.2250541	.2366971	.2434656	.2552066	.2620342	41
20	.1823319	.1883495	.2004248	.2064834	.2186448	.2247485	.2370044	.2431575	.2555165	.2617234	40
21	.1826324	.1880483	.2007274	.2061802	.2189496	.2244429	.2373116	.2428494	.2558264	.2614126	39
22	.1829330	.1877471	.2010300	.2058769	.2192544	.2241374	.2376189	.2425414	.2561363	.2611018	38
23	.1832337	.1874460	.2013327	.2055737	.2195593	.2238319	.2379262	.2422334	.2564463	.2607911	37
24	.1835343	.1871449	.2016354	.2052705	.2198643	.2235265	.2382336	.2419255	.2567564	.2604805	36
25	.1838350	.1868439	.2019381	.2049674	.2201692	.2232211	.2385410	.2416176	.2570664	.2601699	35
26	.1841358	.1865428	.2022409	.2046643	.2204742	.2229157	.2388485	.2413097	.2573766	.2598593	34
27	.1844365	.1862418	.2025437	.2043612	.2207793	.2226104	.2391560	.2410019	.2576868	.2595488	33
28	.1847373	.1859409	.2028465	.2040582	.2210844	.2223051	.2394635	.2406942	.2579970	.2592384	32
29	.1850382	.1856399	.2031494	.2037552	.2213895	.2219999	.2397711	.2403864	.2583073	.2589280	31
30	.1853390	.1853390	.2034523	.2034523	.2216947	.2216947	.2400788	.2400788	.2586176	.2586176	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	10°				11°				12°				13°				14°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	50	50	50	50	50	50	51	51	51	51	51	51	51	51	51	51	52	52	52	52	59
2	100	100	100	101	101	101	101	101	101	102	102	102	102	102	103	103	103	103	103	104	58
3	150	150	150	151	151	151	151	152	152	152	153	153	153	154	154	154	155	155	155	156	57
4	200	200	201	201	201	202	202	202	203	203	204	204	204	205	205	206	206	206	207	207	56
5	250	250	251	251	252	252	252	253	253	254	254	255	255	256	256	257	258	258	259	259	55
6	300	300	301	301	302	302	303	304	304	305	305	306	306	307	308	308	309	310	310	311	54
7	350	351	351	352	352	353	353	354	355	356	356	357	358	358	359	360	361	361	362	363	53
8	400	401	401	402	402	403	404	405	405	406	407	408	409	409	410	411	412	413	414	415	52
9	450	451	451	452	453	454	454	455	456	457	458	459	460	461	462	462	464	465	466	467	51
10	500	501	502	502	503	504	505	506	507	508	509	510	511	512	513	514	515	516	517	518	50
11	550	551	552	553	554	554	555	556	557	558	560	561	562	563	564	565	567	568	569	570	49
12	600	601	602	603	604	605	606	607	608	609	610	612	613	614	615	617	618	619	621	622	48
13	650	651	652	653	654	655	656	658	659	660	661	663	664	665	666	668	670	671	673	674	47
14	700	701	702	703	704	706	707	708	709	711	712	714	715	716	718	719	721	723	724	726	46
15	750	751	752	754	755	756	757	759	760	762	763	765	766	768	769	771	773	774	776	778	45
16	800	801	802	804	805	806	808	809	811	812	814	816	817	819	821	822	824	826	828	830	44
17	850	851	853	854	855	857	858	860	861	863	865	866	868	870	872	874	876	877	879	881	43
18	900	901	903	904	906	907	909	911	912	914	916	917	919	921	923	925	927	929	931	933	42
19	950	951	953	954	956	958	959	961	963	965	967	968	970	972	974	976	979	981	983	985	41
20	1000	1001	1003	1005	1006	1008	1010	1012	1014	1015	1017	1019	1021	1023	1026	1028	1030	1032	1035	1037	40
21	1050	1051	1053	1055	1057	1058	1060	1062	1064	1066	1068	1070	1072	1074	1077	1079	1082	1084	1086	1089	39
22	1100	1102	1103	1105	1107	1109	1111	1113	1115	1117	1119	1121	1124	1126	1128	1131	1133	1136	1138	1141	38
23	1150	1152	1153	1155	1157	1159	1161	1163	1166	1168	1170	1172	1175	1177	1179	1182	1185	1187	1190	1192	37
24	1200	1202	1204	1206	1208	1210	1212	1214	1216	1218	1221	1223	1226	1228	1231	1233	1236	1239	1241	1244	36
25	1250	1252	1254	1256	1258	1260	1262	1265	1267	1269	1272	1274	1277	1279	1282	1285	1288	1290	1293	1296	35
26	1300	1302	1304	1306	1308	1311	1313	1315	1318	1320	1323	1325	1328	1330	1333	1336	1339	1342	1345	1348	34
27	1350	1352	1354	1356	1359	1361	1363	1366	1368	1371	1373	1376	1379	1382	1385	1387	1391	1394	1397	1400	33
28	1400	1402	1404	1407	1409	1411	1414	1416	1419	1422	1424	1427	1430	1433	1436	1439	1442	1445	1448	1452	32
29	1450	1452	1454	1457	1459	1462	1464	1467	1470	1472	1475	1478	1481	1484	1487	1490	1494	1497	1500	1504	31
30	1500	1502	1505	1507	1510	1512	1515	1518	1520	1523	1526	1529	1532	1535	1538	1542	1545	1548	1552	1555	30

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TAN 15°-20°

FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	15°		16°		17°		18°		19°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.2679492	.2867454	.2867454	.3057307	.3057307	.3249197	.3249197	.3443276	.3443276	.3639702	60
1	.2682610	.2864305	.2870602	.3054126	.3060488	.3245981	.3252413	.3440023	.3446530	.3636408	59
2	.2685728	.2861159	.2873751	.3050946	.3063670	.3242766	.3255630	.3436770	.3449785	.3633115	58
3	.2688847	.2858012	.2876900	.3047767	.3066852	.3239552	.3258848	.3433518	.3453040	.3629823	57
4	.2691967	.2854866	.2880050	.3044588	.3070034	.3236338	.3262066	.3430266	.3456296	.3626531	56
5	.2695087	.2851720	.2883201	.3041410	.3073218	.3233125	.3265284	.3427015	.3459553	.3623240	55
6	.2698207	.2848575	.2886352	.3038232	.3076402	.3229912	.3268504	.3423765	.3462810	.3619949	54
7	.2701328	.2845430	.2889503	.3035055	.3079586	.3226700	.3271724	.3420516	.3466068	.3616660	53
8	.2704449	.2842286	.2892655	.3031879	.3082771	.3223489	.3274944	.3417267	.3469327	.3613371	52
9	.2707571	.2839143	.2895808	.3028703	.3085957	.3220278	.3278165	.3414019	.3472586	.3610082	51
10	.2710694	.2835999	.2898961	.3025527	.3089143	.3217067	.3281387	.3410771	.3475846	.3606795	50
11	.2713817	.2832857	.2902114	.3022352	.3092330	.3213858	.3284610	.3407524	.3479107	.3603508	49
12	.2716940	.2829715	.2905269	.3019178	.3095517	.3210649	.3287833	.3404278	.3482368	.3600222	48
13	.2720064	.2826573	.2908423	.3016004	.3098705	.3207440	.3291056	.3401032	.3485630	.3596936	47
14	.2723188	.2823432	.2911578	.3012831	.3101893	.3204232	.3294281	.3397787	.3488893	.3593651	46
15	.2726313	.2820292	.2914734	.3009658	.3105083	.3201025	.3297505	.3394543	.3492156	.3590367	45
16	.2729438	.2817152	.2917890	.3006486	.3108272	.3197819	.3300731	.3391299	.3495420	.3587083	44
17	.2732564	.2814012	.2921047	.3003315	.3111462	.3194613	.3303957	.3388056	.3498685	.3583801	43
18	.2735690	.2810873	.2924205	.3000144	.3114653	.3191407	.3307184	.3384813	.3501950	.3580518	42
19	.2738817	.2807735	.2927363	.2996973	.3117845	.3188202	.3310411	.3381571	.3505216	.3577237	41
20	.2741945	.2804597	.2930521	.2993803	.3121036	.3184998	.3313639	.3378330	.3508483	.3573956	40
21	.2745072	.2801459	.2933680	.2990634	.3124229	.3181794	.3316868	.3375090	.3511750	.3570676	39
22	.2748201	.2798322	.2936839	.2987465	.3127422	.3178591	.3320097	.3371850	.3515018	.3567397	38
23	.2751330	.2795186	.2939999	.2984297	.3130616	.3175389	.3323327	.3368610	.3518287	.3564118	37
24	.2754459	.2792050	.2943160	.2981129	.3133810	.3172187	.3326557	.3365372	.3521556	.3560840	36
25	.2757589	.2788915	.2946321	.2977962	.3137005	.3168986	.3329788	.3362134	.3524826	.3557562	35
26	.2760719	.2785780	.2949483	.2974796	.3140200	.3165785	.3333020	.3358896	.3528096	.3554286	34
27	.2763850	.2782646	.2952645	.2971630	.3143396	.3162585	.3336252	.3355660	.3531368	.3551010	33
28	.2766981	.2779512	.2955808	.2968464	.3146593	.3159385	.3339485	.3352424	.3534640	.3547734	32
29	.2770113	.2776378	.2958971	.2965299	.3149790	.3156186	.3342719	.3349188	.3537912	.3544460	31
30	.2773245	.2773245	.2962135	.2962135	.3152988	.3152988	.3345953	.3345953	.3541186	.3541186	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	15°				16°				17°				18°				19°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	52	52	52	52	53	53	53	53	53	53	53	53	54	54	54	54	54	54	55	55	59
2	104	104	104	105	105	105	105	106	106	106	107	107	107	107	108	108	109	109	109	109	58
3	156	156	157	157	157	158	158	159	159	159	160	160	161	161	162	162	163	163	164	164	57
4	208	208	209	209	210	210	211	212	212	213	213	214	214	215	216	216	217	218	218	219	56
5	260	260	261	262	262	263	264	264	265	266	267	267	268	269	270	270	271	272	273	274	55
6	312	312	313	314	315	316	316	317	318	319	320	321	322	323	324	325	326	326	327	328	54
7	364	365	365	366	367	368	369	370	371	372	373	374	375	376	377	379	380	381	382	383	53
8	416	417	418	419	420	421	422	423	424	425	426	428	429	430	431	433	434	435	437	438	52
9	468	469	470	471	472	473	475	476	477	478	480	481	482	484	485	487	488	490	491	493	51
10	520	521	522	523	525	526	527	529	530	532	533	535	536	538	539	541	542	544	546	547	50
11	572	573	574	576	577	579	580	582	583	585	586	588	590	591	593	595	597	598	600	602	49
12	624	625	627	628	630	631	633	635	636	638	640	641	643	645	647	649	651	653	655	657	48
13	676	677	679	680	682	684	686	687	689	691	693	695	697	699	701	703	705	707	709	712	47
14	728	729	731	733	735	736	738	740	742	744	746	748	750	753	755	757	759	762	764	766	46
15	780	781	783	785	787	789	791	793	795	797	800	802	804	806	809	811	814	816	819	821	45
16	831	833	835	837	839	842	844	846	848	851	853	855	858	860	863	865	868	870	873	876	44
17	883	885	888	890	892	894	897	899	901	904	906	909	911	914	917	919	922	925	928	931	43
18	935	938	940	942	944	947	949	952	954	957	959	962	965	968	971	973	976	979	982	985	42
19	987	990	992	994	997	999	1002	1005	1007	1010	1013	1016	1019	1021	1024	1028	1031	1034	1037	1040	41
20	1039	1042	1044	1047	1049	1052	1055	1058	1060	1063	1066	1069	1072	1075	1078	1082	1085	1088	1091	1095	40
21	1091	1094	1096	1099	1102	1105	1108	1110	1113	1116	1119	1123	1126	1129	1132	1136	1139	1142	1146	1149	39
22	1143	1146	1149	1151	1154	1157	1160	1163	1166	1170	1173	1176	1179	1183	1186	1190	1193	1197	1201	1204	38
23	1195	1198	1201	1204	1207	1210	1213	1216	1219	1223	1226	1229	1233	1236	1240	1244	1247	1251	1255	1259	37
24	1247	1250	1253	1256	1259	1262	1266	1269	1272	1276	1279	1283	1287	1290	1294	1298	1302	1306	1310	1314	36
25	1299	1302	1305	1308	1312	1315	1318	1322	1325	1329	1333	1336	1340	1344	1348	1352	1356	1360	1364	1368	35
26	1351	1354	1358	1361	1364	1368	1371	1375	1378	1382	1386	1390	1394	1398	1402	1406	1410	1414	1419	1423	34
27	1403	1406	1410	1413	1417	1420	1424	1428	1431	1435	1439	1443	1447	1452	1456	1460	1464	1469	1473	1478	33
28	1455	1458	1462	1466	1469	1473	1477	1481	1484	1489	1492	1497	1501	1505	1510	1514	1519	1523	1528	1533	32
29	1507	1511	1514	1518	1522	1525	1529	1533	1537	1542	1546	1550	1554	1559	1564	1568	1573	1578	1582	1587	31
30	1559	1563	1566	1570	1574	1578	1582	1586	1591	1595	1599	1604	1608	1613	1618	1622	1627	1632	1637	1642	30

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FUNCTION FOR EACH MINUTE

SIN 20°-25°

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	20°		21°		22°		23°		24°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.3420201	.3583679	.3583679	.3746066	.3746066	.3907311	.3907311	.4067366	.4067366	.4226183	60
1	.3422935	.3580964	.3586395	.3743369	.3748763	.3904633	.3909989	.4064709	.4070024	.4223546	59
2	.3425668	.3578248	.3589110	.3740671	.3751459	.3901955	.3912666	.4062051	.4072681	.4220909	58
3	.3428400	.3575531	.3591825	.3737973	.3754156	.3899277	.3915343	.4059393	.4075337	.4218272	57
4	.3431133	.3572814	.3594540	.3735275	.3756852	.3896598	.3918019	.4056734	.4077993	.4215634	56
5	.3433865	.3570097	.3597254	.3732577	.3759547	.3893919	.3920695	.4054075	.4080649	.4212996	55
6	.3436597	.3567380	.3599968	.3729878	.3762243	.3891240	.3923371	.4051416	.4083305	.4210358	54
7	.3439329	.3564662	.3602682	.3727179	.3764938	.3888560	.3926047	.4048756	.4085960	.4207719	53
8	.3442060	.3561944	.3605395	.3724479	.3767632	.3885880	.3928722	.4046096	.4088615	.4205080	52
9	.3444791	.3559226	.3608108	.3721779	.3770327	.3883199	.3931397	.4043436	.4091269	.4202441	51
10	.3447521	.3556508	.3610821	.3719079	.3773021	.3880518	.3934071	.4040775	.4093923	.4199801	50
11	.3450252	.3553789	.3613534	.3716379	.3775714	.3877837	.3936745	.4038114	.4096577	.4197161	49
12	.3452982	.3551070	.3616246	.3713678	.3778408	.3875156	.3939419	.4035453	.4099230	.4194521	48
13	.3455712	.3548350	.3618958	.3710977	.3781101	.3872474	.3942093	.4032791	.4101883	.4191880	47
14	.3458441	.3545630	.3621669	.3708276	.3783794	.3869792	.3944766	.4030129	.4104536	.4189239	46
15	.3461171	.3542910	.3624380	.3705574	.3786486	.3867110	.3947439	.4027467	.4107189	.4186597	45
16	.3463900	.3540190	.3627091	.3702872	.3789178	.3864427	.3950111	.4024804	.4109841	.4183956	44
17	.3466628	.3537469	.3629802	.3700170	.3791870	.3861744	.3952783	.4022141	.4112492	.4181313	43
18	.3469357	.3534748	.3632512	.3697468	.3794562	.3859060	.3955455	.4019478	.4115144	.4178671	42
19	.3472085	.3532027	.3635222	.3694765	.3797253	.3856377	.3958127	.4016814	.4117795	.4176028	41
20	.3474812	.3529306	.3637932	.3692061	.3799944	.3853693	.3960798	.4014150	.4120445	.4173385	40
21	.3477540	.3526584	.3640641	.3689358	.3802634	.3851008	.3963468	.4011486	.4123096	.4170741	39
22	.3480267	.3523862	.3643351	.3686654	.3805324	.3848324	.3966139	.4008821	.4125745	.4168097	38
23	.3482994	.3521139	.3646059	.3683950	.3808014	.3845639	.3968809	.4006156	.4128395	.4165453	37
24	.3485720	.3518416	.3648768	.3681246	.3810704	.3842953	.3971479	.4003490	.4131044	.4162808	36
25	.3488447	.3515693	.3651476	.3678541	.3813393	.3840268	.3974148	.4000825	.4133693	.4160163	35
26	.3491173	.3512970	.3654184	.3675836	.3816082	.3837582	.3976818	.3998158	.4136342	.4157517	34
27	.3493898	.3510246	.3656891	.3673130	.3818770	.3834895	.3979486	.3995492	.4138990	.4154872	33
28	.3496624	.3507523	.3659599	.3670425	.3821459	.3832209	.3982155	.3992825	.4141638	.4152226	32
29	.3499349	.3504798	.3662306	.3667719	.3824147	.3829522	.3984823	.3990158	.4144285	.4149579	31
30	.3502074	.3502074	.3665012	.3665012	.3826834	.3826834	.3987491	.3987491	.4146932	.4146932	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	20°				21°				22°				23°				24°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	46	46	45	45	45	45	45	45	45	45	45	45	45	45	44	44	44	44	44	44	59
2	91	91	91	91	91	90	90	90	90	90	90	89	89	89	89	89	89	88	88	88	58
3	137	136	136	136	136	136	135	135	135	135	134	134	134	134	133	133	133	133	132	132	57
4	182	182	182	181	181	181	180	180	180	179	179	179	178	178	178	178	177	177	176	176	56
5	228	227	227	227	226	226	226	225	225	224	224	224	223	223	222	222	221	221	221	220	55
6	273	273	272	272	272	271	271	270	270	269	269	268	268	267	267	266	266	265	265	264	54
7	319	318	318	317	317	316	316	315	315	314	314	313	312	312	311	311	310	309	309	308	53
8	364	364	363	363	362	361	361	360	360	359	358	358	357	356	356	355	354	354	353	352	52
9	410	409	409	408	407	407	406	405	405	404	403	402	402	401	400	399	399	398	397	396	51
10	456	455	454	453	453	452	451	450	450	449	448	447	446	445	445	444	443	442	441	440	50
11	501	500	499	499	498	497	496	495	494	494	493	492	491	490	489	488	487	486	485	484	49
12	547	546	545	544	543	542	541	540	539	538	538	537	536	534	534	533	531	530	529	528	48
13	592	591	590	589	588	587	586	585	584	583	582	581	580	579	578	577	576	575	574	572	47
14	638	637	636	635	634	633	632	631	629	628	627	626	625	624	622	621	620	619	618	616	46
15	683	682	681	680	679	678	677	676	674	673	672	671	669	668	667	666	664	663	662	660	45
16	729	728	727	725	724	723	722	721	719	718	717	715	714	713	711	710	709	707	706	704	44
17	774	773	772	771	769	768	767	766	764	763	761	760	759	757	756	754	753	751	750	748	43
18	820	819	817	816	815	814	812	811	809	808	806	805	803	802	800	799	797	796	794	792	42
19	866	864	863	861	860	859	857	856	854	853	851	850	848	846	845	843	842	840	838	836	41
20	911	910	908	907	905	904	902	901	899	897	896	894	893	891	889	888	886	884	882	881	40
21	957	955	954	952	951	949	947	946	944	942	941	939	937	935	934	932	930	928	926	925	39
22	1002	1001	999	997	996	994	992	991	989	987	985	984	982	980	978	976	974	972	971	969	38
23	1048	1046	1044	1043	1041	1039	1037	1036	1034	1032	1030	1029	1026	1024	1023	1021	1019	1017	1015	1013	37
24	1093	1092	1090	1088	1086	1084	1083	1081	1079	1077	1075	1073	1071	1069	1067	1065	1063	1061	1059	1057	36
25	1139	1137	1135	1133	1132	1130	1128	1126	1124	1122	1120	1118	1116	1113	1112	1110	1107	1105	1103	1101	35
26	1184	1182	1181	1179	1177	1175	1173	1171	1169	1167	1165	1163	1160	1158	1156	1154	1152	1149	1147	1145	34
27	1230	1228	1226	1224	1222	1220	1218	1216	1214	1212	1209	1207	1205	1203	1201	1198	1196	1193	1191	1189	33
28	1275	1273	1271	1269	1267	1265	1263	1261	1259	1256	1254	1252	1250	1247	1245	1243	1240	1238	1235	1233	32
29	1321	1318	1317	1315	1313	1310	1308	1306	1304	1301	1299	1297	1294	1292	1289	1287	1284	1282	1279	1277	31
30	1367	1364	1362	1360	1358	1356	1353	1351	1349	1346	1344	1342	1339	1336	1334	1331	1329	1326	1324	1321	30

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SIN 25°-30°

FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	25°		26°		27°		28°		29°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.4226183	.4383711	.4383711	.4539905	.4539905	.4694716	.4694716	.4848096	.4848096	.5000000	60
1	.4228819	.4381097	.4386326	.4537313	.4542497	.4692147	.4697284	.4845552	.4850640	.4997481	59
2	.4231455	.4378482	.4388940	.4534721	.4545088	.4689578	.4699852	.4843007	.4853184	.4994961	58
3	.4234090	.4375866	.4391553	.4532128	.4547679	.4687009	.4702419	.4840462	.4855727	.4992441	57
4	.4236725	.4373251	.4394166	.4529535	.4550269	.4684439	.4704986	.4837916	.4858270	.4989920	56
5	.4239360	.4370634	.4396779	.4526941	.4552859	.4681869	.4707553	.4835370	.4860812	.4987399	55
6	.4241994	.4368018	.4399392	.4524347	.4555449	.4679298	.4710119	.4832824	.4863354	.4984877	54
7	.4244628	.4365401	.4402004	.4521753	.4558038	.4676727	.4712685	.4830277	.4865895	.4982355	53
8	.4247262	.4362784	.4404615	.4519158	.4560627	.4674156	.4715250	.4827730	.4868436	.4979833	52
9	.4249895	.4360166	.4407227	.4516563	.4563216	.4671584	.4717815	.4825182	.4870977	.4977310	51
10	.4252528	.4357548	.4409838	.4513967	.4565804	.4669012	.4720380	.4822634	.4873517	.4974787	50
11	.4255161	.4354930	.4412448	.4511372	.4568392	.4666439	.4722944	.4820086	.4876057	.4972264	49
12	.4257793	.4352311	.4415059	.4508775	.4570979	.4663866	.4725508	.4817537	.4878597	.4969740	48
13	.4260425	.4349692	.4417668	.4506179	.4573566	.4661293	.4728071	.4814987	.4881136	.4967215	47
14	.4263056	.4347072	.4420278	.4503582	.4576153	.4658719	.4730634	.4812438	.4883674	.4964690	46
15	.4265687	.4344453	.4422887	.4500984	.4578739	.4656145	.4733197	.4809888	.4886212	.4962165	45
16	.4268318	.4341832	.4425496	.4498387	.4581325	.4653571	.4735759	.4807337	.4888750	.4959639	44
17	.4270949	.4339212	.4428104	.4495789	.4583910	.4650996	.4738321	.4804786	.4891288	.4957113	43
18	.4273579	.4336591	.4430712	.4493190	.4586496	.4648420	.4740882	.4802235	.4893825	.4954587	42
19	.4276208	.4333970	.4433319	.4490591	.4589080	.4645845	.4743443	.4799683	.4896361	.4952060	41
20	.4278838	.4331348	.4435927	.4487992	.4591665	.4643269	.4746004	.4797131	.4898897	.4949532	40
21	.4281467	.4328726	.4438534	.4485392	.4594248	.4640692	.4748564	.4794579	.4901433	.4947005	39
22	.4284095	.4326103	.4441140	.4482792	.4596832	.4638115	.4751124	.4792026	.4903968	.4944476	38
23	.4286723	.4323481	.4443746	.4480192	.4599415	.4635538	.4753683	.4789472	.4906503	.4941948	37
24	.4289351	.4320857	.4446352	.4477591	.4601998	.4632960	.4756242	.4786919	.4909038	.4939419	36
25	.4291979	.4318234	.4448957	.4474990	.4604580	.4630382	.4758801	.4784364	.4911572	.4936889	35
26	.4294606	.4315610	.4451562	.4472388	.4607162	.4627804	.4761359	.4781810	.4914105	.4934359	34
27	.4297233	.4312986	.4454167	.4469786	.4609744	.4625225	.4763917	.4779255	.4916638	.4931829	33
28	.4299859	.4310361	.4456771	.4467184	.4612325	.4622646	.4766474	.4776700	.4919171	.4929298	32
29	.4302485	.4307736	.4459375	.4464581	.4614906	.4620066	.4769031	.4774144	.4921704	.4926767	31
30	.4305111	.4305111	.4461978	.4461978	.4617486	.4617486	.4771588	.4771588	.4924236	.4924236	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	25°				26°				27°				28°				29°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	44	44	44	44	44	44	43	43	43	43	43	43	43	43	43	43	42	42	42	42	59
2	88	88	87	87	87	87	87	87	86	86	86	86	86	85	85	85	85	85	84	84	58
3	132	132	131	131	131	130	130	130	130	129	129	129	128	128	128	128	127	127	127	126	57
4	176	175	175	175	174	174	174	173	173	172	172	172	171	171	170	170	170	169	169	168	56
5	220	219	219	218	218	217	217	216	216	216	215	215	214	214	213	213	212	212	211	210	55
6	264	263	263	262	261	261	260	260	259	259	258	257	257	256	256	255	254	254	253	253	54
7	308	307	306	306	305	304	304	303	302	302	301	300	300	299	298	298	297	296	295	295	53
8	352	351	350	349	349	348	347	347	346	345	344	343	342	342	341	340	339	338	338	337	52
9	395	395	394	393	392	391	390	390	389	388	387	386	385	384	383	383	382	381	380	379	51
10	439	439	438	437	436	435	434	433	432	431	430	429	428	427	426	425	424	423	422	421	50
11	483	482	481	480	479	478	477	476	475	474	473	472	471	470	469	468	466	465	464	463	49
12	527	526	525	524	523	522	521	520	518	517	516	515	514	512	511	510	509	508	506	505	48
13	571	570	569	568	566	565	564	563	562	560	559	558	556	555	554	553	551	550	549	547	47
14	615	614	613	611	610	609	607	606	605	603	602	601	599	598	596	595	594	592	591	589	46
15	659	658	656	655	654	652	651	649	648	647	645	644	642	641	639	638	636	635	633	631	45
16	703	702	700	699	697	696	694	693	691	690	688	686	685	683	682	680	678	677	675	673	44
17	747	745	744	742	741	739	738	736	734	733	731	729	728	726	724	723	721	719	717	716	43
18	791	789	788	786	784	783	781	779	778	776	774	772	771	769	767	765	763	761	759	758	42
19	835	833	831	830	828	826	824	823	821	819	817	815	813	811	810	808	806	804	802	800	41
20	879	877	875	873	871	870	868	866	864	862	860	858	856	854	852	850	848	846	844	842	40
21	923	921	919	917	915	913	911	909	907	905	903	901	899	897	895	893	890	888	886	884	39
22	967	965	963	961	959	957	954	952	950	948	946	944	942	940	937	935	933	931	928	926	38
23	1011	1009	1006	1004	1002	1000	998	996	994	991	989	987	985	982	980	978	975	973	970	968	37
24	1055	1052	1050	1048	1046	1044	1041	1039	1037	1034	1032	1030	1027	1025	1023	1020	1018	1015	1013	1010	36
25	1098	1096	1094	1092	1089	1087	1085	1082	1080	1078	1075	1073	1070	1068	1065	1063	1060	1058	1055	1052	35
26	1142	1140	1138	1135	1133	1130	1128	1126	1123	1121	1118	1118	1113	1110	1108	1105	1102	1100	1097	1094	34
27	1186	1184	1182	1179	1176	1174	1171	1169	1166	1164	1161	1158	1156	1153	1150	1148	1145	1142	1139	1136	33
28	1230	1228	1225	1223	1220	1217	1215	1212	1210	1207	1204	1201	1199	1196	1193	1190	1187	1184	1181	1179	32
29	1274	1272	1269	1266	1264	1261	1258	1256	1253	1250	1247	1244	1241	1238	1236	1233	1230	1227	1224	1221	31
30	1318	1316	1313	1310	1307	1304	1302	1299	1296	1293	1290	1287	1284	1281	1278	1275	1272	1269	1266	1263	30

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FUNCTION FOR EACH MINUTE

SIN 30°-35°

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	30°		31°		32°		33°		34°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.5000000	.5150381	.5150381	.5299193	.5299193	.5446390	.5446390	.5591929	.5591929	.5735764	60
1	.5002519	.5147887	.5152874	.5296726	.5301659	.5443951	.5448830	.5589517	.5594340	.5733381	59
2	.5005037	.5145393	.5155367	.5294258	.5304125	.5441510	.5451269	.5587105	.5596751	.5730998	58
3	.5007556	.5142899	.5157859	.5291790	.5306591	.5439069	.5453707	.5584692	.5599162	.5728614	57
4	.5010073	.5140404	.5160351	.5289322	.5309057	.5436628	.5456145	.5582279	.5601572	.5726229	56
5	.5012591	.5137908	.5162842	.5286853	.5311521	.5434187	.5458583	.5579865	.5603981	.5723844	55
6	.5015107	.5135413	.5165333	.5284383	.5313986	.5431744	.5461020	.5577451	.5606390	.5721459	54
7	.5017624	.5132916	.5167824	.5281914	.5316450	.5429302	.5463456	.5575036	.5608798	.5719073	53
8	.5020140	.5130420	.5170314	.5279443	.5318913	.5426859	.5465892	.5572621	.5611206	.5716686	52
9	.5022655	.5127923	.5172804	.5276973	.5321376	.5424415	.5468328	.5570206	.5613614	.5714299	51
10	.5025170	.5125425	.5175293	.5274502	.5323839	.5421971	.5470763	.5567790	.5616021	.5711912	50
11	.5027685	.5122927	.5177782	.5272030	.5326301	.5419527	.5473198	.5565373	.5618428	.5709524	49
12	.5030199	.5120429	.5180270	.5269558	.5328763	.5417082	.5475632	.5562956	.5620834	.5707136	48
13	.5032713	.5117930	.5182758	.5267085	.5331224	.5414637	.5478066	.5560539	.5623239	.5704747	47
14	.5035227	.5115431	.5185246	.5264613	.5333685	.5412191	.5480499	.5558121	.5625645	.5702357	46
15	.5037740	.5112931	.5187733	.5262139	.5336145	.5409745	.5482932	.5555702	.5628049	.5699968	45
16	.5040252	.5110431	.5190219	.5259665	.5338605	.5407298	.5485365	.5553283	.5630453	.5697577	44
17	.5042765	.5107930	.5192705	.5257191	.5341065	.5404851	.5487797	.5550864	.5632857	.5695187	43
18	.5045276	.5105429	.5195191	.5254717	.5343523	.5402403	.5490228	.5548444	.5635260	.5692795	42
19	.5047788	.5102928	.5197676	.5252241	.5345982	.5399955	.5492659	.5546024	.5637663	.5690403	41
20	.5050298	.5100426	.5200161	.5249766	.5348440	.5397507	.5495090	.5543603	.5640066	.5688011	40
21	.5052809	.5097924	.5202646	.5247290	.5350898	.5395058	.5497520	.5541182	.5642467	.5685619	39
22	.5055319	.5095421	.5205130	.5244813	.5353355	.5392608	.5499950	.5538760	.5644869	.5683225	38
23	.5057828	.5092918	.5207613	.5242336	.5355812	.5390158	.5502379	.5536338	.5647270	.5680832	37
24	.5060338	.5090414	.5210096	.5239859	.5358268	.5387708	.5504807	.5533915	.5649670	.5678437	36
25	.5062846	.5087910	.5212579	.5237381	.5360724	.5385257	.5507236	.5531492	.5652070	.5676043	35
26	.5065355	.5085406	.5215061	.5234903	.5363179	.5382806	.5509663	.5529069	.5654469	.5673648	34
27	.5067863	.5082901	.5217543	.5232424	.5365634	.5380354	.5512091	.5526645	.5656868	.5671252	33
28	.5070370	.5080396	.5220024	.5229945	.5368089	.5377902	.5514518	.5524220	.5659267	.5668856	32
29	.5072877	.5077890	.5222505	.5227466	.5370543	.5375449	.5516944	.5521795	.5661665	.5666459	31
30	.5075384	.5075384	.5224986	.5224986	.5372996	.5372996	.5519370	.5519370	.5664062	.5664062	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	30°				31°				32°				33°				34°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	42	42	42	42	42	41	41	41	41	41	41	41	41	41	40	40	40	40	40	40	59
2	84	84	84	83	83	83	83	82	82	82	82	82	81	81	81	81	80	80	80	80	58
3	126	126	125	125	125	124	124	124	123	123	123	122	122	122	121	121	121	120	120	120	57
4	168	168	167	167	166	166	165	165	164	164	164	163	163	162	162	161	161	160	160	159	56
5	210	209	209	208	208	207	207	206	206	205	204	204	203	203	202	202	201	200	200	199	55
6	252	251	251	250	249	249	248	247	247	246	245	245	244	243	243	242	241	240	240	239	54
7	294	293	292	292	291	290	289	289	288	287	286	285	285	284	283	282	281	281	280	279	53
8	336	335	334	333	332	332	331	330	329	328	327	326	325	324	323	323	322	321	320	319	52
9	378	377	376	375	374	373	372	371	370	369	368	367	366	365	364	363	362	361	360	359	51
10	420	419	418	417	416	414	413	412	411	410	409	408	407	406	404	403	402	401	400	398	50
11	462	461	460	458	457	456	455	453	452	451	450	448	447	446	445	444	442	441	440	438	49
12	504	503	501	500	499	497	496	495	493	492	491	489	488	487	485	484	482	481	480	478	48
13	546	544	543	542	540	539	537	536	535	533	531	530	529	527	526	524	523	521	520	518	47
14	588	586	585	583	582	580	579	577	576	574	572	571	569	568	566	565	563	561	559	558	46
15	630	628	627	625	623	622	620	618	617	615	613	612	610	608	606	605	603	601	599	598	45
16	672	670	668	667	665	663	661	660	658	656	654	652	651	649	647	645	643	641	639	637	44
17	714	712	710	708	706	705	703	701	699	697	695	693	691	689	687	685	683	681	679	677	43
18	756	754	752	750	748	746	744	742	740	738	736	734	732	730	728	726	724	721	719	717	42
19	798	796	794	792	790	787	785	783	781	779	777	775	773	770	768	766	764	761	759	757	41
20	840	838	836	833	831	829	827	824	822	820	818	815	813	811	809	806	804	802	799	797	40
21	882	880	877	875	873	870	868	866	863	861	859	856	854	851	849	847	844	842	839	837	39
22	924	921	919	917	914	912	909	907	905	902	899	897	895	892	889	887	884	882	879	876	38
23	966	963	961	958	956	953	951	948	946	943	940	938	935	933	930	927	925	922	919	916	37
24	1008	1005	1003	1000	997	995	992	989	987	984	981	979	976	973	970	968	965	962	959	956	36
25	1050	1047	1044	1042	1039	1036	1033	1031	1028	1025	1022	1019	1017	1014	1011	1008	1005	1002	999	996	35
26	1092	1089	1086	1083	1080	1078	1075	1072	1069	1066	1063	1060	1057	1054	1051	1048	1045	1042	1039	1036	34
27	1134	1131	1128	1125	1122	1119	1116	1113	1110	1107	1104	1101	1098	1095	1092	1089	1085	1082	1079	1076	33
28	1176	1173	1170	1167	1164	1160	1157	1154	1151	1148	1145	1142	1139	1135	1132	1129	1126	1122	1119	1115	32
29	1218	1215	1211	1208	1205	1202	1199	1196	1192	1189	1186	1182	1179	1176	1173	1169	1166	1162	1159	1155	31
30	1260	1257	1253	1250	1247	1243	1240	1237	1234	1230	1227	1223	1220	1216	1213	1210	1206	1202	1199	1195	30

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SIN 35°-40°

FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A
For second half (30' to 60') of each degree read UP in column marked B'

A'	35°		36°		37°		38°		39°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.5735764	.5877853	.5877853	.6018150	.6018150	.6156615	.6156615	.6293204	.6293204	.6427876	60
1	.5738147	.5875499	.5880206	.6015827	.6020473	.6154322	.6158907	.6290943	.6295464	.6425647	59
2	.5740529	.5873145	.5882558	.6013503	.6022795	.6152029	.6161198	.6288682	.6297724	.6423418	58
3	.5742911	.5870790	.5884910	.6011179	.6025117	.6149736	.6163489	.6286420	.6299983	.6421189	57
4	.5745292	.5868435	.5887262	.6008854	.6027439	.6147442	.6165780	.6284157	.6302242	.6418958	56
5	.5747672	.5866080	.5889613	.6006528	.6029760	.6145147	.6168069	.6281894	.6304500	.6416728	55
6	.5750053	.5863724	.5891964	.6004202	.6032080	.6142852	.6170359	.6279631	.6306758	.6414496	54
7	.5752432	.5861367	.5894314	.6001876	.6034400	.6140556	.6172648	.6277366	.6309015	.6412264	53
8	.5754811	.5859010	.5896663	.5999549	.6036719	.6138260	.6174936	.6275102	.6311272	.6410032	52
9	.5757190	.5856652	.5899012	.5997221	.6039038	.6135964	.6177224	.6272837	.6313528	.6407799	51
10	.5759568	.5854294	.5901361	.5994893	.6041356	.6133666	.6179511	.6270571	.6315784	.6405566	50
11	.5761946	.5851936	.5903709	.5992565	.6043674	.6131369	.6181798	.6268305	.6318039	.6403332	49
12	.5764323	.5849577	.5906057	.5990236	.6045991	.6129071	.6184084	.6266038	.6320293	.6401097	48
13	.5766700	.5847217	.5908404	.5987906	.6048308	.6126772	.6186370	.6263771	.6322547	.6398862	47
14	.5769076	.5844857	.5910750	.5985577	.6050624	.6124473	.6188655	.6261503	.6324800	.6396626	46
15	.5771452	.5842497	.5913096	.5983246	.6052940	.6122173	.6190939	.6259235	.6327053	.6394390	45
16	.5773827	.5840136	.5915442	.5980915	.6055255	.6119873	.6193224	.6256966	.6329306	.6392153	44
17	.5776202	.5837774	.5917787	.5978583	.6057570	.6117572	.6195507	.6254696	.6331557	.6389916	43
18	.5778576	.5835412	.5920132	.5976251	.6059884	.6115270	.6197790	.6252427	.6333809	.6387678	42
19	.5780950	.5833050	.5922476	.5973919	.6062198	.6112969	.6200073	.6250156	.6336059	.6385440	41
20	.5783323	.5830687	.5924819	.5971586	.6064511	.6110666	.6202355	.6247885	.6338310	.6383201	40
21	.5785696	.5828323	.5927163	.5969252	.6066824	.6108363	.6204636	.6245614	.6340559	.6380961	39
22	.5788069	.5825959	.5929505	.5966918	.6069136	.6106060	.6206917	.6243342	.6342808	.6378721	38
23	.5790440	.5823595	.5931847	.5964584	.6071447	.6103756	.6209198	.6241069	.6345057	.6376481	37
24	.5792812	.5821230	.5934189	.5962249	.6073758	.6101452	.6211478	.6238796	.6347305	.6374240	36
25	.5795183	.5818864	.5936530	.5959913	.6076069	.6099147	.6213757	.6236522	.6349553	.6371998	35
26	.5797553	.5816498	.5938871	.5957577	.6078379	.6096841	.6216036	.6234248	.6351800	.6369756	34
27	.5799923	.5814132	.5941211	.5955241	.6080689	.6094535	.6218314	.6231974	.6354046	.6367513	33
28	.5802292	.5811765	.5943550	.5952904	.6082998	.6092229	.6220592	.6229698	.6356292	.6365270	32
29	.5804661	.5809397	.5945889	.5950566	.6085306	.6089922	.6222870	.6227423	.6358537	.6363026	31
30	.5807030	.5807030	.5948228	.5948228	.6087614	.6087614	.6225146	.6225146	.6360782	.6360782	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	35°				36°				37°				38°				39°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	40	40	40	39	39	39	39	39	39	39	38	38	38	38	38	38	38	38	37	37	59
2	79	79	79	79	78	78	78	78	77	77	77	77	76	76	76	76	75	75	75	75	58
3	119	119	118	118	118	117	117	117	116	116	115	115	115	114	114	113	113	113	112	112	57
4	159	158	158	157	157	156	156	155	155	154	154	153	153	152	152	151	151	150	150	149	56
5	199	198	197	197	196	195	195	194	194	193	192	192	191	190	190	189	188	188	187	186	55
6	238	238	237	236	235	235	234	233	232	232	231	230	229	228	228	227	226	225	224	224	54
7	278	277	276	275	275	274	273	272	271	270	269	268	267	267	266	265	264	263	262	261	53
8	318	317	316	315	314	313	312	311	310	309	308	307	306	305	304	302	301	300	299	298	52
9	357	356	355	354	353	352	351	350	348	347	346	345	344	343	341	340	339	338	337	335	51
10	397	396	395	393	392	391	390	388	387	386	385	383	382	381	379	378	377	375	374	373	50
11	437	435	434	433	431	430	429	427	426	424	423	422	420	419	417	416	414	413	411	410	49
12	477	475	474	472	471	469	468	466	464	463	462	460	458	457	455	454	452	450	449	447	48
13	516	515	513	511	510	508	507	505	503	502	500	498	497	495	493	491	490	488	486	485	47
14	556	554	552	551	549	547	546	544	542	540	538	537	535	533	531	529	527	526	524	522	46
15	596	594	592	590	588	586	585	583	581	579	577	575	573	571	569	567	565	563	561	559	45
16	635	633	631	629	628	626	624	621	619	617	615	613	611	609	607	605	603	601	599	596	44
17	675	673	671	669	667	665	662	660	658	656	654	652	649	647	645	643	640	638	636	634	43
18	715	713	710	708	706	704	701	699	697	695	692	690	688	685	683	680	678	676	673	671	42
19	755	752	750	748	745	743	740	738	735	733	731	728	726	723	721	718	716	713	711	708	41
20	794	792	789	787	784	782	779	777	774	772	769	767	764	761	759	756	754	751	748	745	40
21	834	832	829	826	824	821	818	816	813	810	808	805	802	800	797	794	791	788	786	783	39
22	874	871	868	866	863	860	857	855	851	849	846	843	840	838	835	832	829	826	823	820	38
23	913	911	908	905	902	899	896	893	890	888	885	882	879	876	873	869	867	863	860	857	37
24	953	950	947	944	941	938	935	932	929	926	923	920	917	914	911	907	904	901	898	895	36
25	993	990	987	984	980	977	974	971	968	965	962	958	955	952	949	945	942	938	935	932	35
26	1033	1029	1026	1023	1020	1016	1013	1010	1006	1003	1000	997	993	990	987	983	980	976	973	969	34
27	1072	1069	1066	1062	1059	1056	1052	1049	1045	1042	1038	1035	1031	1028	1024	1021	1017	1014	1010	1006	33
28	1112	1109	1105	1102	1098	1095	1091	1088	1084	1080	1077	1073	1070	1066	1062	1058	1055	1051	1047	1044	32
29	1152	1148	1145	1141	1137	1134	1130	1126	1122	1119	1115	1112	1108	1104	1100	1096	1093	1089	1085	1081	31
30	1191	1188	1184	1180	1177	1173	1169	1165	1161	1158	1154	1150	1146	1142	1138	1134	1130	1126	1122	1118	30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	40°		41°		42°		43°		44°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.6427876	.6560590	.6560590	.6691306	.6691306	.6819984	.6819984	.6946584	.6946584	.7071068	60
1	.6430104	.6558395	.6562785	.6689144	.6693468	.6817856	.6822111	.6944491	.6948676	.7069011	59
2	.6432332	.6556198	.6564980	.6686981	.6695628	.6815728	.6824237	.6942398	.6950767	.7066953	58
3	.6434559	.6554002	.6567174	.6684818	.6697789	.6813599	.6826363	.6940304	.6952858	.7064894	57
4	.6436785	.6551804	.6569367	.6682655	.6699948	.6811469	.6828489	.6938209	.6954949	.7062835	56
5	.6439011	.6549607	.6571560	.6680490	.6702108	.6809339	.6830613	.6936114	.6957039	.7060776	55
6	.6441236	.6547408	.6573752	.6678326	.6704266	.6807209	.6832738	.6934018	.6959128	.7058716	54
7	.6443461	.6545209	.6575944	.6676160	.6706424	.6805078	.6834861	.6931922	.6961217	.7056655	53
8	.6445685	.6543010	.6578135	.6673994	.6708582	.6802946	.6836984	.6929825	.6963305	.7054594	52
9	.6447909	.6540810	.6580326	.6671828	.6710739	.6800813	.6839107	.6927728	.6965392	.7052532	51
10	.6450132	.6538609	.6582516	.6669661	.6712895	.6798681	.6841229	.6925630	.6967479	.7050469	50
11	.6452355	.6536408	.6584706	.6667493	.6715051	.6796547	.6843350	.6923531	.6969565	.7048406	49
12	.6454577	.6534206	.6586895	.6665325	.6717206	.6794413	.6845471	.6921432	.6971651	.7046342	48
13	.6456798	.6532004	.6589083	.6663156	.6719361	.6792278	.6847591	.6919332	.6973736	.7044278	47
14	.6459019	.6529801	.6591271	.6660987	.6721515	.6790143	.6849711	.6917232	.6975821	.7042213	46
15	.6461240	.6527598	.6593458	.6658817	.6723668	.6788007	.6851830	.6915131	.6977905	.7040147	45
16	.6463460	.6525394	.6595645	.6656646	.6725821	.6785871	.6853948	.6913029	.6979988	.7038081	44
17	.6465679	.6523189	.6597831	.6654475	.6727973	.6783734	.6856066	.6910927	.6982071	.7036014	43
18	.6467898	.6520984	.6600017	.6652304	.6730125	.6781597	.6858184	.6908824	.6984153	.7033947	42
19	.6470116	.6518778	.6602202	.6650131	.6732276	.6779459	.6860300	.6906721	.6986234	.7031879	41
20	.6472334	.6516572	.6604386	.6647959	.6734427	.6777320	.6862416	.6904617	.6988315	.7029811	40
21	.6474551	.6514366	.6606570	.6645785	.6736577	.6775181	.6864532	.6902512	.6990396	.7027741	39
22	.6476767	.6512158	.6608754	.6643612	.6738727	.6773041	.6866647	.6900407	.6992476	.7025672	38
23	.6478984	.6509951	.6610936	.6641437	.6740876	.6770901	.6868761	.6898302	.6994555	.7023601	37
24	.6481199	.6507742	.6613119	.6639262	.6743024	.6768760	.6870875	.6896195	.6996633	.7021531	36
25	.6483414	.6505533	.6615300	.6637087	.6745172	.6766618	.6872988	.6894089	.6998711	.7019459	35
26	.6485628	.6503324	.6617482	.6634910	.6747319	.6764476	.6875101	.6891981	.7000789	.7017387	34
27	.6487842	.6501114	.6619662	.6632734	.6749466	.6762333	.6877213	.6889873	.7002866	.7015314	33
28	.6490056	.6498903	.6621842	.6630557	.6751612	.6760190	.6879325	.6887765	.7004942	.7013241	32
29	.6492268	.6496692	.6624022	.6628379	.6753757	.6758046	.6881435	.6885655	.7007018	.7011167	31
30	.6494480	.6494480	.6626200	.6626200	.6755902	.6755902	.6883546	.6883546	.7009093	.7009093	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	40°				41°				42°				43°				44°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	37	37	37	37	37	37	36	36	36	36	36	36	35	35	35	35	35	35	35	34	59
2	74	74	74	73	73	73	73	72	72	72	71	71	71	71	70	70	70	69	69	69	58
3	111	111	111	110	110	109	109	109	108	108	107	107	106	106	105	105	105	104	104	103	57
4	149	148	148	147	146	146	145	145	144	144	143	142	142	141	141	140	140	139	138	138	56
5	186	185	184	184	183	182	182	181	180	179	179	178	177	177	176	175	174	174	173	172	55
6	223	222	221	220	220	219	218	217	216	215	214	214	213	212	211	210	209	208	207	207	54
7	260	259	258	257	256	255	254	253	252	251	250	249	248	247	246	245	244	243	242	241	53
8	297	296	295	294	293	292	290	289	288	287	286	285	284	282	281	280	279	278	277	275	52
9	334	333	332	331	329	328	327	326	324	323	322	320	319	318	316	315	314	313	311	310	51
10	371	370	369	367	366	364	363	362	360	359	357	356	355	353	352	350	349	347	346	344	50
11	409	407	406	404	402	401	399	398	396	395	393	392	390	388	387	385	384	382	380	379	49
12	446	444	442	441	439	437	436	434	432	431	429	427	425	424	422	420	418	417	415	413	48
13	483	481	479	477	476	474	472	470	468	467	465	463	461	459	457	455	453	452	450	448	47
14	520	518	516	514	512	510	508	506	504	502	500	498	496	494	492	490	488	486	484	482	46
15	557	555	553	551	549	547	545	543	540	538	536	534	532	530	527	525	523	521	519	516	45
16	594	592	590	588	585	583	581	579	576	574	572	570	567	565	563	560	558	556	553	551	44
17	631	629	627	624	622	620	617	615	612	610	608	605	603	600	598	595	593	590	588	585	43
18	669	666	664	661	659	656	653	651	648	646	643	641	638	636	633	630	628	625	622	620	42
19	706	703	701	698	695	693	690	687	684	682	679	676	674	671	668	665	663	660	657	654	41
20	743	740	737	735	732	729	726	723	720	718	715	712	709	706	703	700	697	695	692	689	40
21	780	777	774	771	768	765	762	760	756	754	751	748	745	742	738	735	732	729	726	723	39
22	817	814	811	808	805	802	799	796	792	790	786	783	780	777	774	770	767	764	761	757	38
23	854	851	848	845	842	838	835	832	829	825	822	819	815	812	809	805	802	799	795	792	37
24	891	888	885	881	878	875	871	868	865	861	858	854	851	847	844	840	837	833	830	826	36
25	928	925	922	918	915	911	908	904	901	897	894	890	886	883	879	875	872	868	865	861	35
26	966	962	959	955	951	948	944	940	937	933	929	926	922	918	914	911	907	903	899	895	34
27	1003	999	995	992	988	984	980	977	973	969	965	961	957	953	949	946	942	938	934	930	33
28	1040	1036	1032	1028	1025	1021	1016	1013	1009	1005	1001	997	993	989	985	981	976	972	968	964	32
29	1077	1073	1069	1065	1061	1057	1053	1049	1045	1041	1037	1032	1028	1024	1020	1016	1011	1007	1003	998	31
30	1114	1110	1106	1102	1098	1094	1089	1085	1081	1077	1072	1068	1064	1059	1055	1051	1046	1042	1037	1033	30

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SIN 45°-50°

FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	45°		46°		47°		48°		49°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.7071068	.7193398	.7193398	.7313537	.7313537	.7431448	.7431448	.7547096	.7547096	.7660444	60
1	.7073124	.7191377	.7195418	.7311553	.7315521	.7429502	.7433394	.7545187	.7549004	.7658574	59
2	.7075180	.7189355	.7197438	.7309568	.7317503	.7427554	.7435340	.7543278	.7550911	.7656704	58
3	.7077236	.7187333	.7199457	.7307583	.7319486	.7425606	.7437285	.7541368	.7552818	.7654832	57
4	.7079291	.7185310	.7201476	.7305597	.7321467	.7423658	.7439229	.7539457	.7554724	.7652960	56
5	.7081345	.7183287	.7203494	.7303610	.7323449	.7421708	.7441173	.7537546	.7556630	.7651087	55
6	.7083398	.7181263	.7205511	.7301623	.7325429	.7419758	.7443115	.7535634	.7558535	.7649214	54
7	.7085451	.7179238	.7207528	.7299635	.7327409	.7417808	.7445058	.7533721	.7560439	.7647340	53
8	.7087504	.7177213	.7209544	.7297646	.7329388	.7415857	.7446999	.7531808	.7562343	.7645465	52
9	.7089556	.7175187	.7211559	.7295657	.7331367	.7413905	.7448941	.7529894	.7564246	.7643590	51
10	.7091607	.7173161	.7213574	.7293668	.7333345	.7411953	.7450881	.7527980	.7566148	.7641714	50
11	.7093657	.7171134	.7215589	.7291677	.7335322	.7410000	.7452821	.7526065	.7568050	.7639838	49
12	.7095707	.7169106	.7217602	.7289686	.7337299	.7408046	.7454760	.7524149	.7569951	.7637960	48
13	.7097757	.7167078	.7219615	.7287695	.7339275	.7406092	.7456699	.7522233	.7571851	.7636082	47
14	.7099806	.7165049	.7221628	.7285703	.7341250	.7404137	.7458636	.7520316	.7573751	.7634204	46
15	.7101854	.7163019	.7223640	.7283710	.7343225	.7402181	.7460574	.7518398	.7575650	.7632325	45
16	.7103901	.7160989	.7225651	.7281716	.7345199	.7400225	.7462510	.7516480	.7577548	.7630445	44
17	.7105948	.7158959	.7227661	.7279722	.7347173	.7398268	.7464446	.7514561	.7579446	.7628564	43
18	.7107995	.7156927	.7229671	.7277728	.7349146	.7396311	.7466382	.7512641	.7581343	.7626683	42
19	.7110041	.7154895	.7231681	.7275732	.7351118	.7394353	.7468317	.7510721	.7583240	.7624802	41
20	.7112086	.7152863	.7233690	.7273736	.7353090	.7392394	.7470251	.7508800	.7585136	.7622919	40
21	.7114130	.7150830	.7235698	.7271740	.7355061	.7390435	.7472184	.7506879	.7587031	.7621036	39
22	.7116174	.7148796	.7237705	.7269743	.7357032	.7388475	.7474117	.7504957	.7588926	.7619152	38
23	.7118218	.7146762	.7239712	.7267745	.7359002	.7386515	.7476049	.7503034	.7590820	.7617268	37
24	.7120260	.7144727	.7241719	.7265747	.7360971	.7384553	.7477981	.7501111	.7592713	.7615383	36
25	.7122303	.7142691	.7243724	.7263748	.7362940	.7382592	.7479912	.7499187	.7594606	.7613497	35
26	.7124344	.7140655	.7245729	.7261748	.7364908	.7380629	.7481842	.7497262	.7596498	.7611611	34
27	.7126385	.7138618	.7247734	.7259748	.7366875	.7378666	.7483772	.7495337	.7598389	.7609724	33
28	.7128426	.7136581	.7249738	.7257747	.7368842	.7376703	.7485701	.7493411	.7600280	.7607837	32
29	.7130465	.7134543	.7251741	.7255746	.7370808	.7374738	.7487629	.7491484	.7602170	.7605949	31
30	.7132504	.7132504	.7253744	.7253744	.7372773	.7372773	.7489557	.7489557	.7604060	.7604060	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	45°				46°				47°				48°				49°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	34	34	34	34	34	34	33	33	33	33	33	33	32	32	32	32	32	32	31	31	59
2	69	68	68	68	67	67	67	66	66	66	66	65	65	65	64	64	64	63	63	63	58
3	103	102	102	101	101	101	100	100	99	99	98	98	97	97	96	96	95	95	94	94	57
4	137	136	136	135	135	134	134	133	132	132	131	130	130	129	128	128	127	127	126	125	56
5	171	171	170	169	168	168	167	166	165	165	164	163	162	161	161	160	159	158	157	157	55
6	206	205	204	203	202	201	200	199	198	198	197	196	195	194	193	192	191	190	189	188	54
7	240	239	238	237	236	235	234	233	231	230	229	228	227	226	225	224	223	221	220	219	53
8	274	273	272	271	269	268	267	266	264	263	262	261	260	258	257	256	254	253	252	251	52
9	309	307	306	304	303	302	300	299	298	296	295	293	292	290	289	288	286	285	283	282	51
10	343	341	340	338	337	335	334	332	331	329	328	326	324	323	321	320	318	316	315	313	50
11	377	375	374	372	370	369	367	365	364	362	360	359	357	355	353	352	350	348	346	345	49
12	411	410	408	406	404	402	400	399	397	395	393	391	389	387	386	384	382	380	378	376	48
13	446	444	442	440	438	436	434	432	430	428	426	424	422	420	418	416	413	411	409	407	47
14	480	478	476	474	472	469	467	465	463	461	459	456	454	452	450	448	445	443	441	439	46
15	514	512	510	508	505	503	501	498	496	494	491	489	487	484	482	480	477	475	472	470	45
16	549	546	544	541	539	536	534	531	529	527	524	522	519	516	514	512	509	506	504	501	44
17	583	580	578	575	572	570	567	565	562	559	557	554	552	549	546	543	541	538	535	532	43
18	617	614	612	609	606	603	601	598	595	592	590	587	584	581	578	575	573	570	567	564	42
19	651	648	646	643	640	637	634	631	628	625	622	619	616	613	610	607	604	601	598	595	41
20	686	683	680	677	674	670	667	664	661	658	655	652	649	646	643	639	636	633	630	626	40
21	720	717	714	710	707	704	701	698	694	691	688	685	681	678	675	671	668	665	661	658	39
22	754	751	748	744	741	737	734	731	727	724	721	717	714	710	707	703	700	696	693	689	38
23	788	785	782	778	775	771	768	764	760	757	753	750	746	742	739	735	732	728	724	720	37
24	823	819	816	812	808	805	801	797	794	790	786	782	779	775	771	767	763	760	756	752	36
25	857	853	850	846	842	838	834	830	827	823	819	815	811	807	803	799	795	791	787	783	35
26	891	887	883	880	876	872	868	864	860	856	852	847	843	839	835	831	827	823	819	814	34
27	926	922	917	913	909	905	901	897	893	889	884	880	876	871	867	863	859	854	850	846	33
28	960	956	951	947	943	939	934	930	926	921	917	913	908	904	900	895	891	886	882	878	32
29	994	990	985	981	977	972	968	963	959	954	950	945	941	936	932	927	922	918	913	908	31
30	1028	1024	1019	1015	1010	1006	1001	997	992	987	983	978	973	968	964	959	954	949	945	940	30

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FUNCTION FOR EACH MINUTE

SIN 50°-55°

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	50°		51°		52°		53°		54°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.7660444	.7771460	.7771460	.7880108	.7880108	.7986355	.7986355	.8090170	.8090170	.8191520	60
1	.7662314	.7769629	.7773290	.7878316	.7881898	.7984604	.7988105	.8088460	.8091879	.8189852	59
2	.7664183	.7767797	.7775120	.7876524	.7883688	.7982853	.7989855	.8086749	.8093588	.8188182	58
3	.7666051	.7765965	.7776949	.7874732	.7885477	.7981100	.7991604	.8085037	.8095296	.8186512	57
4	.7667918	.7764132	.7778777	.7872939	.7887266	.7979347	.7993352	.8083325	.8097004	.8184841	56
5	.7669785	.7762298	.7780604	.7871145	.7889054	.7977594	.7995100	.8081612	.8098710	.8183169	55
6	.7671652	.7760464	.7782431	.7869350	.7890841	.7975839	.7996847	.8079899	.8100416	.8181497	54
7	.7673517	.7758629	.7784258	.7867555	.7892627	.7974084	.7998593	.8078185	.8102122	.8179824	53
8	.7675382	.7756794	.7786084	.7865759	.7894413	.7972329	.8000338	.8076470	.8103826	.8178151	52
9	.7677246	.7754957	.7787909	.7863963	.7896198	.7970572	.8002083	.8074754	.8105530	.8176476	51
10	.7679110	.7753121	.7789733	.7862165	.7897983	.7968815	.8003827	.8073038	.8107234	.8174801	50
11	.7680973	.7751283	.7791557	.7860367	.7899767	.7967058	.8005571	.8071321	.8108936	.8173125	49
12	.7682835	.7749445	.7793380	.7858569	.7901550	.7965299	.8007314	.8069603	.8110638	.8171449	48
13	.7684697	.7747606	.7795202	.7856770	.7903333	.7963540	.8009056	.8067885	.8112339	.8169772	47
14	.7686558	.7745767	.7797024	.7854970	.7905115	.7961780	.8010797	.8066166	.8114040	.8168094	46
15	.7688418	.7743926	.7798845	.7853169	.7906896	.7960020	.8012538	.8064446	.8115740	.8166416	45
16	.7690278	.7742086	.7800665	.7851368	.7908676	.7958259	.8014278	.8062726	.8117439	.8164736	44
17	.7692137	.7740244	.7802485	.7849566	.7910456	.7956497	.8016018	.8061005	.8119137	.8163056	43
18	.7693996	.7738402	.7804304	.7847764	.7912235	.7954735	.8017756	.8059283	.8120835	.8161376	42
19	.7695853	.7736559	.7806123	.7845961	.7914014	.7952972	.8019495	.8057560	.8122532	.8159695	41
20	.7697710	.7734716	.7807940	.7844157	.7915792	.7951208	.8021232	.8055837	.8124229	.8158013	40
21	.7699567	.7732872	.7809757	.7842352	.7917569	.7949444	.8022969	.8054113	.8125925	.8156330	39
22	.7701423	.7731027	.7811574	.7840547	.7919345	.7947678	.8024705	.8052389	.8127620	.8154647	38
23	.7703278	.7729182	.7813390	.7838741	.7921121	.7945913	.8026440	.8050664	.8129314	.8152963	37
24	.7705132	.7727336	.7815205	.7836935	.7922896	.7944146	.8028175	.8048938	.8131008	.8151278	36
25	.7706986	.7725489	.7817019	.7835127	.7924671	.7942379	.8029909	.8047211	.8132701	.8149593	35
26	.7708840	.7723642	.7818833	.7833320	.7926445	.7940611	.8031642	.8045484	.8134393	.8147906	34
27	.7710692	.7721794	.7820646	.7831511	.7928218	.7938843	.8033375	.8043756	.8136084	.8146220	33
28	.7712544	.7719945	.7822459	.7829702	.7929990	.7937074	.8035107	.8042028	.8137775	.8144532	32
29	.7714395	.7718096	.7824270	.7827892	.7931762	.7935304	.8036838	.8040299	.8139466	.8142844	31
30	.7716246	.7716246	.7826082	.7826082	.7933533	.7933533	.8038569	.8038569	.8141155	.8141155	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	50°				51°				52°				53°				54°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	31	31	31	31	31	30	30	30	30	30	30	29	29	29	29	29	29	28	28	28	59
2	62	62	62	61	61	61	60	60	60	59	59	59	58	58	58	57	57	57	56	56	58
3	93	93	93	92	92	91	91	90	90	89	89	88	88	87	87	86	85	85	84	84	57
4	125	124	123	123	122	121	121	120	119	119	118	117	117	116	115	115	114	113	113	112	56
5	156	155	154	153	153	152	151	150	149	148	148	147	146	145	144	143	142	142	141	140	55
6	187	186	185	184	183	182	181	180	179	178	177	176	175	174	173	172	171	170	169	168	54
7	218	217	216	215	214	212	211	210	209	208	207	205	204	203	202	201	199	198	197	196	53
8	249	248	247	245	244	243	241	240	239	237	236	235	233	232	231	229	228	227	225	224	52
9	280	279	278	276	275	273	272	270	269	267	266	264	263	261	260	258	256	255	253	252	51
10	312	310	308	307	305	303	302	300	298	297	295	293	292	290	288	287	285	283	282	280	50
11	343	341	339	337	336	334	332	330	328	326	325	323	321	319	317	315	313	312	310	308	49
12	374	372	370	368	366	364	362	360	358	356	354	352	350	348	346	344	342	340	338	336	48
13	405	403	401	399	397	394	392	390	388	386	384	381	379	377	375	373	370	368	366	364	47
14	436	434	432	429	427	425	422	420	418	416	413	411	408	406	404	401	399	397	394	392	46
15	467	465	463	460	458	455	453	450	448	445	443	440	438	435	433	430	427	425	422	420	45
16	499	496	493	491	488	485	483	480	478	475	472	469	467	464	461	459	456	453	450	448	44
17	530	527	524	521	519	516	513	510	507	505	502	499	496	493	490	487	484	482	479	476	43
18	561	558	555	552	549	546	543	540	537	534	531	528	525	522	519	516	513	510	507	504	42
19	592	589	586	583	580	577	573	570	567	564	561	558	554	551	548	545	541	538	535	532	41
20	623	620	617	613	610	607	604	600	597	594	590	587	583	580	577	573	570	566	563	560	40
21	654	651	648	644	641	637	634	630	627	623	620	616	613	609	606	602	598	595	591	588	39
22	685	682	678	675	671	668	664	660	657	653	649	646	642	638	634	631	627	623	619	616	38
23	717	713	709	705	702	698	694	690	686	683	679	675	671	667	663	659	655	651	648	644	37
24	748	744	740	736	732	728	724	720	716	712	708	704	700	696	692	688	684	680	676	672	36
25	779	775	771	767	763	759	754	750	746	742	738	734	729	725	721	717	712	708	704	700	35
26	810	806	802	797	793	789	785	780	776	772	767	763	758	754	750	745	741	736	732	727	34
27	841	837	833	828	824	819	815	810	806	801	797	792	788	783	779	774	769	765	760	755	33
28	872	868	863	859	854	850	845	840	836	831	826	822	817	812	807	803	798	793	788	783	32
29	904	899	894	889	885	880	875	870	866	861	856	851	846	841	836	831	826	821	816	811	31
30	935	930	925	920	915	910	905	900	895	890	885	880	875	870	865	860	855	850	845	839	30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	55°		56°		57°		58°		59°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.8191520	.8290376	.8290376	.8386706	.8386706	.8480481	.8480481	.8571673	.8571673	.8660254	60
1	.8193189	.8288749	.8292002	.8385121	.8388290	.8478939	.8482022	.8570174	.8573171	.8658799	59
2	.8194856	.8287121	.8293628	.8383536	.8389873	.8477397	.8483562	.8568675	.8574668	.8657344	58
3	.8196523	.8285493	.8295252	.8381950	.8391455	.8475853	.8485102	.8567175	.8576164	.8655887	57
4	.8198189	.8283864	.8296877	.8380363	.8393037	.8474309	.8486641	.8565674	.8577660	.8654430	56
5	.8199854	.8282234	.8298500	.8378775	.8394618	.8472765	.8488179	.8564173	.8579155	.8652973	55
6	.8201519	.8280603	.8300123	.8377187	.8396199	.8471219	.8489717	.8562671	.8580649	.8651514	54
7	.8203183	.8278972	.8301745	.8375598	.8397778	.8469673	.8491254	.8561168	.8582143	.8650055	53
8	.8204846	.8277340	.8303366	.8374009	.8399357	.8468126	.8492790	.8559664	.8583635	.8648595	52
9	.8206509	.8275708	.8304987	.8372418	.8400936	.8466579	.8494325	.8558160	.8585127	.8647134	51
10	.8208170	.8274074	.8306607	.8370827	.8402513	.8465030	.8495860	.8556655	.8586619	.8645673	50
11	.8209832	.8272440	.8308226	.8369236	.8404090	.8463481	.8497394	.8555149	.8588109	.8644211	49
12	.8211492	.8270806	.8309845	.8367643	.8405666	.8461932	.8498927	.8553643	.8589599	.8642748	48
13	.8213152	.8269170	.8311463	.8366050	.8407241	.8460381	.8500459	.8552135	.8591088	.8641284	47
14	.8214811	.8267534	.8313080	.8364456	.8408816	.8458830	.8501991	.8550627	.8592576	.8639820	46
15	.8216469	.8265897	.8314696	.8362862	.8410390	.8457278	.8503522	.8549119	.8594064	.8638355	45
16	.8218127	.8264260	.8316312	.8361266	.8411963	.8455726	.8505053	.8547609	.8595551	.8636889	44
17	.8219784	.8262622	.8317927	.8359670	.8413536	.8454172	.8506582	.8546099	.8597037	.8635423	43
18	.8221440	.8260983	.8319541	.8358074	.8415108	.8452618	.8508111	.8544588	.8598523	.8633956	42
19	.8223096	.8259343	.8321155	.8356476	.8416679	.8451064	.8509639	.8543077	.8600007	.8632488	41
20	.8224751	.8257703	.8322768	.8354878	.8418249	.8449508	.8511167	.8541564	.8601491	.8631019	40
21	.8226405	.8256062	.8324380	.8353279	.8419819	.8447952	.8512693	.8540051	.8602975	.8629549	39
22	.8228059	.8254420	.8325991	.8351680	.8421388	.8446395	.8514219	.8538538	.8604457	.8628079	38
23	.8229712	.8252778	.8327602	.8350080	.8422956	.8444838	.8515745	.8537023	.8605939	.8626608	37
24	.8231364	.8251135	.8329212	.8348479	.8424524	.8443279	.8517269	.8535508	.8607420	.8625137	36
25	.8233015	.8249491	.8330822	.8346877	.8426091	.8441720	.8518793	.8533992	.8608901	.8623664	35
26	.8234666	.8247847	.8332430	.8345275	.8427657	.8440161	.8520316	.8532475	.8610380	.8622191	34
27	.8236316	.8246202	.8334038	.8343672	.8429222	.8438600	.8521839	.8530958	.8611859	.8620717	33
28	.8237965	.8244556	.8335646	.8342068	.8430787	.8437039	.8523360	.8529440	.8613337	.8619243	32
29	.8239614	.8242909	.8337252	.8340463	.8432351	.8435477	.8524881	.8527921	.8614815	.8617768	31
30	.8241262	.8241262	.8338858	.8338858	.8433914	.8433914	.8526402	.8526402	.8616292	.8616292	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	55°				56°				57°				58°				59°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	28	28	27	27	27	27	27	27	26	26	26	26	26	26	25	25	25	25	25	24	59
2	56	55	55	55	54	54	54	53	53	52	52	52	51	51	51	50	50	50	49	49	58
3	83	83	82	82	81	81	80	80	79	79	78	78	77	77	76	75	75	74	74	73	57
4	111	111	110	109	108	108	107	106	106	105	104	104	103	102	101	101	100	99	98	98	56
5	139	138	137	136	136	135	134	133	132	131	130	129	128	128	127	126	125	124	123	122	55
6	167	166	165	164	163	162	161	159	158	157	156	155	154	153	152	151	150	149	148	147	54
7	195	193	192	191	190	189	187	186	185	184	182	181	180	179	177	176	175	174	172	171	53
8	222	221	220	218	217	216	214	213	211	210	208	207	206	204	203	201	200	198	197	195	52
9	250	249	247	246	244	242	241	239	238	236	234	233	231	230	228	226	225	223	221	220	51
10	278	276	275	273	271	269	268	266	264	262	260	259	257	255	253	251	250	248	246	244	50
11	306	304	302	300	298	296	294	292	290	288	287	285	283	281	279	277	275	273	271	269	49
12	334	332	330	327	325	323	321	319	317	315	313	310	308	306	304	302	300	297	295	293	48
13	361	359	357	355	352	350	348	346	343	341	339	336	334	332	329	327	325	322	320	317	47
14	389	387	384	382	380	377	375	372	370	367	365	362	360	357	355	352	350	347	344	342	46
15	417	415	412	409	407	404	401	399	396	393	391	388	385	383	380	377	375	372	369	366	45
16	445	442	439	437	434	431	428	425	422	420	417	414	411	408	405	402	400	397	394	391	44
17	473	470	467	464	461	458	455	452	449	446	443	440	437	434	431	428	424	421	418	415	43
18	501	497	494	491	488	485	482	478	475	472	469	466	462	460	456	453	449	446	443	440	42
19	528	525	522	518	515	512	508	505	502	498	495	492	488	485	481	478	474	471	467	464	41
20	556	553	549	546	542	539	535	532	528	524	521	517	514	511	507	503	499	496	492	488	40
21	584	580	577	573	569	566	562	558	554	551	547	543	539	536	532	528	524	521	517	513	39
22	612	608	604	600	596	593	589	585	581	577	573	569	565	562	557	553	549	545	541	537	38
23	640	636	632	628	624	620	615	611	607	603	599	595	591	587	583	578	574	570	566	562	37
24	667	663	659	655	651	646	642	638	634	629	625	621	617	613	608	604	599	595	590	586	36
25	695	691	687	682	678	673	669	664	660	656	651	647	642	638	633	629	624	620	615	611	35
26	723	718	714	709	705	700	696	691	687	682	677	673	668	664	659	654	649	644	640	635	34
27	751	746	741	737	732	727	722	718	713	708	703	698	694	689	684	679	674	669	664	659	33
28	779	774	769	764	759	754	749	744	739	734	729	724	719	715	709	704	699	694	689	684	32
29	806	801	796	791	786	781	776	771	766	761	755	750	745	740	735	729	724	719	714	708	31
30	834	829	824	819	813	808	803	797	792	787	781	776	771	766	760	754	749	744	738	733	30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	60°		61°		62°		63°		64°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.8660254	.8746197	.8746197	.8829476	.8829476	.8910065	.8910065	.8987940	.8987940	.9063078	60
1	.8661708	.8744786	.8747607	.8828110	.8830841	.8908744	.8911385	.8986665	.8989215	.9061848	59
2	.8663161	.8743375	.8749016	.8826743	.8832206	.8907423	.8912705	.8985389	.8990489	.9060618	58
3	.8664614	.8741963	.8750425	.8825376	.8833569	.8906100	.8914024	.8984112	.8991763	.9059386	57
4	.8666066	.8740550	.8751832	.8824007	.8834933	.8904777	.8915342	.8982834	.8993035	.9058154	56
5	.8667517	.8739137	.8753239	.8822638	.8836295	.8903453	.8916659	.8981555	.8994307	.9056922	55
6	.8668967	.8737722	.8754645	.8821269	.8837656	.8902128	.8917975	.8980276	.8995578	.9055688	54
7	.8670417	.8736307	.8756051	.8819898	.8839017	.8900803	.8919291	.8978996	.8996848	.9054454	53
8	.8671866	.8734891	.8757455	.8818527	.8840377	.8899476	.8920606	.8977715	.8998117	.9053219	52
9	.8673314	.8733475	.8758859	.8817155	.8841736	.8898149	.8921920	.8976433	.8999386	.9051983	51
10	.8674762	.8732058	.8760263	.8815782	.8843095	.8896822	.8923234	.8975151	.9000654	.9050746	50
11	.8676209	.8730640	.8761665	.8814409	.8844453	.8895493	.8924546	.8973868	.9001921	.9049509	49
12	.8677655	.8729221	.8763067	.8813035	.8845810	.8894164	.8925858	.8972584	.9003188	.9048271	48
13	.8679100	.8727801	.8764468	.8811660	.8847166	.8892834	.8927169	.8971299	.9004453	.9047032	47
14	.8680544	.8726381	.8765868	.8810284	.8848522	.8891503	.8928480	.8970014	.9005718	.9045792	46
15	.8681988	.8724960	.8767268	.8808907	.8849876	.8890171	.8929789	.8968727	.9006982	.9044551	45
16	.8683431	.8723538	.8768666	.8807530	.8851230	.8888839	.8931098	.8967440	.9008246	.9043310	44
17	.8684874	.8722116	.8770064	.8806152	.8852584	.8887506	.8932406	.8966153	.9009508	.9042068	43
18	.8686315	.8720693	.8771462	.8804774	.8853936	.8886172	.8933714	.8964864	.9010770	.9040825	42
19	.8687756	.8719269	.8772858	.8803394	.8855288	.8884838	.8935021	.8963575	.9012031	.9039582	41
20	.8689196	.8717844	.8774254	.8802014	.8856639	.8883503	.8936326	.8962285	.9013292	.9038338	40
21	.8690636	.8716419	.8775649	.8800633	.8857989	.8882166	.8937632	.8960994	.9014551	.9037093	39
22	.8692074	.8714993	.8777043	.8799251	.8859339	.8880830	.8938936	.8959703	.9015810	.9035847	38
23	.8693512	.8713566	.8778437	.8797869	.8860688	.8879492	.8940240	.8958411	.9017068	.9034600	37
24	.8694949	.8712138	.8779830	.8796486	.8862036	.8878154	.8941542	.8957118	.9018325	.9033353	36
25	.8696386	.8710710	.8781222	.8795102	.8863383	.8876815	.8942844	.8955824	.9019582	.9032105	35
26	.8697821	.8709281	.8782613	.8793717	.8864730	.8875475	.8944146	.8954529	.9020838	.9030856	34
27	.8699256	.8707851	.8784004	.8792332	.8866075	.8874134	.8945446	.8953234	.9022092	.9029606	33
28	.8700691	.8706420	.8785394	.8790946	.8867420	.8872793	.8946746	.8951938	.9023347	.9028356	32
29	.8702124	.8704989	.8786783	.8789559	.8868765	.8871451	.8948045	.8950641	.9024600	.9027105	31
30	.8703557	.8703557	.8788171	.8788171	.8870108	.8870108	.8949344	.8949344	.9025853	.9025853	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	60°				61°				62°				63°				64°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	24	24	24	24	24	23	23	23	23	23	22	22	22	22	22	21	21	21	21	21	59
2	49	48	48	47	47	47	46	46	46	45	45	44	44	44	43	43	43	42	42	41	58
3	73	72	72	71	71	70	69	69	68	68	67	67	66	65	65	64	64	63	63	62	57
4	97	96	96	95	94	93	93	92	91	90	90	89	88	87	86	86	85	84	84	83	56
5	121	120	119	118	118	117	116	115	114	113	112	111	110	109	108	107	106	105	104	103	55
6	145	144	143	142	141	140	139	138	137	135	134	133	132	131	130	129	128	126	125	124	54
7	170	168	167	166	165	163	162	161	159	158	157	155	154	153	151	150	149	147	146	145	53
8	194	192	191	190	188	187	185	184	182	181	179	178	176	175	173	172	170	168	167	165	52
9	218	217	215	213	212	210	208	207	205	203	201	200	198	197	195	193	191	190	188	186	51
10	242	241	239	237	235	233	231	229	228	226	224	222	220	218	216	214	213	211	209	207	50
11	267	265	263	261	259	257	255	252	250	248	246	244	242	240	238	236	234	232	230	227	49
12	291	289	287	284	282	280	278	275	273	271	269	266	264	262	260	257	255	253	250	248	48
13	315	313	310	308	306	303	301	298	296	293	291	289	286	284	281	279	276	274	271	269	47
14	339	337	334	332	329	327	324	321	319	316	313	311	308	306	303	300	298	295	292	290	46
15	364	361	358	355	353	350	347	344	341	339	336	333	330	327	324	322	319	316	313	310	45
16	388	385	382	379	376	373	370	367	364	361	358	355	352	349	346	343	340	337	334	331	44
17	412	409	406	403	400	396	393	390	387	384	381	377	374	371	368	365	361	358	355	352	43
18	436	433	430	426	423	420	416	413	410	406	403	400	396	393	389	386	383	379	376	372	42
19	461	457	454	450	447	443	440	436	432	429	425	422	418	415	411	407	404	400	397	393	41
20	485	481	477	474	470	466	463	459	455	451	448	444	440	436	433	429	425	421	417	414	40
21	509	505	501	497	494	490	486	482	478	474	470	466	462	458	454	450	446	442	438	434	39
22	533	529	525	521	517	513	509	505	501	497	492	488	484	480	476	472	468	463	459	455	38
23	558	553	549	545	541	536	532	528	523	519	515	511	506	502	497	493	489	484	480	476	37
24	582	577	573	569	564	560	555	551	546	542	537	533	528	524	519	515	510	506	501	496	36
25	606	601	597	592	588	583	578	574	569	564	560	555	550	546	541	536	531	527	522	517	35
26	630	625	621	616	611	606	602	597	592	587	582	577	572	567	562	558	553	548	543	538	34
27	654	650	645	640	635	630	625	620	615	609	604	599	594	589	584	579	574	569	563	558	33
28	679	674	668	663	658	653	648	643	637	632	627	622	616	611	606	600	595	590	584	579	32
29	703	698	692	687	682	676	671	665	660	655	649	644	638	633	627	622	616	611	605	600	31
30	727	722	716	711	705	700	694	688	683	677	672	666	660	655	649	643	638	632	626	620	30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	65°		66°		67°		68°		69°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.9063078	.9135455	.9135455	.9205049	.9205049	.9271839	.9271839	.9335804	.9335804	.9396926	60
1	.9064307	.9134271	.9136637	.9203912	.9206185	.9270748	.9272928	.9334761	.9336846	.9395931	59
2	.9065535	.9133087	.9137819	.9202774	.9207320	.9269658	.9274016	.9333718	.9337888	.9394935	58
3	.9066762	.9131902	.9139001	.9201635	.9208455	.9268566	.9275104	.9332673	.9338928	.9393938	57
4	.9067989	.9130716	.9140181	.9200496	.9209589	.9267474	.9276191	.9331628	.9339968	.9392940	56
5	.9069215	.9129529	.9141361	.9199356	.9210722	.9266380	.9277277	.9330582	.9341007	.9391942	55
6	.9070440	.9128342	.9142540	.9198215	.9211854	.9265286	.9278363	.9329535	.9342045	.9390943	54
7	.9071665	.9127154	.9143718	.9197073	.9212986	.9264192	.9279447	.9328488	.9343082	.9389942	53
8	.9072888	.9125965	.9144895	.9195931	.9214116	.9263096	.9280531	.9327439	.9344119	.9388942	52
9	.9074111	.9124775	.9146072	.9194788	.9215246	.9262000	.9281614	.9326390	.9345154	.9387940	51
10	.9075333	.9123584	.9147247	.9193644	.9216375	.9260902	.9282696	.9325340	.9346189	.9386938	50
11	.9076554	.9122393	.9148422	.9192499	.9217504	.9259805	.9283778	.9324290	.9347223	.9385934	49
12	.9077775	.9121201	.9149597	.9191353	.9218632	.9258706	.9284858	.9323238	.9348257	.9384930	48
13	.9078995	.9120008	.9150770	.9190207	.9219758	.9257606	.9285938	.9322186	.9349289	.9383925	47
14	.9080214	.9118815	.9151943	.9189060	.9220884	.9256506	.9287017	.9321133	.9350321	.9382920	46
15	.9081432	.9117620	.9153115	.9187912	.9222010	.9255405	.9288096	.9320079	.9351352	.9381913	45
16	.9082649	.9116425	.9154286	.9186763	.9223134	.9254303	.9289173	.9319024	.9352382	.9380906	44
17	.9083866	.9115229	.9155456	.9185614	.9224258	.9253201	.9290250	.9317969	.9353412	.9379898	43
18	.9085082	.9114033	.9156626	.9184464	.9225381	.9252097	.9291326	.9316912	.9354440	.9378889	42
19	.9086297	.9112835	.9157795	.9183313	.9226503	.9250993	.9292401	.9315855	.9355468	.9377880	41
20	.9087511	.9111637	.9158963	.9182161	.9227624	.9249888	.9293475	.9314797	.9356495	.9376869	40
21	.9088725	.9110438	.9160130	.9181009	.9228745	.9248782	.9294549	.9313739	.9357521	.9375858	39
22	.9089938	.9109238	.9161297	.9179855	.9229865	.9247676	.9295622	.9312679	.9358547	.9374846	38
23	.9091150	.9108038	.9162462	.9178701	.9230984	.9246568	.9296694	.9311619	.9359571	.9373833	37
24	.9092361	.9106837	.9163627	.9177546	.9232102	.9245460	.9297765	.9310558	.9360595	.9372820	36
25	.9093572	.9105635	.9164791	.9176391	.9233220	.9244351	.9298835	.9309496	.9361618	.9371806	35
26	.9094781	.9104432	.9165955	.9175234	.9234336	.9243242	.9299905	.9308434	.9362641	.9370790	34
27	.9095990	.9103228	.9167118	.9174077	.9235452	.9242131	.9300974	.9307370	.9363662	.9369774	33
28	.9097199	.9102024	.9168279	.9172919	.9236567	.9241020	.9302042	.9306306	.9364683	.9368758	32
29	.9098406	.9100819	.9169440	.9171760	.9237682	.9239908	.9303109	.9305241	.9365703	.9367740	31
30	.9099613	.9099613	.9170601	.9170601	.9238795	.9238795	.9304176	.9304176	.9366722	.9366722	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	65°				66°				67°				68°				69°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	21	20	20	20	20	19	19	19	19	19	19	18	18	18	18	18	17	17	17	17	60
2	41	41	40	40	39	39	39	38	38	38	37	37	36	36	36	35	35	34	34	34	59
3	61	61	60	60	59	59	58	57	57	56	56	55	54	54	53	53	52	52	51	50	58
4	82	81	80	80	79	78	77	77	76	75	74	73	73	72	71	70	69	69	68	67	57
5	102	101	101	100	99	98	97	96	95	94	93	92	91	90	89	88	87	86	85	84	56
6	123	122	121	120	118	117	116	115	114	113	111	110	109	108	107	105	104	103	102	101	55
7	143	142	141	139	138	137	135	134	133	131	130	129	127	126	124	123	122	120	119	118	54
8	164	162	161	159	158	156	155	153	152	150	148	147	145	144	142	141	139	137	136	134	53
9	184	183	181	179	177	176	174	172	171	169	167	165	163	162	160	158	156	155	153	151	52
10	205	203	201	199	197	195	193	191	189	188	186	184	182	180	178	176	174	172	170	168	51
11	225	223	221	219	217	215	213	211	208	206	204	202	200	198	195	193	191	189	187	185	50
12	246	244	241	239	237	234	232	230	227	225	223	220	218	216	213	211	208	206	204	201	49
13	266	264	261	259	256	254	251	249	246	244	241	239	236	234	231	228	226	223	221	218	48
14	287	284	282	279	276	273	271	268	265	263	260	257	254	251	249	246	243	240	238	235	47
15	307	304	302	299	296	293	290	287	284	281	278	275	272	269	267	264	261	258	255	252	46
16	328	325	322	319	316	312	309	306	303	300	297	294	291	287	284	281	278	275	272	269	45
17	348	345	342	339	335	332	329	325	322	319	315	312	309	305	302	299	295	292	289	285	44
18	369	365	362	358	355	351	348	345	341	338	334	331	327	323	320	316	313	309	306	302	43
19	389	386	382	378	375	371	367	364	360	356	352	349	345	341	338	334	330	326	323	319	42
20	410	406	402	398	394	391	387	383	379	375	371	367	363	359	355	351	347	344	340	336	41
21	430	426	422	418	414	410	406	402	398	394	390	386	381	377	373	369	365	361	357	352	40
22	451	447	442	438	434	430	425	421	417	413	408	404	400	395	391	387	382	378	374	369	39
23	471	467	462	458	454	449	445	440	436	431	427	422	418	413	409	404	400	395	390	386	38
24	492	487	483	478	473	469	464	459	455	450	445	441	436	429	426	422	417	412	407	403	37
25	512	507	503	498	493	488	483	478	474	469	464	459	454	449	444	439	434	429	424	420	36
26	533	528	523	518	513	508	503	498	493	488	482	478	472	467	462	457	452	447	441	436	35
27	553	548	543	538	532	527	522	517	511	506	501	496	490	485	480	474	469	464	458	453	34
28	574	568	563	558	552	547	541	536	530	525	520	514	509	503	498	492	486	481	475	470	33
29	594	589	583	578	572	566	561	555	549	544	538	532	527	521	515	510	504	498	492	487	32
30	615	609	603	597	592	586	580	574	568	563	557	551	545	539	533	527	521	515	509	504	31
																					30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	70°		71°		72°		73°		74°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.9396926	.9455186	.9455186	.9510565	.9510565	.9563048	.9563048	.9612617	.9612617	.9659258	60
1	.9397921	.9454238	.9456132	.9509666	.9511464	.9562197	.9563898	.9611815	.9613418	.9658505	59
2	.9398914	.9453290	.9457078	.9508766	.9512361	.9561345	.9564747	.9611012	.9614219	.9657751	58
3	.9399907	.9452341	.9458023	.9507865	.9513258	.9560492	.9565595	.9610208	.9615019	.9656996	57
4	.9400899	.9451391	.9458968	.9506963	.9514154	.9559639	.9566443	.9609403	.9615818	.9656240	56
5	.9401891	.9450441	.9459911	.9506061	.9515050	.9558785	.9567290	.9608598	.9616616	.9655484	55
6	.9402881	.9449489	.9460854	.9505157	.9515944	.9557930	.9568136	.9607792	.9617413	.9654726	54
7	.9403871	.9448537	.9461795	.9504253	.9516838	.9557074	.9568981	.9606984	.9618210	.9653968	53
8	.9404860	.9447584	.9462736	.9503348	.9517731	.9556218	.9569825	.9606177	.9619005	.9653209	52
9	.9405848	.9446630	.9463677	.9502443	.9518623	.9555361	.9570669	.9605368	.9619800	.9652449	51
10	.9406835	.9445675	.9464616	.9501536	.9519514	.9554502	.9571512	.9604558	.9620594	.9651689	50
11	.9407822	.9444720	.9465555	.9500629	.9520404	.9553643	.9572354	.9603748	.9621387	.9650927	49
12	.9408808	.9443764	.9466493	.9499721	.9521294	.9552784	.9573195	.9602937	.9622180	.9650165	48
13	.9409793	.9442807	.9467430	.9498812	.9522183	.9551923	.9574035	.9602125	.9622972	.9649402	47
14	.9410777	.9441849	.9468366	.9497902	.9523071	.9551062	.9574875	.9601312	.9623762	.9648638	46
15	.9411760	.9440890	.9469301	.9496991	.9523958	.9550199	.9575714	.9600499	.9624552	.9647873	45
16	.9412743	.9439931	.9470236	.9496080	.9524844	.9549336	.9576552	.9599684	.9625342	.9647108	44
17	.9413724	.9438971	.9471170	.9495168	.9525730	.9548473	.9577389	.9598869	.9626130	.9646341	43
18	.9414705	.9438010	.9472103	.9494255	.9526615	.9547608	.9578225	.9598053	.9626917	.9645574	42
19	.9415686	.9437048	.9473035	.9493341	.9527499	.9546743	.9579060	.9597236	.9627704	.9644806	41
20	.9416665	.9436085	.9473966	.9492426	.9528382	.9545876	.9579895	.9596418	.9628490	.9644037	40
21	.9417644	.9435122	.9474897	.9491511	.9529264	.9545009	.9580729	.9595600	.9629275	.9643268	39
22	.9418621	.9434157	.9475827	.9490595	.9530146	.9544141	.9581562	.9594781	.9630060	.9642497	38
23	.9419598	.9433192	.9476756	.9489678	.9531027	.9543273	.9582394	.9593961	.9630843	.9641726	37
24	.9420575	.9432227	.9477684	.9488760	.9531907	.9542403	.9583226	.9593140	.9631626	.9640954	36
25	.9421550	.9431260	.9478612	.9487842	.9532786	.9541533	.9584056	.9592318	.9632408	.9640181	35
26	.9422525	.9430293	.9479538	.9486922	.9533664	.9540662	.9584886	.9591496	.9633189	.9639407	34
27	.9423498	.9429324	.9480464	.9486002	.9534542	.9539790	.9585715	.9590672	.9633969	.9638633	33
28	.9424471	.9428355	.9481389	.9485081	.9535418	.9538917	.9586543	.9589848	.9634748	.9637858	32
29	.9425444	.9427386	.9482313	.9484159	.9536294	.9538044	.9587371	.9589023	.9635527	.9637081	31
30	.9426415	.9426415	.9483237	.9483237	.9537170	.9537170	.9588197	.9588197	.9636305	.9636305	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	70°				71°				72°				73°				74°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	17	16	16	16	16	16	15	15	15	15	15	14	14	14	14	14	13	13	13	13	59
2	33	33	32	32	32	31	31	30	30	30	29	29	28	28	28	27	27	26	26	26	58
3	50	49	49	48	47	47	46	46	45	44	44	43	43	42	41	41	40	39	39	38	57
4	66	66	65	64	63	62	62	61	60	59	58	58	57	56	55	54	53	53	52	51	56
5	83	82	81	80	79	78	77	76	75	74	73	72	71	70	69	68	67	66	65	64	55
6	100	98	97	96	95	94	92	91	90	89	87	86	85	84	83	81	80	79	78	77	54
7	116	115	113	112	111	109	108	106	105	103	102	101	99	98	96	95	94	92	91	89	53
8	133	131	129	128	126	125	123	121	120	118	117	115	113	112	110	109	107	105	104	102	52
9	149	147	146	144	142	140	139	137	135	133	131	129	128	126	124	122	120	118	117	115	51
10	166	164	162	160	158	156	154	152	150	148	146	144	142	140	138	136	134	132	130	128	50
11	182	180	178	176	174	172	169	167	165	163	160	158	156	154	151	149	147	145	142	140	49
12	199	197	194	192	189	187	185	182	180	177	175	172	170	168	165	163	160	158	155	153	48
13	216	213	210	208	205	202	200	197	195	192	190	187	184	182	179	176	174	171	168	166	47
14	232	229	227	224	221	218	215	213	210	207	204	201	198	196	193	190	187	184	181	179	46
15	249	246	243	240	237	234	231	228	225	222	219	216	213	210	207	204	200	197	194	191	45
16	265	262	259	256	253	249	246	243	240	237	233	230	227	224	220	217	214	211	207	204	44
17	282	279	275	272	268	265	262	258	255	251	248	244	241	237	234	231	227	224	220	217	43
18	299	295	291	288	284	281	277	273	270	266	262	259	255	251	248	244	241	237	233	230	42
19	315	311	308	304	300	296	292	289	285	281	277	273	269	265	262	258	254	250	246	242	41
20	332	328	324	320	316	312	308	304	300	296	292	287	283	279	275	271	267	263	259	255	40
21	348	344	340	336	331	327	323	319	315	310	306	302	298	293	289	285	281	276	272	268	39
22	365	360	356	352	347	343	338	334	330	325	321	316	312	307	303	299	294	290	285	281	38
23	381	377	372	368	363	358	354	349	345	340	335	331	326	321	317	312	307	303	298	293	37
24	398	393	388	384	379	374	369	364	360	355	350	345	340	335	330	326	321	316	311	306	36
25	415	410	405	400	395	390	385	380	375	370	364	359	354	349	344	339	334	329	324	319	35
26	431	426	421	416	410	405	400	395	389	384	379	374	368	363	358	353	348	342	337	332	34
27	448	442	437	432	426	421	415	410	404	399	394	388	383	377	372	366	361	355	350	344	33
28	464	459	453	448	442	436	431	425	419	414	408	402	397	391	386	380	374	369	363	357	32
29	481	475	469	464	458	452	446	440	434	429	423	417	411	405	399	394	388	382	376	370	31
30	498	492	486	480	474	468	462	456	449	443	437	431	425	419	413	407	401	395	389	383	30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	75°		76°		77°		78°		79°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.9659258	.9702957	.9702957	.9743701	.9743701	.9781476	.9781476	.9816272	.9816272	.9848078	60
1	.9660011	.9702253	.9703661	.9743046	.9744355	.9780871	.9782080	.9815716	.9816826	.9847572	59
2	.9660762	.9701548	.9704363	.9742390	.9745008	.9780265	.9782684	.9815160	.9817380	.9847066	58
3	.9661513	.9700842	.9705065	.9741734	.9745660	.9779658	.9783287	.9814603	.9817933	.9846558	57
4	.9662263	.9700136	.9705766	.9741077	.9746311	.9779050	.9783889	.9814045	.9818485	.9846050	56
5	.9663012	.9699428	.9706466	.9740419	.9746962	.9778442	.9784490	.9813486	.9819037	.9845542	55
6	.9663761	.9698720	.9707165	.9739760	.9747612	.9777832	.9785090	.9812927	.9819587	.9845032	54
7	.9664508	.9698011	.9707863	.9739100	.9748261	.9777222	.9785689	.9812366	.9820137	.9844521	53
8	.9665255	.9697301	.9708561	.9738439	.9748909	.9776611	.9786288	.9811805	.9820686	.9844010	52
9	.9666001	.9696591	.9709258	.9737778	.9749556	.9775999	.9786886	.9811243	.9821234	.9843498	51
10	.9666746	.9695879	.9709953	.9737116	.9750203	.9775387	.9787483	.9810680	.9821781	.9842985	50
11	.9667490	.9695167	.9710649	.9736453	.9750849	.9774773	.9788079	.9810116	.9822327	.9842471	49
12	.9668234	.9694453	.9711343	.9735789	.9751494	.9774159	.9788674	.9809552	.9822873	.9841956	48
13	.9668977	.9693740	.9712036	.9735124	.9752138	.9773544	.9789268	.9808986	.9823417	.9841441	47
14	.9669718	.9693025	.9712729	.9734459	.9752781	.9772928	.9789862	.9808420	.9823961	.9840924	46
15	.9670459	.9692309	.9713421	.9733793	.9753423	.9772311	.9790455	.9807853	.9824504	.9840407	45
16	.9671200	.9691593	.9714112	.9733125	.9754065	.9771693	.9791047	.9807285	.9825046	.9839889	44
17	.9671939	.9690875	.9714802	.9732458	.9754706	.9771075	.9791638	.9806716	.9825587	.9839370	43
18	.9672678	.9690157	.9715491	.9731789	.9755345	.9770456	.9792228	.9806147	.9826128	.9838850	42
19	.9673415	.9689438	.9716180	.9731119	.9755985	.9769836	.9792818	.9805576	.9826668	.9838330	41
20	.9674152	.9688719	.9716867	.9730449	.9756623	.9769215	.9793406	.9805005	.9827206	.9837808	40
21	.9674888	.9687998	.9717554	.9729777	.9757260	.9768593	.9793994	.9804433	.9827744	.9837286	39
22	.9675624	.9687277	.9718240	.9729105	.9757897	.9767970	.9794581	.9803860	.9828282	.9836763	38
23	.9676358	.9686555	.9718926	.9728432	.9758533	.9767347	.9795167	.9803286	.9828818	.9836239	37
24	.9677092	.9685832	.9719610	.9727759	.9759168	.9766723	.9795752	.9802712	.9829353	.9835715	36
25	.9677825	.9685108	.9720294	.9727084	.9759802	.9766098	.9796337	.9802136	.9829888	.9835189	35
26	.9678557	.9684383	.9720976	.9726409	.9760435	.9765472	.9796921	.9801560	.9830422	.9834663	34
27	.9679288	.9683658	.9721658	.9725733	.9761068	.9764845	.9797504	.9800983	.9830955	.9834136	33
28	.9680018	.9682931	.9722339	.9725056	.9761699	.9764218	.9798086	.9800405	.9831487	.9833608	32
29	.9680748	.9682204	.9723020	.9724378	.9762330	.9763589	.9798667	.9799827	.9832019	.9833079	31
30	.9681476	.9681476	.9723699	.9723699	.9762960	.9762960	.9799247	.9799247	.9832549	.9832549	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	75°				76°				77°				78°				79°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	13	12	12	12	12	12	11	11	11	11	11	10	10	10	10	9	9	9	9	9	59
2	25	25	24	24	23	23	22	22	22	21	21	21	20	20	19	19	19	18	18	17	58
3	38	37	36	36	35	35	34	33	33	32	31	31	30	30	29	28	28	27	27	26	57
4	50	49	49	48	47	46	45	44	44	43	42	41	40	40	39	38	37	36	35	34	56
5	63	62	61	60	59	58	57	56	55	54	52	51	50	49	48	47	46	45	44	43	55
6	75	74	73	72	70	69	68	67	65	64	63	62	60	59	58	57	56	54	53	52	54
7	88	86	85	84	82	81	79	78	76	75	73	72	71	69	68	66	65	63	62	60	53
8	100	99	97	96	94	92	91	89	87	86	84	82	81	79	77	76	74	72	71	69	52
9	113	111	109	107	106	104	102	100	98	96	94	93	91	89	87	85	83	81	80	78	51
10	125	123	121	119	117	115	113	111	109	107	105	103	101	99	97	95	93	90	88	86	50
11	138	136	134	131	129	127	124	122	120	118	115	113	111	109	106	104	102	100	97	95	49
12	151	148	146	143	141	138	136	133	131	128	126	123	121	118	116	114	111	109	106	104	48
13	163	160	158	155	153	150	147	144	142	139	136	134	131	128	126	123	120	118	115	112	47
14	176	173	170	167	164	161	159	156	153	150	147	144	141	138	135	132	130	127	124	121	46
15	188	185	182	179	176	173	170	167	164	161	157	154	151	148	145	142	139	136	133	129	45
16	201	197	194	191	188	184	181	178	175	171	168	165	161	158	155	151	148	145	141	138	44
17	213	210	206	203	199	196	192	189	185	182	178	175	171	168	164	161	157	154	150	147	43
18	226	222	218	215	211	207	204	200	196	193	189	185	181	178	174	170	167	163	159	155	42
19	238	235	231	227	223	219	215	211	207	203	199	195	192	188	184	180	176	172	168	164	41
20	251	247	243	239	235	231	226	222	218	214	210	206	202	197	193	189	185	181	177	173	40
21	263	259	255	251	246	242	238	233	229	225	220	216	212	207	203	199	194	190	186	181	39
22	276	272	267	263	258	254	249	244	240	235	231	226	222	217	213	208	204	199	194	190	38
23	289	284	279	275	270	265	260	256	251	246	241	237	232	227	222	218	213	208	203	198	37
24	301	296	291	286	281	277	272	267	262	257	252	247	242	237	232	227	222	217	212	207	36
25	314	309	303	298	293	288	283	278	273	268	262	257	252	247	242	237	231	226	221	216	35
26	326	321	316	310	305	300	294	289	284	278	273	267	262	257	251	246	241	235	230	224	34
27	339	333	328	322	317	311	306	300	294	289	283	278	272	267	261	256	250	244	239	233	33
28	351	346	340	334	328	323	317	311	305	300	294	288	282	276	271	265	259	253	247	242	32
29	364	358	352	346	340	334	328	322	316	310	304	298	292	286	280	274	268	262	256	250	31
30	376	370	364	358	352	346	340	333	327	321	315	309	302	296	290	284	278	271	265	259	30

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FUNCTION FOR EACH MINUTE

SIN 80°-85°

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	80°		81°		82°		83°		84°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.9848078	.9876883	.9876883	.9902681	.9902681	.9925462	.9925462	.9945219	.9945219	.9961947	60
1	.9848582	.9876428	.9877338	.9902275	.9903085	.9925107	.9925816	.9944914	.9945523	.9961693	59
2	.9849086	.9875972	.9877792	.9901869	.9903489	.9924751	.9926169	.9944609	.9945825	.9961438	58
3	.9849589	.9875514	.9878245	.9901462	.9903891	.9924394	.9926521	.9944303	.9946127	.9961183	57
4	.9850091	.9875057	.9878697	.9901055	.9904293	.9924037	.9926873	.9943996	.9946428	.9960926	56
5	.9850593	.9874598	.9879148	.9900646	.9904694	.9923679	.9927224	.9943688	.9946729	.9960669	55
6	.9851093	.9874138	.9879599	.9900237	.9905095	.9923319	.9927573	.9943379	.9947028	.9960411	54
7	.9851593	.9873678	.9880048	.9899826	.9905494	.9922959	.9927922	.9943070	.9947327	.9960152	53
8	.9852092	.9873216	.9880497	.9899415	.9905893	.9922599	.9928271	.9942760	.9947625	.9959892	52
9	.9852590	.9872754	.9880945	.9899003	.9906290	.9922237	.9928618	.9942448	.9947921	.9959631	51
10	.9853087	.9872291	.9881392	.9898590	.9906687	.9921874	.9928965	.9942136	.9948217	.9959370	50
11	.9853583	.9871827	.9881838	.9898177	.9907083	.9921511	.9929310	.9941823	.9948513	.9959107	49
12	.9854079	.9871363	.9882284	.9897762	.9907478	.9921147	.9929655	.9941510	.9948807	.9958844	48
13	.9854574	.9870897	.9882728	.9897347	.9907873	.9920782	.9929999	.9941195	.9949101	.9958580	47
14	.9855068	.9870431	.9883172	.9896931	.9908266	.9920416	.9930342	.9940880	.9949393	.9958315	46
15	.9855561	.9869964	.9883615	.9896514	.9908659	.9920049	.9930685	.9940563	.9949685	.9958049	45
16	.9856053	.9869496	.9884057	.9896096	.9909051	.9919682	.9931026	.9940246	.9949976	.9957783	44
17	.9856544	.9869027	.9884498	.9895677	.9909442	.9919314	.9931367	.9939928	.9950266	.9957515	43
18	.9857035	.9868557	.9884939	.9895258	.9909832	.9918944	.9931706	.9939610	.9950556	.9957247	42
19	.9857524	.9868087	.9885378	.9894838	.9910221	.9918574	.9932045	.9939290	.9950844	.9956978	41
20	.9858013	.9867615	.9885817	.9894416	.9910610	.9918204	.9932384	.9938969	.9951132	.9956708	40
21	.9858501	.9867143	.9886255	.9893994	.9910997	.9917832	.9932721	.9938648	.9951419	.9956437	39
22	.9858988	.9866670	.9886692	.9893572	.9911384	.9917459	.9933057	.9938326	.9951705	.9956165	38
23	.9859475	.9866196	.9887128	.9893148	.9911770	.9917086	.9933393	.9938003	.9951990	.9955893	37
24	.9859960	.9865722	.9887564	.9892723	.9912155	.9916712	.9933728	.9937679	.9952274	.9955620	36
25	.9860445	.9865246	.9887998	.9892298	.9912540	.9916337	.9934062	.9937355	.9952557	.9955345	35
26	.9860929	.9864770	.9888432	.9891872	.9912923	.9915961	.9934395	.9937029	.9952840	.9955070	34
27	.9861412	.9864293	.9888865	.9891445	.9913306	.9915584	.9934727	.9936703	.9953122	.9954795	33
28	.9861894	.9863815	.9889297	.9891017	.9913688	.9915206	.9935058	.9936375	.9953403	.9954518	32
29	.9862375	.9863336	.9889728	.9890588	.9914069	.9914828	.9935389	.9936047	.9953683	.9954240	31
30	.9862856	.9862856	.9890159	.9890159	.9914449	.9914449	.9935719	.9935719	.9953962	.9953962	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	80°				81°				82°				83°				84°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	8	8	8	8	8	7	7	7	7	7	6	6	6	6	6	5	5	5	5	4	59
2	17	16	16	16	15	15	14	14	13	13	13	12	12	11	11	11	10	10	9	9	58
3	25	25	24	23	23	22	22	21	20	20	19	18	18	17	16	16	15	15	14	13	57
4	34	33	32	31	30	30	29	28	27	26	25	24	24	23	22	21	20	19	19	18	56
5	42	41	40	39	38	37	36	35	34	33	32	31	30	28	27	26	25	24	23	22	55
6	51	49	48	47	46	44	43	42	40	39	38	37	35	34	33	32	30	29	28	27	54
7	59	57	56	55	53	52	50	49	47	46	44	43	41	40	38	37	36	34	33	31	53
8	67	66	64	62	61	59	57	56	54	52	51	49	47	46	44	42	41	39	37	36	52
9	76	74	72	70	68	66	65	63	61	59	57	55	53	51	49	48	46	44	42	40	51
10	84	82	80	78	76	74	72	70	67	65	63	61	59	57	55	53	51	49	46	44	50
11	93	90	88	86	83	81	79	76	74	72	70	67	65	63	60	58	56	53	51	49	49
12	101	99	96	94	91	89	86	83	81	78	76	73	71	68	66	63	61	58	56	53	48
13	109	107	104	101	99	96	93	90	88	85	82	80	77	74	71	69	66	63	60	58	47
14	118	115	112	109	106	103	100	97	94	92	89	86	83	80	77	74	71	68	65	62	46
15	126	123	120	117	114	111	108	104	101	98	95	92	89	85	82	79	76	73	70	67	45
16	135	131	128	125	121	118	115	111	108	105	101	98	95	91	88	84	81	78	74	71	44
17	143	140	136	132	129	125	122	118	115	111	108	104	100	97	93	90	86	83	79	75	43
18	152	148	144	140	136	133	129	125	121	118	114	110	106	103	99	95	91	87	84	80	42
19	160	156	152	148	144	140	136	132	128	124	120	116	112	108	104	100	96	92	88	84	41
20	168	164	160	156	152	148	143	139	135	131	127	122	118	114	110	106	101	97	93	89	40
21	177	172	168	164	159	155	151	146	142	137	133	128	124	120	115	111	106	102	98	93	39
22	185	181	176	171	167	162	158	153	148	144	139	135	130	125	121	116	112	107	102	98	38
23	194	189	184	179	174	170	165	160	155	150	146	141	136	131	126	121	117	112	107	102	37
24	202	197	192	187	182	177	172	167	162	157	152	147	142	137	132	127	122	117	112	106	36
25	210	205	200	195	190	185	179	174	169	163	158	153	148	142	137	132	127	121	116	111	35
26	219	213	208	203	197	192	186	181	175	170	165	159	154	148	143	137	132	126	121	115	34
27	227	222	216	210	205	199	194	188	182	176	171	165	159	154	148	142	137	131	125	120	33
28	236	230	224	218	213	207	201	195	189	183	178	171	165	159	154	147	142	136	130	124	32
29	244	238	232	226	221	214	208	202	196	190	184	177	171	165	159	152	147	141	135	129	31
30	253	246	240	234	227	221	215	209	202	196	190	184	177	171	165	158	152	146	139	133	30

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SIN 85°-90°

FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	85°		86°		87°		88°		89°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.9961947	.9975641	.9975641	.9986295	.9986295	.9993908	.9993908	.9998477	.9998477	1.0000000	60
1	.9962200	.9975437	.9975843	.9986143	.9986447	.9993806	.9994009	.9998426	.9998527	1.0000000	59
2	.9962452	.9975233	.9976045	.9985989	.9986598	.9993704	.9994110	.9998374	.9998577	.9999998	58
3	.9962704	.9975028	.9976245	.9985835	.9986748	.9993600	.9994209	.9998321	.9998625	.9999996	57
4	.9962954	.9974822	.9976445	.9985680	.9986898	.9993495	.9994308	.9998267	.9998673	.9999993	56
5	.9963204	.9974615	.9976645	.9985524	.9987046	.9993390	.9994405	.9998213	.9998720	.9999989	55
6	.9963453	.9974408	.9976843	.9985367	.9987194	.9993284	.9994502	.9998157	.9998766	.9999985	54
7	.9963701	.9974199	.9977040	.9985209	.9987340	.9993177	.9994598	.9998101	.9998812	.9999979	53
8	.9963948	.9973990	.9977237	.9985050	.9987486	.9993069	.9994693	.9998044	.9998856	.9999973	52
9	.9964195	.9973780	.9977433	.9984891	.9987631	.9992960	.9994788	.9997986	.9998900	.9999966	51
10	.9964440	.9973569	.9977627	.9984731	.9987775	.9992851	.9994881	.9997927	.9998942	.9999958	50
11	.9964685	.9973357	.9977821	.9984570	.9987919	.9992740	.9994974	.9997867	.9998984	.9999949	49
12	.9964929	.9973145	.9978015	.9984408	.9988061	.9992629	.9995066	.9997807	.9999025	.9999939	48
13	.9965172	.9972931	.9978207	.9984245	.9988203	.9992517	.9995157	.9997745	.9999065	.9999928	47
14	.9965414	.9972717	.9978399	.9984081	.9988344	.9992404	.9995247	.9997683	.9999105	.9999917	46
15	.9965655	.9972502	.9978589	.9983917	.9988484	.9992290	.9995336	.9997620	.9999143	.9999905	45
16	.9965895	.9972286	.9978779	.9983751	.9988623	.9992176	.9995424	.9997556	.9999181	.9999892	44
17	.9966135	.9972069	.9978968	.9983585	.9988761	.9992060	.9995512	.9997492	.9999218	.9999878	43
18	.9966374	.9971851	.9979156	.9983418	.9988899	.9991944	.9995599	.9997426	.9999254	.9999863	42
19	.9966612	.9971633	.9979343	.9983250	.9989035	.9991827	.9995684	.9997360	.9999289	.9999847	41
20	.9966849	.9971413	.9979530	.9983082	.9989171	.9991709	.9995770	.9997292	.9999323	.9999831	40
21	.9967085	.9971193	.9979716	.9982912	.9989306	.9991590	.9995854	.9997224	.9999357	.9999813	39
22	.9967321	.9970972	.9979900	.9982742	.9989440	.9991470	.9995937	.9997155	.9999389	.9999795	38
23	.9967555	.9970750	.9980084	.9982570	.9989573	.9991350	.9996020	.9997086	.9999421	.9999776	37
24	.9967789	.9970528	.9980267	.9982398	.9989706	.9991228	.9996101	.9997015	.9999452	.9999756	36
25	.9968022	.9970304	.9980450	.9982225	.9989837	.9991106	.9996182	.9996943	.9999482	.9999736	35
26	.9968254	.9970080	.9980631	.9982052	.9989968	.9990983	.9996262	.9996871	.9999511	.9999714	34
27	.9968485	.9969854	.9980811	.9981877	.9990098	.9990859	.9996341	.9996798	.9999539	.9999692	33
28	.9968715	.9969628	.9980991	.9981701	.9990227	.9990734	.9996419	.9996724	.9999567	.9999668	32
29	.9968945	.9969401	.9981170	.9981525	.9990355	.9990609	.9996497	.9996649	.9999593	.9999644	31
30	.9969173	.9969173	.9981348	.9981348	.9990482	.9990482	.9996573	.9996573	.9999619	.9999619	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	85°				86°				87°				88°				89°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	4	4	4	4	3	3	3	3	3	2	2	2	2	2	1	1	1	1	0	0	59
2	8	8	8	7	7	6	6	5	5	5	4	4	3	3	3	2	2	1	1	0	58
3	13	12	11	11	10	10	9	8	8	7	6	6	5	4	4	3	3	2	1	1	57
4	17	16	15	14	13	13	12	11	10	9	8	8	7	6	5	4	3	3	2	1	56
5	21	20	19	18	17	16	15	14	13	12	11	10	8	7	6	5	4	3	2	1	55
6	25	24	23	22	20	19	18	16	15	14	13	11	10	9	8	6	5	4	3	1	54
7	30	28	27	25	24	22	20	19	18	16	15	13	12	10	9	7	6	4	3	2	53
8	34	32	30	29	27	25	23	22	20	19	17	15	14	12	10	8	7	5	3	2	52
9	38	36	34	32	30	29	26	25	23	21	19	17	15	13	11	10	8	6	4	2	51
10	42	40	38	36	34	32	29	27	25	23	21	19	17	15	13	11	8	6	4	2	50
11	46	44	42	40	37	35	32	30	28	26	23	21	19	16	14	12	9	7	5	2	49
12	51	48	46	43	41	38	35	33	30	28	25	23	20	18	15	13	10	8	5	3	48
13	55	52	49	47	44	41	38	36	33	30	28	25	22	19	16	14	11	8	5	3	47
14	59	56	53	50	47	44	41	38	36	33	30	27	24	21	18	15	12	9	6	3	46
15	63	60	57	54	51	48	44	41	38	35	32	29	25	22	19	16	13	10	6	3	45
16	68	64	61	57	54	51	47	44	41	37	34	30	27	24	20	17	14	10	7	3	44
17	72	68	65	61	57	54	50	47	43	39	36	32	29	25	22	18	14	11	7	4	43
18	76	72	68	65	61	57	53	49	46	42	38	34	30	27	23	19	15	11	8	4	42
19	80	76	72	68	64	60	56	52	48	44	40	36	32	28	24	20	16	12	8	4	41
20	84	80	76	72	68	63	59	55	51	46	42	38	34	30	25	21	17	13	8	4	40
21	89	84	80	75	71	67	61	58	53	49	44	40	35	31	27	22	18	13	9	5	39
22	93	88	84	79	74	70	64	60	56	51	47	42	37	33	28	23	19	14	9	5	38
23	97	92	87	83	78	73	67	63	58	53	49	44	39	34	29	24	19	14	10	5	37
24	101	96	91	86	81	76	70	66	61	56	51	46	41	35	30	25	20	15	10	5	36
25	105	100	95	90	85	79	73	69	63	58	53	48	42	37	32	27	21	16	11	5	35
26	110	104	99	93	88	82	76	71	66	60	55	49	44	38	33	28	22	16	11	6	34
27	114	108	102	97	91	86	79	74	68	63	57	51	46	40	34	29	23	17	11	6	33
28	118	112	106	101	95	89	82	77	71	65	59	53	47	41	36	30	24	18	12	6	32
29	123	116	110	104	98	92	85	80	74	67	61	55	49	43	37	31	25	18	12	6	31
30	127	120	114	108	101	95	88	82	76	70	64	57	51	44	38	32	25	19	13	6	30

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FUNCTION FOR EACH MINUTE

TAN 0°-5°

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	0°		1°		2°		3°		4°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.0000000	.0174551	.0174551	.0349208	.0349208	.0524078	.0524078	.0699268	.0699268	.0874887	60
1	.0002909	.0171641	.0177460	.0346295	.0352120	.0521161	.0526995	.0696345	.0702191	.0871956	59
2	.0005818	.0168731	.0180370	.0343383	.0355033	.0518244	.0529912	.0693422	.0705115	.0869025	58
3	.0008727	.0165821	.0183280	.0340471	.0357945	.0515328	.0532829	.0690499	.0708038	.0866094	57
4	.0011636	.0162912	.0186190	.0337558	.0360858	.0512411	.0535746	.0687577	.0710961	.0863163	56
5	.0014544	.0160002	.0189100	.0334646	.0363771	.0509495	.0538663	.0684654	.0713885	.0860233	55
6	.0017453	.0157093	.0192010	.0331734	.0366683	.0506578	.0541581	.0681732	.0716809	.0857302	54
7	.0020362	.0154183	.0194920	.0328822	.0369596	.0503662	.0544498	.0678809	.0719733	.0854372	53
8	.0023271	.0151273	.0197830	.0325910	.0372509	.0500746	.0547416	.0675887	.0722657	.0851442	52
9	.0026180	.0148364	.0200740	.0322998	.0375422	.0497829	.0550333	.0672965	.0725581	.0848512	51
10	.0029089	.0145454	.0203650	.0320086	.0378335	.0494913	.0553251	.0670043	.0728505	.0845583	50
11	.0031998	.0142545	.0206560	.0317174	.0381248	.0491997	.0556169	.0667121	.0731430	.0842653	49
12	.0034907	.0139635	.0209470	.0314263	.0384161	.0489082	.0559087	.0664199	.0734354	.0839723	48
13	.0037816	.0136726	.0212380	.0311351	.0387074	.0486166	.0562005	.0661278	.0737279	.0836794	47
14	.0040725	.0133817	.0215291	.0308439	.0389988	.0483250	.0564923	.0658356	.0740203	.0833865	46
15	.0043634	.0130907	.0218201	.0305528	.0392901	.0480334	.0567841	.0655435	.0743128	.0830936	45
16	.0046542	.0127998	.0221111	.0302616	.0395814	.0477419	.0570759	.0652513	.0746053	.0828007	44
17	.0049451	.0125088	.0224021	.0299705	.0398728	.0474503	.0573678	.0649592	.0748979	.0825078	43
18	.0052360	.0122179	.0226932	.0296793	.0401641	.0471588	.0576596	.0646671	.0751904	.0822150	42
19	.0055269	.0119270	.0229842	.0293882	.0404555	.0468673	.0579515	.0643750	.0754829	.0819221	41
20	.0058178	.0116361	.0232753	.0290970	.0407469	.0465757	.0582434	.0640829	.0757755	.0816293	40
21	.0061087	.0113451	.0235663	.0288059	.0410383	.0462842	.0585352	.0637908	.0760680	.0813365	39
22	.0063996	.0110542	.0238574	.0285148	.0413296	.0459927	.0588271	.0634988	.0763606	.0810437	38
23	.0066905	.0107633	.0241484	.0282236	.0416210	.0457012	.0591190	.0632067	.0766532	.0807509	37
24	.0069814	.0104724	.0244395	.0279325	.0419124	.0454097	.0594109	.0629147	.0769458	.0804581	36
25	.0072723	.0101814	.0247305	.0276414	.0422038	.0451183	.0597029	.0626226	.0772384	.0801653	35
26	.0075632	.0098905	.0250216	.0273503	.0424952	.0448268	.0599948	.0623306	.0775311	.0798726	34
27	.0078541	.0095996	.0253127	.0270592	.0427866	.0445353	.0602867	.0620386	.0778237	.0795798	33
28	.0081450	.0093087	.0256038	.0267681	.0430781	.0442438	.0605787	.0617466	.0781164	.0792871	32
29	.0084360	.0090178	.0258948	.0264770	.0433695	.0439524	.0608706	.0614546	.0784090	.0789944	31
30	.0087269	.0087269	.0261859	.0261859	.0436609	.0436609	.0611626	.0611626	.0787017	.0787017	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	0°				1°				2°				3°				4°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	59
2	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	97	98	98	58
3	145	145	145	145	145	146	146	146	146	146	146	146	146	146	146	146	146	146	146	146	57
4	194	194	194	194	194	194	194	194	194	194	194	194	194	195	195	195	195	195	195	195	56
5	242	242	242	242	242	243	243	243	243	243	243	243	243	243	243	243	244	244	244	244	55
6	291	291	291	291	291	291	291	291	291	291	291	292	292	292	292	292	292	293	293	293	54
7	339	339	339	339	339	340	340	340	340	340	340	340	340	340	341	341	341	341	342	342	53
8	388	388	388	388	388	388	388	388	388	388	389	389	389	389	389	390	390	390	390	391	52
9	436	436	436	436	436	437	437	437	437	437	437	437	438	438	438	438	438	439	439	439	51
10	485	485	485	485	485	485	485	485	485	486	486	486	486	486	487	487	487	488	488	488	50
11	533	533	533	533	533	534	534	534	534	534	534	535	535	535	535	536	536	536	537	537	49
12	582	582	582	582	582	582	582	582	582	583	583	583	584	584	584	584	585	585	585	586	48
13	630	630	630	630	630	631	631	631	631	631	631	632	632	632	633	633	634	634	634	635	47
14	679	679	679	679	679	679	679	679	680	680	680	680	681	681	681	682	682	683	683	683	46
15	727	727	727	727	727	728	728	728	728	728	729	729	729	730	730	730	731	731	732	732	45
16	776	776	776	776	776	776	776	776	777	777	777	778	778	778	779	779	780	780	781	781	44
17	824	824	824	824	824	825	825	825	825	826	826	826	826	827	827	828	828	829	829	830	43
18	873	873	873	873	873	873	873	873	874	874	874	875	875	875	876	876	877	878	878	879	42
19	921	921	921	921	921	922	922	922	922	923	923	923	924	924	925	925	926	926	927	928	41
20	970	970	970	970	970	970	970	971	971	971	972	972	972	973	973	974	974	975	976	976	40
21	1018	1018	1018	1018	1018	1019	1019	1019	1019	1020	1020	1020	1021	1021	1022	1022	1023	1024	1024	1025	39
22	1067	1067	1067	1067	1067	1067	1067	1068	1068	1068	1069	1069	1069	1070	1071	1071	1072	1072	1073	1074	38
23	1115	1115	1115	1115	1115	1116	1116	1116	1116	1117	1117	1118	1118	1119	1119	1120	1121	1121	1122	1123	37
24	1164	1164	1164	1164	1164	1164	1164	1165	1165	1165	1166	1166	1167	1167	1168	1169	1169	1170	1171	1172	36
25	1212	1212	1212	1212	1212	1213	1213	1213	1213	1214	1214	1215	1215	1216	1217	1217	1218	1219	1220	1220	35
26	1260	1260	1261	1261	1261	1261	1261	1262	1262	1262	1263	1263	1264	1265	1265	1266	1267	1268	1268	1269	34
27	1309	1309	1309	1309	1309	1310	1310	1310	1311	1311	1311	1312	1313	1313	1314	1315	1315	1316	1317	1318	33
28	1357	1357	1358	1358	1358	1358	1358	1359	1359	1360	1360	1361	1361	1362	1363	1363	1364	1365	1366	1367	32
29	1406	1406	1406	1406	1406	1407	1407	1407	1408	1408	1409	1409	1410	1410	1411	1412	1413	1414	1415	1416	31
30	1454	1454	1455	1455	1455	1455	1455	1456	1456	1457	1457	1458	1458	1459	1460	1461	1462	1463	1464	1465	30

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TAN 5°-10°

FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	5°		6°		7°		8°		9°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.0874887	.1051042	.1051042	.1227846	.1227846	.1405408	.1405408	.1583844	.1583844	.1763270	60
1	.0877818	.1048101	.1053983	.1224893	.1230798	.1402442	.1408375	.1580863	.1586826	.1760271	59
2	.0880749	.1045161	.1056925	.1221941	.1233752	.1399476	.1411342	.1577881	.1589809	.1757272	58
3	.0883681	.1042220	.1059866	.1218988	.1236705	.1396510	.1414308	.1574900	.1592791	.1754273	57
4	.0886612	.1039280	.1062808	.1216036	.1239658	.1393545	.1417276	.1571919	.1595774	.1751275	56
5	.0889544	.1036340	.1065750	.1213085	.1242612	.1390580	.1420243	.1568939	.1598757	.1748277	55
6	.0892476	.1033400	.1068692	.1210133	.1245566	.1387615	.1423211	.1565958	.1601740	.1745279	54
7	.0895408	.1030460	.1071634	.1207182	.1248520	.1384650	.1426179	.1562978	.1604724	.1742282	53
8	.0898341	.1027520	.1074576	.1204230	.1251474	.1381685	.1429147	.1559998	.1607708	.1739285	52
9	.0901273	.1024580	.1077519	.1201279	.1254429	.1378721	.1432115	.1557019	.1610692	.1736288	51
10	.0904206	.1021641	.1080462	.1198329	.1257384	.1375757	.1435084	.1554040	.1613677	.1733292	50
11	.0907138	.1018702	.1083405	.1195378	.1260339	.1372793	.1438053	.1551061	.1616662	.1730296	49
12	.0910071	.1015763	.1086348	.1192428	.1263294	.1369830	.1441022	.1548082	.1619647	.1727300	48
13	.0913004	.1012824	.1089291	.1189478	.1266249	.1366866	.1443991	.1545103	.1622632	.1724304	47
14	.0915938	.1009886	.1092234	.1186528	.1269205	.1363903	.1446961	.1542125	.1625618	.1721309	46
15	.0918871	.1006947	.1095178	.1183578	.1272161	.1360940	.1449931	.1539147	.1628603	.1718314	45
16	.0921804	.1004009	.1098122	.1180628	.1275117	.1357978	.1452901	.1536170	.1631590	.1715320	44
17	.0924738	.1001071	.1101066	.1177679	.1278073	.1355015	.1455872	.1533192	.1634576	.1712325	43
18	.0927672	.0998133	.1104010	.1174730	.1281030	.1352053	.1458842	.1530215	.1637563	.1709331	42
19	.0930606	.0995195	.1106955	.1171781	.1283986	.1349091	.1461813	.1527238	.1640550	.1706338	41
20	.0933540	.0992257	.1109899	.1168832	.1286943	.1346129	.1464784	.1524262	.1643537	.1703344	40
21	.0936474	.0989320	.1112844	.1165884	.1289900	.1343168	.1467756	.1521285	.1646525	.1700351	39
22	.0939409	.0986383	.1115789	.1162936	.1292858	.1340207	.1470727	.1518309	.1649513	.1697358	38
23	.0942344	.0983446	.1118734	.1159987	.1295815	.1337246	.1473699	.1515333	.1652501	.1694366	37
24	.0945278	.0980509	.1121680	.1157039	.1298773	.1334285	.1476672	.1512358	.1655489	.1691373	36
25	.0948213	.0977572	.1124625	.1154092	.1301731	.1331324	.1479644	.1509383	.1658478	.1688381	35
26	.0951148	.0974635	.1127571	.1151144	.1304690	.1328364	.1482617	.1506408	.1661467	.1685390	34
27	.0954084	.0971699	.1130517	.1148197	.1307648	.1325404	.1485590	.1503433	.1664456	.1682398	33
28	.0957019	.0968763	.1133463	.1145250	.1310607	.1322444	.1488563	.1500458	.1667446	.1679407	32
29	.0959955	.0965826	.1136410	.1142303	.1313566	.1319484	.1491536	.1497484	.1670436	.1676417	31
30	.0962890	.0962890	.1139356	.1139356	.1316525	.1316525	.1494510	.1494510	.1673426	.1673426	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	5°				6°				7°				8°				9°				B".
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	49	49	49	49	49	49	49	49	49	49	49	49	49	49	50	50	50	50	50	50	59
2	98	98	98	98	98	98	98	98	98	99	99	99	99	99	99	99	99	100	100	100	58
3	147	147	147	147	147	147	147	147	148	148	148	148	148	149	149	149	149	149	150	150	57
4	195	196	196	196	196	196	196	197	197	197	197	198	198	198	198	199	199	199	199	200	56
5	244	244	245	245	245	245	246	246	246	246	247	247	247	248	248	248	249	249	249	250	55
6	293	293	294	294	294	294	295	295	295	296	296	296	297	297	297	298	298	299	299	300	54
7	342	342	342	343	343	343	344	344	345	345	345	346	346	347	347	347	348	348	349	349	53
8	391	391	391	392	392	392	393	393	394	394	395	395	396	396	397	397	398	398	399	399	52
9	440	440	440	441	441	442	442	442	443	443	444	444	445	446	446	447	447	448	449	449	51
10	489	489	489	490	490	491	491	492	492	493	493	494	494	495	496	496	497	498	498	499	50
11	537	538	538	539	539	540	540	541	541	542	543	543	544	545	545	546	547	547	548	549	49
12	586	587	587	588	588	589	589	590	591	591	592	593	593	594	595	596	596	597	598	599	48
13	635	636	636	637	637	638	638	639	640	640	641	642	643	644	644	645	646	647	648	649	47
14	684	684	685	686	686	687	688	688	689	690	691	691	692	693	694	695	696	697	698	699	46
15	733	733	734	735	735	736	737	737	738	739	740	741	742	743	744	745	746	747	748	749	45
16	782	782	783	784	784	785	786	787	787	788	789	790	791	792	793	794	795	796	797	799	44
17	831	831	832	833	833	834	835	836	837	838	839	840	841	842	843	844	845	846	847	849	43
18	879	880	881	882	882	883	884	885	886	887	888	889	890	891	892	893	895	896	897	899	42
19	928	929	930	931	931	932	933	934	935	936	937	938	939	941	942	943	944	946	947	948	41
20	977	978	979	979	980	981	982	983	984	985	986	988	989	990	991	993	994	995	997	998	40
21	1026	1027	1027	1028	1029	1030	1031	1032	1034	1035	1036	1037	1038	1040	1041	1042	1044	1045	1047	1048	39
22	1075	1076	1076	1077	1078	1079	1080	1082	1083	1084	1085	1086	1088	1089	1091	1092	1093	1095	1097	1098	38
23	1124	1124	1125	1126	1127	1128	1130	1131	1132	1133	1134	1136	1137	1139	1140	1142	1143	1145	1146	1148	37
24	1172	1173	1174	1175	1176	1178	1179	1180	1181	1182	1184	1185	1187	1188	1190	1191	1193	1194	1196	1198	36
25	1221	1222	1223	1224	1225	1227	1228	1229	1230	1232	1233	1235	1236	1238	1239	1241	1243	1244	1246	1248	35
26	1270	1271	1272	1273	1274	1276	1277	1278	1280	1281	1282	1284	1286	1287	1289	1291	1292	1294	1296	1298	34
27	1319	1320	1321	1322	1323	1325	1326	1327	1329	1330	1332	1333	1335	1337	1338	1340	1342	1344	1346	1348	33
28	1368	1369	1370	1371	1372	1374	1375	1377	1378	1380	1381	1383	1384	1386	1388	1390	1392	1394	1396	1398	32
29	1417	1418	1419	1420	1421	1423	1424	1426	1427	1429	1430	1432	1434	1436	1437	1439	1441	1443	1445	1448	31
30	1466	1467	1468	1469	1471	1472	1473	1475	1477	1478	1480	1482	1483	1485	1487	1489	1491	1493	1495	1498	30

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FUNCTION FOR EACH MINUTE

SIN 10°-15°

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	10°		11°		12°		13°		14°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.1736482	.1908090	.1908090	.2079117	.2079117	.2249511	.2249511	.2419219	.2419219	.2588190	60
1	.1739346	.1905234	.1910945	.2076272	.2081962	.2246676	.2252345	.2416396	.2422041	.2585381	59
2	.1742211	.1902379	.1913801	.2073426	.2084807	.2243842	.2255179	.2413574	.2424863	.2582570	58
3	.1745075	.1899523	.1916656	.2070580	.2087652	.2241007	.2258013	.2410751	.2427685	.2579760	57
4	.1747939	.1896667	.1919510	.2067734	.2090497	.2238172	.2260846	.2407927	.2430507	.2576950	56
5	.1750803	.1893811	.1922365	.2064888	.2093341	.2235337	.2263680	.2405104	.2433329	.2574139	55
6	.1753667	.1890954	.1925220	.2062042	.2096186	.2232501	.2266513	.2402280	.2436150	.2571328	54
7	.1756531	.1888098	.1928074	.2059195	.2099030	.2229666	.2269346	.2399457	.2438971	.2568517	53
8	.1759395	.1885241	.1930928	.2056349	.2101874	.2226830	.2272179	.2396633	.2441792	.2565705	52
9	.1762258	.1882385	.1933782	.2053502	.2104718	.2223994	.2275012	.2393808	.2444613	.2562894	51
10	.1765121	.1879528	.1936636	.2050655	.2107561	.2221158	.2277844	.2390984	.2447433	.2560082	50
11	.1767984	.1876670	.1939490	.2047808	.2110405	.2218321	.2280677	.2388159	.2450254	.2557270	49
12	.1770847	.1873813	.1942344	.2044961	.2113248	.2215485	.2283509	.2385335	.2453074	.2554458	48
13	.1773710	.1870956	.1945197	.2042113	.2116091	.2212648	.2286341	.2382510	.2455894	.2551645	47
14	.1776573	.1868098	.1948050	.2039265	.2118934	.2209811	.2289172	.2379684	.2458713	.2548832	46
15	.1779435	.1865240	.1950903	.2036418	.2121777	.2206974	.2292004	.2376859	.2461533	.2546019	45
16	.1782298	.1862382	.1953756	.2033569	.2124619	.2204137	.2294835	.2374033	.2464352	.2543206	44
17	.1785160	.1859524	.1956609	.2030721	.2127462	.2201300	.2297666	.2371207	.2467171	.2540393	43
18	.1788022	.1856666	.1959461	.2027873	.2130304	.2198462	.2300497	.2368381	.2469990	.2537579	42
19	.1790884	.1853808	.1962314	.2025024	.2133146	.2195624	.2303328	.2365555	.2472809	.2534766	41
20	.1793746	.1850949	.1965166	.2022176	.2135988	.2192786	.2306159	.2362729	.2475627	.2531952	40
21	.1796607	.1848091	.1968018	.2019327	.2138829	.2189948	.2308989	.2359902	.2478445	.2529137	39
22	.1799469	.1845232	.1970870	.2016478	.2141671	.2187110	.2311819	.2357075	.2481263	.2526323	38
23	.1802330	.1842373	.1973722	.2013629	.2144512	.2184271	.2314649	.2354248	.2484081	.2523508	37
24	.1805191	.1839514	.1976573	.2010779	.2147353	.2181432	.2317479	.2351421	.2486899	.2520694	36
25	.1808052	.1836654	.1979425	.2007930	.2150194	.2178593	.2320309	.2348594	.2489716	.2517879	35
26	.1810913	.1833795	.1982276	.2005080	.2153035	.2175754	.2323138	.2345766	.2492533	.2515063	34
27	.1813774	.1830935	.1985127	.2002230	.2155876	.2172915	.2325967	.2342938	.2495350	.2512248	33
28	.1816635	.1828075	.1987978	.1999380	.2158716	.2170076	.2328796	.2340110	.2498167	.2509432	32
29	.1819495	.1825215	.1990829	.1996530	.2161556	.2167236	.2331625	.2337282	.2500984	.2506616	31
30	.1822355	.1822355	.1993679	.1993679	.2164396	.2164396	.2334454	.2334454	.2503800	.2503800	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	10°				11°				12°				13°				14°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	48	48	48	48	48	48	48	48	48	48	47	47	47	47	47	47	47	47	47	47	59
2	95	95	95	95	95	95	95	95	95	95	95	95	95	94	94	94	94	94	94	94	58
3	143	143	143	143	143	143	143	142	142	142	142	142	142	142	141	141	141	141	141	141	57
4	191	191	191	190	190	190	190	190	190	190	189	189	189	189	189	188	188	188	188	188	56
5	239	238	238	238	238	238	238	237	237	237	237	236	236	236	236	235	235	235	235	234	55
6	286	286	286	286	286	285	285	285	285	284	284	284	284	283	283	283	282	282	282	281	54
7	334	334	334	333	333	333	333	332	332	332	331	331	331	330	330	330	329	329	328	328	53
8	382	382	381	381	381	380	380	380	379	379	379	378	378	378	377	377	376	376	375	375	52
9	430	429	429	429	428	428	428	427	427	426	426	426	425	425	424	424	423	423	422	422	51
10	477	477	477	476	476	475	475	475	474	474	473	473	473	472	471	471	470	470	469	469	50
11	525	525	524	524	523	523	523	522	521	521	521	520	520	519	519	518	517	517	516	516	49
12	573	572	572	571	571	571	570	570	569	569	568	567	567	566	566	565	564	564	563	563	48
13	621	620	620	619	619	618	618	617	616	616	615	615	614	614	613	612	612	611	610	609	47
14	668	668	667	667	666	665	665	665	664	663	663	662	662	661	660	659	659	658	657	656	46
15	716	716	715	714	714	713	713	712	711	711	710	709	709	708	707	706	706	705	704	703	45
16	764	763	763	762	761	761	760	759	759	758	757	757	756	755	754	753	753	752	751	750	44
17	812	811	810	810	809	808	808	807	806	805	805	804	803	802	801	801	800	799	798	797	43
18	859	859	858	857	857	856	855	854	854	853	852	851	851	849	849	848	847	846	845	844	42
19	907	906	906	905	904	903	903	902	901	900	899	899	898	897	896	895	894	893	892	891	41
20	955	954	953	953	952	951	950	949	949	948	947	946	945	944	943	942	941	940	939	938	40
21	1003	1002	1001	1001	999	998	998	997	996	995	994	993	992	991	990	989	988	987	986	985	39
22	1050	1049	1049	1048	1047	1046	1045	1044	1043	1042	1041	1040	1040	1038	1037	1036	1035	1034	1033	1031	38
23	1098	1097	1096	1095	1095	1094	1093	1092	1091	1090	1089	1088	1087	1085	1084	1083	1082	1081	1079	1078	37
24	1146	1145	1144	1143	1142	1141	1140	1139	1138	1137	1136	1135	1134	1132	1131	1130	1129	1128	1126	1125	36
25	1194	1193	1192	1191	1190	1189	1188	1187	1186	1185	1183	1182	1181	1180	1179	1177	1176	1175	1173	1172	35
26	1241	1240	1239	1238	1237	1236	1235	1234	1233	1232	1231	1229	1228	1227	1226	1224	1223	1222	1220	1219	34
27	1289	1288	1287	1286	1285	1284	1283	1282	1280	1279	1278	1277	1276	1274	1273	1271	1270	1269	1267	1266	33
28	1337	1336	1335	1334	1332	1331	1330	1329	1328	1327	1325	1324	1323	1321	1320	1319	1317	1316	1314	1313	32
29	1385	1384	1382	1381	1380	1379	1378	1377	1375	1374	1373	1371	1370	1368	1367	1366	1364	1363	1361	1360	31
30	1432	1431	1430	1429	1428	1427	1425	1424	1423	1421	1420	1419	1417	1416	1414	1413	1411	1410	1408	1407	30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	15°		16°		17°		18°		19°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.2588190	.2756374	.2756374	.2923717	.2923717	.3090170	.3090170	.3255682	.3255682	.3420201	60
1	.2591000	.2753577	.2759170	.2920935	.2926499	.3087403	.3092936	.3252931	.3258432	.3417468	59
2	.2593810	.2750781	.2761965	.2918153	.2929280	.3084636	.3095702	.3250180	.3261182	.3414734	58
3	.2596619	.2747984	.2764761	.2915371	.2932061	.3081869	.3098468	.3247429	.3263932	.3412000	57
4	.2599428	.2745187	.2767556	.2912588	.2934842	.3079102	.3101234	.3244678	.3266681	.3409265	56
5	.2602237	.2742390	.2770352	.2909805	.2937623	.3076334	.3103999	.3241926	.3269430	.3406531	55
6	.2605045	.2739592	.2773147	.2907022	.2940403	.3073566	.3106764	.3239174	.3272179	.3403796	54
7	.2607853	.2736794	.2775941	.2904239	.2943183	.3070798	.3109529	.3236422	.3274928	.3401060	53
8	.2610662	.2733997	.2778736	.2901455	.2945963	.3068030	.3112294	.3233670	.3277676	.3398325	52
9	.2613469	.2731198	.2781530	.2898671	.2948743	.3065261	.3115058	.3230917	.3280424	.3395589	51
10	.2616277	.2728400	.2784324	.2895887	.2951522	.3062492	.3117822	.3228164	.3283172	.3392852	50
11	.2619085	.2725601	.2787118	.2893103	.2954302	.3059723	.3120586	.3225411	.3285919	.3390116	49
12	.2621892	.2722803	.2789911	.2890318	.2957081	.3056953	.3123349	.3222657	.3288666	.3387379	48
13	.2624699	.2720004	.2792704	.2887533	.2959859	.3054183	.3126112	.3219903	.3291413	.3384642	47
14	.2627506	.2717204	.2795497	.2884748	.2962638	.3051413	.3128875	.3217149	.3294160	.3381905	46
15	.2630312	.2714405	.2798290	.2881963	.2965416	.3048643	.3131638	.3214395	.3296906	.3379167	45
16	.2633118	.2711605	.2801083	.2879177	.2968194	.3045872	.3134400	.3211640	.3299653	.3376429	44
17	.2635925	.2708805	.2803875	.2876391	.2970971	.3043102	.3137163	.3208885	.3302398	.3373691	43
18	.2638730	.2706005	.2806667	.2873605	.2973749	.3040331	.3139925	.3206130	.3305144	.3370953	42
19	.2641536	.2703204	.2809459	.2870819	.2976526	.3037559	.3142686	.3203374	.3307889	.3368214	41
20	.2644342	.2700403	.2812251	.2868032	.2979303	.3034788	.3145448	.3200619	.3310634	.3365475	40
21	.2647147	.2697602	.2815042	.2865246	.2982079	.3032016	.3148209	.3197863	.3313379	.3362735	39
22	.2649952	.2694801	.2817833	.2862458	.2984856	.3029244	.3150970	.3195106	.3316123	.3359996	38
23	.2652757	.2692000	.2820624	.2859671	.2987632	.3026471	.3153730	.3192350	.3318867	.3357256	37
24	.2655561	.2689198	.2823415	.2856884	.2990408	.3023699	.3156490	.3189593	.3321611	.3354516	36
25	.2658366	.2686396	.2826205	.2854096	.2993184	.3020926	.3159250	.3186836	.3324355	.3351775	35
26	.2661170	.2683594	.2828995	.2851308	.2995959	.3018153	.3162010	.3184079	.3327098	.3349034	34
27	.2663973	.2680792	.2831785	.2848520	.2998734	.3015380	.3164770	.3181321	.3329841	.3346293	33
28	.2666777	.2677989	.2834575	.2845731	.3001509	.3012606	.3167529	.3178563	.3332584	.3343552	32
29	.2669581	.2675187	.2837364	.2842942	.3004284	.3009832	.3170288	.3175805	.3335326	.3340810	31
30	.2672384	.2672384	.2840153	.2840153	.3007058	.3007058	.3173047	.3173047	.3338069	.3338069	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	15°				16°				17°				18°				19°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	47	47	47	47	47	47	47	46	46	46	46	46	46	46	46	46	46	46	46	46	59
2	94	94	93	93	93	93	93	93	93	93	93	92	92	92	92	92	92	92	91	91	58
3	141	140	140	140	140	140	139	139	139	139	139	139	138	138	138	138	138	137	137	137	57
4	187	187	187	187	186	186	186	186	185	185	185	185	184	184	184	184	183	183	183	182	56
5	234	234	234	233	233	233	232	232	232	232	231	231	231	230	230	230	229	229	228	228	55
6	281	281	280	280	280	279	279	279	278	278	277	277	277	276	276	275	275	275	274	274	54
7	328	327	327	327	326	326	325	325	325	324	324	323	323	322	322	321	321	320	320	319	53
8	375	374	374	373	373	372	372	371	371	370	370	369	369	368	368	367	367	366	366	365	52
9	421	421	421	420	419	419	418	418	417	417	416	416	415	414	414	413	413	412	411	411	51
10	468	468	467	467	466	465	465	464	464	463	462	462	461	460	460	459	458	458	457	456	50
11	515	514	514	513	513	512	511	511	510	509	509	508	507	507	506	505	504	503	503	502	49
12	562	561	561	560	559	558	558	557	556	556	555	554	553	553	552	551	550	549	548	548	48
13	609	608	607	607	606	605	604	604	603	602	601	600	599	599	598	597	596	595	594	593	47
14	656	655	654	653	652	652	651	650	649	648	647	646	646	645	644	643	642	641	640	639	46
15	703	702	701	700	699	698	697	696	695	695	694	693	692	691	690	689	688	687	685	684	45
16	749	748	747	746	746	745	744	743	742	741	740	739	738	737	736	735	733	732	731	730	44
17	796	795	794	793	792	791	790	789	788	787	786	785	784	783	782	780	779	778	777	776	43
18	843	842	841	840	839	838	837	836	835	833	832	831	830	829	828	826	825	824	823	821	42
19	890	889	888	886	885	884	883	882	881	880	878	877	876	875	874	872	871	870	868	867	41
20	937	935	934	933	932	931	930	929	927	926	925	923	922	921	920	918	917	915	914	913	40
21	984	982	981	980	979	977	976	975	974	972	971	970	968	967	966	964	963	961	960	958	39
22	1030	1029	1028	1026	1025	1024	1023	1021	1020	1019	1017	1016	1014	1013	1012	1010	1008	1007	1005	1004	38
23	1077	1076	1074	1073	1072	1070	1069	1068	1066	1065	1064	1062	1061	1059	1057	1056	1054	1053	1051	1049	37
24	1124	1122	1121	1120	1118	1117	1116	1114	1113	1111	1110	1108	1107	1105	1103	1102	1100	1098	1097	1095	36
25	1171	1169	1168	1166	1165	1163	1162	1161	1159	1158	1156	1154	1153	1151	1149	1148	1146	1144	1142	1141	35
26	1218	1216	1215	1213	1212	1210	1209	1207	1206	1204	1202	1200	1199	1197	1195	1194	1192	1190	1188	1186	34
27	1265	1263	1261	1260	1258	1257	1255	1254	1252	1250	1248	1247	1245	1243	1241	1240	1238	1236	1234	1232	33
28	1311	1310	1308	1306	1305	1303	1301	1300	1298	1296	1295	1293	1291	1289	1287	1285	1283	1282	1280	1278	32
29	1358	1357	1355	1353	1351	1350	1348	1347	1345	1343	1341	1339	1337	1335	1333	1331	1329	1327	1325	1323	31
30	1405	1403	1402	1400	1398	1396	1394	1393	1391	1389	1387	1385	1383	1381	1379	1377	1375	1373	1371	1369	30

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FUNCTION FOR EACH MINUTE

TAN 20°-25°

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	20°		21°		22°		23°		24°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.3639702	.3838640	.3838640	.4040262	.4040262	.4244748	.4244748	.4452287	.4452287	.4663077	60
1	.3642997	.3835303	.3841978	.4036879	.4043646	.4241316	.4248182	.4448802	.4455773	.4659536	59
2	.3646292	.3831967	.3845317	.4033496	.4047031	.4237884	.4251616	.4445318	.4459260	.4655996	58
3	.3649588	.3828631	.3848656	.4030115	.4050417	.4234453	.4255051	.4441834	.4462747	.4652457	57
4	.3652885	.3825296	.3851996	.4026734	.4053804	.4231023	.4258487	.4438352	.4466236	.4648919	56
5	.3656182	.3821962	.3855337	.4023354	.4057191	.4227594	.4261924	.4434871	.4469726	.4645382	55
6	.3659480	.3818629	.3858679	.4019974	.4060579	.4224165	.4265361	.4431390	.4473216	.4641845	54
7	.3662779	.3815296	.3862021	.4016596	.4063968	.4220738	.4268800	.4427910	.4476708	.4638310	53
8	.3666079	.3811964	.3865364	.4013218	.4067358	.4217311	.4272239	.4424432	.4480200	.4634776	52
9	.3669379	.3808633	.3868708	.4009841	.4070748	.4213885	.4275680	.4420954	.4483693	.4631243	51
10	.3672680	.3805302	.3872053	.4006465	.4074139	.4210460	.4279121	.4417477	.4487187	.4627710	50
11	.3675981	.3801973	.3875398	.4003089	.4077531	.4207036	.4282563	.4414001	.4490682	.4624179	49
12	.3679284	.3798644	.3878744	.3999715	.4080924	.4203613	.4286005	.4410526	.4494178	.4620649	48
13	.3682587	.3795315	.3882091	.3996341	.4084318	.4200190	.4289449	.4407051	.4497675	.4617119	47
14	.3685890	.3791988	.3885439	.3992968	.4087713	.4196769	.4292894	.4403578	.4501173	.4613591	46
15	.3689195	.3788661	.3888787	.3989595	.4091108	.4193348	.4296339	.4400105	.4504672	.4610063	45
16	.3692500	.3785335	.3892136	.3986224	.4094504	.4189928	.4299785	.4396634	.4508171	.4606537	44
17	.3695806	.3782010	.3895486	.3982853	.4097901	.4186509	.4303232	.4393163	.4511672	.4603011	43
18	.3699112	.3778685	.3898837	.3979483	.4101299	.4183091	.4306680	.4389693	.4515173	.4599486	42
19	.3702420	.3775361	.3902189	.3976114	.4104697	.4179673	.4310129	.4386224	.4518676	.4595962	41
20	.3705728	.3772038	.3905541	.3972746	.4108097	.4176257	.4313579	.4382756	.4522179	.4592439	40
21	.3709036	.3768716	.3908894	.3969378	.4111497	.4172841	.4317030	.4379289	.4525683	.4588918	39
22	.3712346	.3765394	.3912247	.3966011	.4114898	.4169426	.4320481	.4375823	.4529188	.4585397	38
23	.3715656	.3762073	.3915602	.3962645	.4118300	.4166012	.4323933	.4372357	.4532694	.4581877	37
24	.3718967	.3758753	.3918957	.3959280	.4121703	.4162598	.4327386	.4368893	.4536201	.4578357	36
25	.3722278	.3755433	.3922313	.3955916	.4125106	.4159186	.4330840	.4365429	.4539709	.4574839	35
26	.3725590	.3752115	.3925670	.3952552	.4128510	.4155774	.4334295	.4361966	.4543218	.4571322	34
27	.3728903	.3748797	.3929027	.3949189	.4131915	.4152363	.4337751	.4358504	.4546728	.4567806	33
28	.3732217	.3745479	.3932386	.3945827	.4135321	.4148953	.4341208	.4355043	.4550238	.4564290	32
29	.3735532	.3742163	.3935745	.3942465	.4138728	.4145544	.4344665	.4351583	.4553750	.4560776	31
30	.3738847	.3738847	.3939105	.3939105	.4142136	.4142136	.4348124	.4348124	.4557263	.4557263	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	20°				21°				22°				23°				24°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	55	55	55	55	56	56	56	56	56	57	57	57	57	57	58	58	58	58	59	59	59
2	110	110	111	111	111	112	112	112	113	113	114	114	114	115	115	116	116	117	117	118	58
3	165	165	166	166	167	167	168	169	169	170	170	171	172	172	173	174	174	175	176	176	57
4	220	220	221	222	223	223	224	225	226	226	227	228	229	230	231	232	232	233	234	235	56
5	275	275	276	277	278	279	280	281	282	283	284	285	286	287	288	289	290	292	293	294	55
6	329	331	332	333	334	335	336	337	338	340	341	342	343	345	346	347	349	350	351	353	54
7	384	386	387	388	389	391	392	393	395	396	398	399	401	402	404	405	407	408	410	412	53
8	439	441	442	444	445	447	448	450	451	453	454	456	458	460	461	463	465	467	468	470	52
9	494	496	497	499	501	502	504	506	508	509	511	513	515	517	519	521	523	525	527	529	51
10	549	551	553	554	556	558	560	562	564	566	568	570	572	574	577	579	581	583	586	588	50
11	604	606	608	609	612	614	616	618	620	623	625	627	629	632	634	637	639	642	644	647	49
12	659	661	663	665	668	670	672	674	677	679	682	684	687	689	692	695	697	700	703	705	48
13	714	715	718	721	723	726	728	731	733	736	738	741	744	747	750	752	755	758	761	764	47
14	769	771	774	776	779	782	784	787	790	792	795	798	801	804	807	810	813	817	820	823	46
15	824	826	829	832	835	837	840	843	846	849	852	855	858	862	865	868	871	875	878	882	45
16	879	881	884	887	890	893	896	899	902	906	909	912	916	919	923	926	930	933	937	941	44
17	933	936	940	943	946	949	952	955	959	962	966	969	973	976	980	984	988	992	995	999	43
18	988	992	995	998	1001	1005	1008	1012	1015	1019	1023	1026	1030	1034	1038	1042	1046	1050	1054	1058	42
19	1043	1047	1050	1053	1057	1061	1064	1068	1072	1075	1079	1083	1087	1091	1095	1100	1104	1108	1113	1117	41
20	1098	1102	1105	1109	1113	1116	1120	1124	1128	1132	1136	1140	1144	1149	1153	1157	1162	1167	1171	1176	40
21	1153	1157	1161	1164	1168	1172	1176	1180	1184	1189	1193	1197	1202	1206	1211	1215	1220	1225	1230	1235	39
22	1208	1212	1216	1220	1224	1228	1232	1236	1241	1245	1250	1254	1259	1264	1268	1273	1278	1283	1288	1293	38
23	1263	1267	1271	1275	1280	1284	1288	1293	1297	1302	1307	1311	1316	1321	1326	1331	1336	1342	1347	1352	37
24	1318	1322	1326	1331	1335	1340	1344	1349	1354	1358	1363	1368	1373	1378	1384	1389	1394	1400	1405	1411	36
25	1373	1377	1382	1386	1391	1395	1400	1405	1410	1415	1420	1425	1431	1436	1441	1447	1452	1458	1464	1470	35
26	1428	1432	1437	1442	1447	1451	1456	1461	1466	1472	1477	1482	1488	1493	1499	1505	1511	1517	1522	1529	34
27	1483	1487	1492	1497	1502	1507	1512	1517	1523	1528	1534	1539	1545	1551	1557	1563	1569	1575	1581	1587	33
28	1537	1542	1548	1553	1558	1563	1568	1574	1579	1585	1591	1596	1602	1608	1614	1620	1627	1633	1640	1646	32
29	1592	1598	1603	1608	1613	1619	1624	1630	1636	1641	1647	1653	1660	1666	1672	1678	1685	1692	1698	1705	31
30	1647	1653	1658	1663	1669	1675	1680	1686	1692	1698	1704	1710	1717	1723	1730	1736	1743	1750	1757	1764	30

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TAN 25°-30°

FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	25°		26°		27°		28°		29°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.4663077	.4877326	.4877326	.5095254	.5095254	.5317094	.5317094	.5543091	.5543091	.5773503	60
1	.4666618	.4873726	.4880927	.5091591	.5098919	.5313364	.5320826	.5539288	.5546894	.5769625	59
2	.4670161	.4870126	.4884530	.5087929	.5102585	.5309634	.5324559	.5535488	.5550698	.5765748	58
3	.4673705	.4866528	.4888133	.5084267	.5106252	.5305906	.5328293	.5531688	.5554504	.5761873	57
4	.4677250	.4862931	.4891737	.5080607	.5109919	.5302178	.5332029	.5527890	.5558311	.5757999	56
5	.4680796	.4859334	.4895343	.5076948	.5113588	.5298452	.5335765	.5524093	.5562119	.5754126	55
6	.4684342	.4855739	.4898949	.5073290	.5117259	.5294727	.5339503	.5520297	.5565929	.5750255	54
7	.4687890	.4852145	.4902557	.5069633	.5120930	.5291004	.5343242	.5516502	.5569739	.5746385	53
8	.4691439	.4848552	.4906166	.5065977	.5124602	.5287281	.5346981	.5512708	.5573551	.5742516	52
9	.4694988	.4844959	.4909775	.5062322	.5128275	.5283560	.5350723	.5508916	.5577364	.5738649	51
10	.4698539	.4841368	.4913386	.5058668	.5131950	.5279839	.5354465	.5505125	.5581179	.5734783	50
11	.4702090	.4837778	.4916997	.5055015	.5135625	.5276120	.5358208	.5501335	.5584994	.5730918	49
12	.4705643	.4834189	.4920610	.5051363	.5139302	.5272402	.5361953	.5497547	.5588811	.5727054	48
13	.4709196	.4830601	.4924224	.5047713	.5142980	.5268685	.5365699	.5493759	.5592629	.5723192	47
14	.4712751	.4827014	.4927838	.5044063	.5146658	.5264969	.5369446	.5489973	.5596449	.5719331	46
15	.4716306	.4823427	.4931454	.5040415	.5150338	.5261255	.5373194	.5486188	.5600269	.5715471	45
16	.4719863	.4819842	.4935071	.5036768	.5154019	.5257541	.5376943	.5482404	.5604091	.5711612	44
17	.4723420	.4816258	.4938689	.5033121	.5157702	.5253829	.5380694	.5478621	.5607914	.5707755	43
18	.4726978	.4812675	.4942308	.5029476	.5161385	.5250117	.5384445	.5474840	.5611738	.5703899	42
19	.4730538	.4809093	.4945928	.5025832	.5165069	.5246407	.5388198	.5471060	.5615564	.5700045	41
20	.4734098	.4805512	.4949549	.5022189	.5168755	.5242698	.5391952	.5467281	.5619391	.5696191	40
21	.4737659	.4801932	.4953171	.5018547	.5172441	.5238990	.5395707	.5463503	.5623219	.5692339	39
22	.4741222	.4798352	.4956794	.5014906	.5176129	.5235284	.5399464	.5459727	.5627048	.5688488	38
23	.4744785	.4794774	.4960418	.5011266	.5179818	.5231578	.5403221	.5455951	.5630879	.5684639	37
24	.4748349	.4791197	.4964043	.5007627	.5183508	.5227874	.5406980	.5452177	.5634711	.5680791	36
25	.4751914	.4787621	.4967669	.5003989	.5187199	.5224170	.5410740	.5448404	.5638543	.5676944	35
26	.4755481	.4784046	.4971297	.5000352	.5190891	.5220468	.5414501	.5444632	.5642378	.5673098	34
27	.4759048	.4780472	.4974925	.4996717	.5194584	.5216767	.5418263	.5440862	.5646213	.5669254	33
28	.4762616	.4776899	.4978554	.4993082	.5198278	.5213067	.5422027	.5437092	.5650050	.5665410	32
29	.4766185	.4773326	.4982185	.4989449	.5201974	.5209368	.5425791	.5433324	.5653888	.5661568	31
30	.4769755	.4769755	.4985816	.4985816	.5205671	.5205671	.5429557	.5429557	.5657728	.5657728	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	25°				26°				27°				28°				29°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	59	59	59	60	60	60	61	61	61	61	62	62	62	63	63	63	63	64	64	64	59
2	118	119	119	120	120	121	121	122	122	123	123	124	124	125	126	126	127	127	128	129	58
3	177	178	179	179	180	181	182	182	183	184	185	186	187	187	188	189	190	191	192	193	57
4	236	237	238	239	240	241	242	243	244	245	246	248	249	250	251	252	254	255	256	257	56
5	295	296	298	299	300	301	303	304	305	307	308	310	311	312	314	315	317	318	320	322	55
6	354	356	357	359	360	362	363	365	366	368	370	371	373	375	377	379	380	382	384	386	54
7	413	415	417	418	420	422	424	426	428	429	431	433	435	437	440	442	444	446	448	450	53
8	472	474	476	478	480	482	484	486	489	491	493	495	498	500	502	505	507	510	512	515	52
9	531	533	536	538	540	543	545	547	550	552	555	557	560	562	565	568	571	573	576	579	51
10	590	593	595	598	600	603	605	608	611	614	616	619	622	625	628	631	634	637	640	643	50
11	649	652	655	657	660	663	666	669	672	675	678	681	684	687	691	694	697	701	704	708	49
12	708	711	714	717	720	723	727	730	733	736	740	743	746	750	753	757	761	764	768	772	48
13	767	771	774	777	780	784	787	791	794	798	801	805	809	812	816	820	824	828	832	836	47
14	826	830	833	837	840	844	848	851	855	859	863	867	871	875	879	883	887	892	896	901	46
15	885	889	893	897	900	904	908	912	916	920	924	929	933	937	942	946	951	955	960	965	45
16	944	948	952	956	960	965	969	973	977	982	986	991	995	1000	1005	1009	1014	1019	1024	1029	44
17	1004	1008	1012	1016	1020	1025	1029	1034	1038	1043	1048	1052	1057	1062	1067	1072	1078	1083	1088	1094	43
18	1063	1067	1071	1076	1080	1085	1090	1095	1099	1104	1109	1114	1120	1125	1130	1136	1141	1147	1152	1158	42
19	1122	1126	1131	1136	1140	1145	1150	1155	1160	1166	1171	1176	1182	1187	1193	1199	1204	1210	1216	1222	41
20	1181	1185	1190	1195	1200	1206	1211	1216	1222	1227	1233	1238	1244	1250	1256	1262	1268	1274	1280	1287	40
21	1240	1245	1250	1255	1260	1266	1271	1277	1283	1288	1294	1300	1306	1312	1318	1325	1331	1338	1344	1351	39
22	1299	1304	1309	1315	1320	1326	1332	1338	1344	1350	1356	1362	1368	1375	1381	1388	1395	1401	1408	1415	38
23	1358	1363	1369	1375	1381	1387	1393	1399	1405	1411	1418	1424	1431	1437	1444	1451	1458	1465	1472	1480	37
24	1417	1423	1428	1434	1441	1447	1453	1459	1466	1472	1479	1486	1493	1500	1507	1514	1521	1529	1536	1544	36
25	1476	1482	1488	1494	1501	1507	1514	1520	1527	1534	1541	1548	1555	1562	1570	1577	1585	1592	1600	1608	35
26	1535	1541	1548	1554	1561	1567	1574	1581	1588	1595	1602	1610	1617	1625	1632	1640	1648	1656	1664	1673	34
27	1594	1600	1607	1614	1621	1628	1635	1642	1649	1656	1664	1672	1679	1687	1695	1703	1712	1720	1728	1738	33
28	1653	1660	1667	1674	1681	1688	1695	1703	1710	1718	1726	1733	1742	1750	1758	1766	1775	1784	1792	1802	32
29	1712	1719	1726	1733	1741	1748	1756	1764	1771	1779	1787	1795	1804	1812	1821	1829	1838	1847	1856	1866	31
30	1771	1778	1786	1793	1801	1809	1816	1824	1832	1841	1849	1857	1866	1875	1884	1893	1902	1911	1920	1930	30

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FUNCTION FOR EACH MINUTE

For first half (0′ to 30′) of each degree read DOWN in column marked A′
For second half (30′ to 60′) of each degree read UP in column marked B′

A′	30°		31°		32°		33°		34°		B′
	A′	B′	A′	B′	A′	B′	A′	B′	A′	B′	
0	.5773503	.6008606	.6008606	.6248694	.6248694	.6494076	.6494076	.6745085	.6745085	.7002075	60
1	.5777382	.6004648	.6012566	.6244650	.6252739	.6489941	.6498212	.6740854	.6749318	.6997741	59
2	.5781262	.6000691	.6016527	.6240607	.6256786	.6485808	.6502350	.6736624	.6753553	.6993409	58
3	.5785144	.5996735	.6020490	.6236566	.6260834	.6481676	.6506490	.6732396	.6757790	.6989078	57
4	.5789027	.5992781	.6024454	.6232527	.6264884	.6477546	.6510631	.6728169	.6762028	.6984749	56
5	.5792912	.5988828	.6028419	.6228488	.6268935	.6473417	.6514774	.6723944	.6766288	.6980422	55
6	.5796797	.5984877	.6032386	.6224452	.6272988	.6469290	.6518918	.6719721	.6770509	.6976097	54
7	.5800684	.5980926	.6036354	.6220417	.6277042	.6465165	.6523064	.6715500	.6774752	.6971773	53
8	.5804573	.5976978	.6040323	.6216383	.6281098	.6461041	.6527211	.6711280	.6778997	.6967451	52
9	.5808462	.5973030	.6044294	.6212351	.6285155	.6456918	.6531360	.6707061	.6783243	.6963131	51
10	.5812353	.5969084	.6048266	.6208320	.6289214	.6452797	.6535511	.6702845	.6787492	.6958813	50
11	.5816245	.5965140	.6052240	.6204291	.6293274	.6448678	.6539663	.6698630	.6791741	.6954496	49
12	.5820139	.5961196	.6056215	.6200263	.6297336	.6444560	.6543817	.6694417	.6795993	.6950181	48
13	.5824034	.5957255	.6060192	.6196236	.6301399	.6440444	.6547972	.6690205	.6800246	.6945868	47
14	.5827930	.5953314	.6064170	.6192211	.6305464	.6436329	.6552129	.6685995	.6804501	.6941557	46
15	.5831828	.5949375	.6068149	.6188188	.6309530	.6432216	.6556287	.6681786	.6808758	.6937247	45
16	.5835726	.5945437	.6072130	.6184166	.6313598	.6428105	.6560447	.6677580	.6813016	.6932939	44
17	.5839627	.5941501	.6076112	.6180145	.6317667	.6423994	.6564609	.6673374	.6817276	.6928633	43
18	.5843528	.5937565	.6080095	.6176126	.6321738	.6419886	.6568772	.6669171	.6821537	.6924328	42
19	.5847431	.5933632	.6084080	.6172108	.6325810	.6415779	.6572937	.6664969	.6825801	.6920026	41
20	.5851335	.5929699	.6088067	.6168092	.6329883	.6411673	.6577103	.6660769	.6830066	.6915725	40
21	.5855241	.5925768	.6092054	.6164077	.6333959	.6407569	.6581271	.6656570	.6834333	.6911425	39
22	.5859148	.5921839	.6096043	.6160064	.6338035	.6403467	.6585441	.6652373	.6838601	.6907128	38
23	.5863056	.5917910	.6100034	.6156052	.6342113	.6399366	.6589612	.6648178	.6842871	.6902832	37
24	.5866965	.5913984	.6104026	.6152041	.6346193	.6395267	.6593785	.6643984	.6847143	.6898538	36
25	.5870876	.5910058	.6108019	.6148032	.6350274	.6391169	.6597960	.6639792	.6851416	.6894246	35
26	.5874788	.5906134	.6112014	.6144024	.6354357	.6387073	.6602136	.6635601	.6855692	.6889955	34
27	.5878702	.5902211	.6116011	.6140018	.6358441	.6382978	.6606313	.6631413	.6859969	.6885666	33
28	.5882616	.5898289	.6120008	.6136014	.6362527	.6378885	.6610492	.6627225	.6864247	.6881379	32
29	.5886533	.5894369	.6124007	.6132010	.6366614	.6374793	.6614673	.6623040	.6868528	.6877093	31
30	.5890450	.5890450	.6128008	.6128008	.6370703	.6370703	.6618856	.6618856	.6872810	.6872810	30

INCREMENT FOR SECONDS

For A″ add increment to function in above table. For B″ subtract increment from function for next higher minute in above table.

A″	30°				31°				32°				33°				34°				B″
	0′	15′	30′	45′	0′	15′	30′	45′	0′	15′	30′	45′	0′	15′	30′	45′	0′	15′	30′	45′	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	65	65	65	66	66	66	67	67	67	68	68	69	69	69	70	70	71	71	71	72	59
2	129	130	131	131	132	133	133	134	135	136	136	137	138	139	140	140	141	142	143	144	58
3	194	195	196	197	198	199	200	201	202	203	205	206	207	208	209	210	212	213	214	215	57
4	259	260	261	263	264	265	267	268	270	271	273	274	276	277	279	281	282	284	286	287	56
5	323	325	327	328	330	332	334	335	337	339	341	343	345	347	349	351	353	355	357	359	55
6	388	390	392	394	396	398	400	402	405	407	409	411	414	416	418	421	423	426	428	431	54
7	453	455	457	460	462	464	467	469	472	475	477	480	483	485	488	491	494	497	500	503	53
8	517	520	523	525	528	531	534	536	539	542	545	548	552	555	558	561	564	568	571	575	52
9	582	585	588	591	594	597	600	604	607	610	613	617	620	624	628	631	635	639	643	646	51
10	647	650	653	657	660	663	667	671	674	678	682	686	689	693	697	701	706	710	714	718	50
11	711	715	718	722	726	730	734	738	742	746	750	754	758	763	767	772	776	781	785	790	49
12	776	780	784	788	792	796	800	805	809	814	818	823	827	832	837	842	847	852	857	862	48
13	840	845	849	853	858	863	867	872	877	881	886	891	896	901	907	912	917	923	928	934	47
14	905	910	914	919	924	929	934	939	944	949	954	960	965	971	976	982	988	994	1000	1006	46
15	970	975	980	985	990	995	1001	1006	1011	1017	1023	1028	1034	1040	1046	1052	1058	1065	1071	1077	45
16	1034	1040	1045	1050	1056	1062	1067	1073	1079	1085	1091	1097	1103	1109	1116	1122	1129	1136	1142	1149	44
17	1099	1105	1110	1116	1122	1128	1134	1140	1146	1153	1159	1165	1172	1179	1186	1192	1199	1206	1214	1221	43
18	1164	1170	1176	1182	1188	1194	1201	1207	1214	1220	1227	1234	1241	1248	1255	1263	1270	1277	1285	1293	42
19	1228	1235	1241	1247	1254	1261	1267	1274	1281	1288	1295	1303	1310	1317	1325	1333	1341	1348	1356	1365	41
20	1293	1300	1306	1313	1320	1327	1334	1341	1349	1356	1363	1371	1379	1387	1395	1403	1411	1419	1428	1437	40
21	1358	1365	1372	1379	1386	1393	1401	1408	1416	1424	1432	1440	1448	1456	1464	1473	1482	1490	1499	1508	39
22	1422	1430	1437	1444	1452	1460	1468	1475	1483	1492	1500	1508	1517	1525	1534	1543	1552	1561	1571	1580	38
23	1487	1495	1502	1510	1518	1526	1534	1542	1551	1559	1568	1577	1586	1595	1604	1613	1623	1632	1642	1652	37
24	1552	1560	1568	1576	1584	1592	1601	1609	1618	1627	1636	1645	1655	1664	1674	1683	1693	1703	1713	1724	36
25	1616	1625	1633	1641	1650	1659	1668	1677	1686	1695	1704	1714	1723	1733	1743	1753	1764	1774	1785	1796	35
26	1681	1689	1698	1707	1716	1725	1734	1744	1753	1763	1772	1782	1792	1803	1813	1824	1834	1845	1856	1868	34
27	1746	1754	1763	1773	1782	1791	1801	1811	1821	1831	1841	1851	1861	1872	1883	1894	1905	1916	1928	1939	33
28	1810	1819	1829	1838	1848	1858	1868	1878	1888	1898	1909	1920	1930	1941	1953	1964	1976	1987	1999	2011	32
29	1875	1884	1894	1904	1914	1924	1934	1945	1955	1966	1977	1988	1999	2011	2022	2034	2046	2058	2071	2083	31
30	1940	1949	1959	1970	1980	1990	2001	2012	2023	2034	2045	2057	2068	2080	2092	2104	2117	2129	2142	2155	30

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FUNCTION FOR EACH MINUTE

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	35°		36°		37°		38°		39°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.7002075	.7265425	.7265425	.7535541	.7535541	.7812856	.7812856	.8097840	.8097840	.8390996	60
1	.7006411	.7260982	.7269871	.7530981	.7540102	.7808173	.7817542	.8093025	.8102658	.8386041	59
2	.7010749	.7256540	.7274318	.7526423	.7544666	.7803492	.7822229	.8088212	.8107478	.8381087	58
3	.7015089	.7252101	.7278767	.7521867	.7549232	.7798812	.7826919	.8083401	.8112300	.8376136	57
4	.7019430	.7247663	.7283218	.7517314	.7553799	.7794135	.7831611	.8078593	.8117124	.8371188	56
5	.7023773	.7243227	.7287671	.7512762	.7558369	.7789460	.7836305	.8073787	.8121951	.8366242	55
6	.7028118	.7238793	.7292125	.7508212	.7562941	.7784788	.7841002	.8068983	.8126780	.8361298	54
7	.7032464	.7234361	.7296582	.7503665	.7567514	.7780117	.7845700	.8064181	.8131611	.8356357	53
8	.7036813	.7229930	.7301041	.7499119	.7572090	.7775448	.7850400	.8059382	.8136444	.8351418	52
9	.7041163	.7225502	.7305501	.7494575	.7576668	.7770782	.7855103	.8054584	.8141280	.8346481	51
10	.7045515	.7221075	.7309963	.7490033	.7581248	.7766118	.7859808	.8049790	.8146118	.8341547	50
11	.7049869	.7216650	.7314428	.7485494	.7585829	.7761455	.7864515	.8044997	.8150958	.8336615	49
12	.7054224	.7212227	.7318894	.7480956	.7590413	.7756795	.7869224	.8040206	.8155801	.8331686	48
13	.7058581	.7207806	.7323362	.7476420	.7594999	.7752137	.7873935	.8035418	.8160646	.8326759	47
14	.7062940	.7203387	.7327832	.7471886	.7599587	.7747481	.7878649	.8030632	.8165493	.8321834	46
15	.7067301	.7198970	.7332303	.7467354	.7604177	.7742827	.7883364	.8025849	.8170343	.8316912	45
16	.7071664	.7194554	.7336777	.7462824	.7608769	.7738176	.7888082	.8021067	.8175195	.8311992	44
17	.7076028	.7190141	.7341253	.7458296	.7613363	.7733526	.7892802	.8016288	.8180049	.8307075	43
18	.7080395	.7185729	.7345730	.7453770	.7617959	.7728878	.7897524	.8011511	.8184905	.8302160	42
19	.7084763	.7181319	.7350210	.7449246	.7622557	.7724233	.7902248	.8006736	.8189764	.8297247	41
20	.7089133	.7176911	.7354691	.7444724	.7627157	.7719589	.7906975	.8001963	.8194625	.8292337	40
21	.7093504	.7172505	.7359174	.7440204	.7631759	.7714948	.7911703	.7997193	.8199488	.8287429	39
22	.7097878	.7168100	.7363660	.7435686	.7636363	.7710309	.7916434	.7992425	.8204354	.8282523	38
23	.7102253	.7163698	.7368147	.7431170	.7640969	.7705672	.7921167	.7987659	.8209222	.8277620	37
24	.7106630	.7159297	.7372636	.7426655	.7645577	.7701037	.7925902	.7982895	.8214093	.8272719	36
25	.7111009	.7154898	.7377127	.7422143	.7650188	.7696404	.7930640	.7978134	.8218965	.8267821	35
26	.7115390	.7150501	.7381620	.7417633	.7654800	.7691773	.7935379	.7973374	.8223840	.8262925	34
27	.7119772	.7146106	.7386115	.7413124	.7659414	.7687144	.7940121	.7968617	.8228718	.8258031	33
28	.7124157	.7141712	.7390611	.7408618	.7664031	.7682517	.7944865	.7963862	.8233597	.8253140	32
29	.7128543	.7137320	.7395110	.7404113	.7668649	.7677893	.7949611	.7959110	.8238479	.8248251	31
30	.7132931	.7132931	.7399611	.7399611	.7673270	.7673270	.7954359	.7954359	.8243364	.8243364	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	35°				36°				37°				38°				39°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	72	73	73	74	74	75	75	76	76	77	77	78	78	79	79	80	80	81	81	82	59
2	145	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	161	162	163	164	58
3	217	218	219	221	222	224	225	227	228	230	231	233	234	236	238	239	241	243	244	246	57
4	289	291	293	294	296	298	300	302	304	306	308	310	312	314	317	319	321	324	326	328	56
5	361	363	366	368	370	373	375	378	380	383	385	388	390	393	396	399	401	404	407	410	55
6	433	436	439	442	445	447	450	453	456	459	462	465	469	472	475	478	482	485	489	492	54
7	506	509	512	515	519	522	525	529	532	536	539	543	547	550	554	558	562	566	570	574	53
8	578	582	585	589	593	596	600	604	608	612	616	620	625	629	633	638	642	647	652	656	52
9	650	654	658	663	667	671	675	680	684	689	693	698	703	708	713	718	723	728	733	738	51
10	723	727	732	736	741	746	750	755	760	765	770	776	781	786	792	797	803	809	814	820	50
11	795	800	805	810	815	820	826	831	836	842	848	853	859	865	871	877	883	890	896	902	49
12	867	872	878	883	889	895	901	906	912	918	925	931	937	944	950	957	964	970	977	984	48
13	939	945	951	957	963	969	976	982	988	995	1002	1008	1015	1022	1029	1037	1044	1051	1059	1066	47
14	1012	1018	1024	1031	1037	1044	1051	1058	1064	1072	1079	1086	1093	1101	1108	1116	1124	1132	1140	1149	46
15	1084	1090	1097	1104	1111	1118	1126	1133	1140	1148	1156	1164	1171	1179	1188	1196	1204	1213	1222	1231	45
16	1156	1163	1171	1178	1185	1193	1201	1209	1216	1225	1233	1241	1249	1258	1267	1276	1285	1294	1303	1313	44
17	1228	1236	1244	1252	1260	1268	1276	1284	1292	1301	1310	1319	1328	1337	1346	1355	1365	1375	1385	1395	43
18	1301	1309	1317	1325	1334	1342	1351	1360	1368	1378	1387	1396	1406	1415	1425	1435	1445	1456	1466	1477	42
19	1373	1381	1390	1399	1408	1417	1426	1435	1444	1454	1464	1474	1484	1494	1504	1515	1526	1537	1547	1559	41
20	1445	1454	1463	1472	1482	1491	1501	1511	1521	1531	1541	1551	1562	1573	1583	1595	1606	1617	1629	1641	40
21	1518	1527	1536	1546	1556	1566	1576	1586	1597	1607	1618	1629	1640	1651	1663	1674	1686	1698	1710	1723	39
22	1590	1599	1610	1620	1630	1640	1651	1662	1673	1684	1695	1706	1718	1730	1742	1754	1766	1779	1792	1805	38
23	1662	1672	1683	1693	1704	1715	1726	1737	1749	1760	1772	1784	1796	1809	1821	1834	1847	1860	1873	1887	37
24	1734	1745	1756	1767	1778	1790	1801	1813	1825	1837	1849	1862	1874	1887	1900	1914	1927	1941	1955	1969	36
25	1807	1817	1829	1841	1852	1864	1876	1888	1901	1913	1926	1939	1952	1966	1979	1993	2007	2022	2036	2051	35
26	1879	1890	1902	1914	1926	1939	1951	1964	1977	1990	2003	2017	2030	2044	2058	2073	2088	2103	2118	2133	34
27	1951	1963	1975	1988	2000	2013	2026	2039	2053	2066	2080	2094	2108	2123	2138	2153	2168	2183	2199	2215	33
28	2023	2036	2049	2061	2075	2088	2101	2115	2129	2143	2157	2172	2187	2202	2217	2232	2248	2264	2281	2297	32
29	2096	2108	2122	2135	2149	2162	2176	2190	2205	2219	2234	2249	2265	2280	2296	2312	2329	2345	2362	2379	31
30	2168	2181	2195	2209	2223	2237	2251	2266	2281	2296	2311	2327	2343	2359	2375	2392	2409	2426	2443	2461	30

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FUNCTION FOR EACH MINUTE

TAN 40°-45°

For first half (0' to 30') of each degree read DOWN in column marked A'
For second half (30' to 60') of each degree read UP in column marked B'

A'	40°		41°		42°		43°		44°		B'
	A'	B'	A'	B'	A'	B'	A'	B'	A'	B'	
0	.8390996	.8692867	.8692867	.9004040	.9004040	.9325151	.9325151	.9656888	.9656888	1.0000000	60
1	.8395955	.8687762	.8697976	.8998775	.9009309	.9319714	.9330591	.9651268	.9662511	.9994184	59
2	.8400915	.8682659	.8703087	.8993512	.9014580	.9314280	.9336034	.9645651	.9668137	.9988371	58
3	.8405878	.8677558	.8708200	.8988251	.9019854	.9308849	.9341479	.9640037	.9673767	.9982562	57
4	.8410844	.8672460	.8713316	.8982994	.9025131	.9303421	.9346928	.9634427	.9679399	.9976756	56
5	.8415812	.8667365	.8718435	.8977739	.9030411	.9297996	.9352380	.9628819	.9685035	.9970953	55
6	.8420782	.8662272	.8723556	.8972487	.9035693	.9292573	.9357834	.9623215	.9690674	.9965154	54
7	.8425755	.8657181	.8728680	.8967238	.9040979	.9287154	.9363292	.9617614	.9696316	.9959358	53
8	.8430730	.8652094	.8733806	.8961991	.9046267	.9281738	.9368753	.9612016	.9701962	.9953566	52
9	.8435708	.8647009	.8738935	.8956747	.9051557	.9276324	.9374216	.9606421	.9707610	.9947777	51
10	.8440688	.8641926	.8744067	.8951506	.9056851	.9270914	.9379683	.9600829	.9713262	.9941991	50
11	.8445670	.8636846	.8749201	.8946268	.9062147	.9265506	.9385153	.9595241	.9718917	.9936208	49
12	.8450655	.8631768	.8754338	.8941032	.9067446	.9260102	.9390625	.9589655	.9724575	.9930429	48
13	.8455643	.8626694	.8759478	.8935799	.9072748	.9254700	.9396101	.9584073	.9730236	.9924654	47
14	.8460633	.8621621	.8764620	.8930569	.9078053	.9249301	.9401579	.9578494	.9735901	.9918881	46
15	.8465625	.8616551	.8769765	.8925341	.9083360	.9243905	.9407061	.9572917	.9741569	.9913112	45
16	.8470620	.8611484	.8774912	.8920116	.9088671	.9238512	.9412545	.9567344	.9747240	.9907346	44
17	.8475617	.8606419	.8780062	.8914894	.9093984	.9233122	.9418033	.9561774	.9752914	.9901584	43
18	.8480617	.8601357	.8785215	.8909675	.9099300	.9227734	.9423523	.9556208	.9758591	.9895825	42
19	.8485619	.8596297	.8790370	.8904458	.9104619	.9222350	.9429017	.9550644	.9764272	.9890069	41
20	.8490624	.8591240	.8795528	.8899244	.9109940	.9216969	.9434513	.9545083	.9769956	.9884316	40
21	.8495631	.8586185	.8800688	.8894033	.9115265	.9211590	.9440013	.9539526	.9775643	.9878567	39
22	.8500640	.8581133	.8805852	.8888825	.9120592	.9206214	.9445516	.9533971	.9781333	.9872821	38
23	.8505653	.8576084	.8811017	.8883619	.9125922	.9200841	.9451021	.9528420	.9787027	.9867079	37
24	.8510667	.8571037	.8816186	.8878415	.9131255	.9195471	.9456530	.9522871	.9792724	.9861339	36
25	.8515684	.8565992	.8821357	.8873215	.9136591	.9190104	.9462042	.9517326	.9798424	.9855603	35
26	.8520704	.8560950	.8826531	.8868017	.9141929	.9184740	.9467556	.9511784	.9804127	.9849871	34
27	.8525726	.8555910	.8831707	.8862822	.9147270	.9179379	.9473074	.9506245	.9809833	.9844141	33
28	.8530750	.8550873	.8836886	.8857630	.9152615	.9174020	.9478595	.9500709	.9815543	.9838415	32
29	.8535777	.8545839	.8842068	.8852440	.9157962	.9168665	.9484119	.9495176	.9821256	.9832692	31
30	.8540807	.8540807	.8847253	.8847253	.9163312	.9163312	.9489646	.9489646	.9826973	.9826973	30

INCREMENT FOR SECONDS

For A" add increment to function in above table. For B" subtract increment from function for next higher minute in above table.

A"	40°				41°				42°				43°				44°				B"
	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	0'	15'	30'	45'	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	60
1	83	83	84	84	85	86	86	87	88	88	89	90	91	91	92	93	94	94	95	96	59
2	165	166	168	169	170	172	173	174	176	177	178	180	181	183	184	186	187	189	191	192	58
3	248	250	252	253	255	257	259	261	263	266	268	270	272	274	277	279	281	284	286	288	57
4	331	333	336	338	341	343	346	348	351	354	357	360	363	366	369	372	375	378	381	385	56
5	413	416	419	422	426	429	432	436	439	443	446	450	453	457	461	465	469	473	477	481	55
6	496	499	503	507	511	515	519	523	527	531	535	540	544	548	553	558	562	567	572	577	54
7	578	582	587	591	596	601	605	610	615	620	624	630	635	640	645	651	656	662	667	673	53
8	661	666	671	676	681	686	692	697	702	708	714	719	725	731	737	744	750	756	763	769	52
9	744	749	755	760	766	772	778	784	790	797	803	809	816	823	830	836	843	851	858	865	51
10	826	832	839	845	851	858	865	871	878	885	892	899	907	914	922	929	937	945	953	962	50
11	909	916	923	929	937	944	951	958	966	974	981	989	997	1005	1014	1022	1031	1040	1049	1058	49
12	992	999	1006	1014	1022	1029	1037	1046	1054	1062	1071	1079	1088	1097	1106	1115	1125	1134	1144	1154	48
13	1074	1082	1090	1098	1107	1115	1124	1133	1141	1151	1160	1169	1179	1188	1198	1208	1218	1229	1239	1250	47
14	1157	1165	1174	1183	1192	1201	1210	1220	1229	1239	1249	1259	1269	1280	1290	1301	1312	1323	1335	1346	46
15	1240	1248	1258	1267	1277	1287	1297	1307	1317	1328	1338	1349	1360	1371	1383	1394	1406	1418	1430	1442	45
16	1322	1332	1342	1352	1362	1373	1383	1394	1405	1416	1427	1439	1451	1463	1475	1487	1500	1512	1525	1538	44
17	1405	1415	1426	1436	1447	1458	1470	1481	1493	1505	1517	1529	1541	1554	1567	1580	1593	1607	1621	1635	43
18	1487	1498	1510	1521	1532	1544	1556	1568	1581	1593	1606	1619	1632	1645	1659	1673	1687	1701	1716	1731	42
19	1570	1581	1594	1605	1618	1630	1643	1655	1668	1682	1695	1709	1723	1737	1751	1766	1781	1796	1811	1827	41
20	1653	1665	1677	1690	1703	1716	1729	1743	1756	1770	1784	1799	1813	1828	1843	1859	1874	1890	1907	1923	40
21	1735	1748	1761	1774	1788	1802	1816	1830	1844	1859	1873	1889	1904	1920	1936	1952	1968	1985	2002	2019	39
22	1818	1831	1845	1859	1873	1887	1902	1917	1932	1947	1963	1979	1995	2011	2028	2045	2062	2079	2097	2115	38
23	1901	1914	1929	1943	1958	1973	1988	2004	2020	2036	2052	2068	2085	2102	2120	2138	2156	2174	2193	2211	37
24	1983	1997	2013	2028	2043	2059	2075	2091	2107	2124	2141	2158	2176	2194	2212	2230	2249	2268	2288	2308	36
25	2066	2081	2097	2112	2128	2145	2161	2178	2195	2213	2230	2248	2267	2285	2304	2323	2343	2363	2383	2404	35
26	2149	2164	2181	2197	2214	2230	2248	2265	2283	2301	2320	2338	2357	2377	2396	2416	2437	2457	2479	2500	34
27	2231	2247	2264	2281	2299	2316	2334	2352	2371	2390	2409	2428	2448	2468	2489	2509	2530	2552	2574	2596	33
28	2314	2330	2348	2366	2384	2402	2421	2440	2459	2478	2498	2518	2539	2559	2581	2602	2624	2646	2669	2692	32
29	2396	2413	2432	2450	2469	2488	2507	2527	2546	2567	2587	2608	2629	2651	2673	2695	2718	2741	2765	2788	31
30	2479	2497	2516	2535	2554	2574	2594	2614	2634	2655	2676	2698	2720	2742	2765	2788	2812	2836	2860	2885	30

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To order, request TABLE 99

Price 25 Cents

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R. A. Davis' Table of Natural Sines and Cosines

By permission of the author, we are privileged to supply this table as a part of our series of Marchant Methods.

The table is useful for the solution of many problems involving angles because it enables quick determination of the cosine when the sine is given (or vice-versa), without the necessity of determining the angle. It is also useful for solving problems that may be expressed in the form $\sqrt{1 - x^2}$, *because* $\sin^2 a + \cos^2 a = 1$.

The range in which straight-line interpolation provides last-place accuracy (subject to rounding-off errors) is:

Sine 0 to .800	7	place accuracy of interpolated cosine
.8 to .960	6	" " " " "
.99 to .998	5	" " " " "

In the higher ranges of the table, interpolation by Marchant Method MM-64 or MM-189 will provide seven place accuracy.

DIRECT INTERPOLATION

A rapid method of straight-line interpolation is illustrated in the following example:

Find the cosine corresponding to sin = 0.2394725.

Decimals: Upper Dial 4, Middle Dial 11, Keyboard Dial 7.

(1) Set up in Keyboard Dial the cosine corresponding to the angle whose sine is .240 (.9707729) and multiply by the ratio that the desired interpolate is "on the way" toward .240 (.4725).

(2) Change Keyboard Dial to read the cosine corresponding to the angle whose sine is .239 (.9710196) and, without clearing any dials, multiply by such amount as will cause Upper Dial to read 1.0000 (In this case the multiplier is .5275).
The desired interpolate (.9709030) appears in Middle Dial.

INVERSE INTERPOLATION

Find the sine corresponding to cos = 0.9709030.

(1) Inspection of table shows that the desired sine lies between .239 and .240. Set up in Keyboard Dial the cosine (.9710196) corresponding to the lower value of these sines (.239) and, with carriage in 5th position, depress Add Bar.

(2) Set up in Keyboard Dial the difference (.0002467). Clear Upper Dial and, with Manual Counter Control toward the operator and by using reverse multiplication, build up the Upper Dial until the Middle Dial is reduced to the cosine to which it is desired to interpolate (.9709030).
Upper Dial shows the four figures that are to be annexed to the lower value of the sine (.4725). The desired sine is thus, 0.2394725.

On M models this process is most easily performed if the Non-Shift Key is depressed. Successive depressions of the No. 1 key of the Single Row Keyboard with carriage in 4th position will soon cause the Middle Dial to show an amount which is slightly less than the desired amount. Next, multiply by "1", then shift to 3rd position and repeat, etc.

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R. A. Davis' Table of Natural Sines and Cosines

SINE	COSINE	DIFF.	SINE	COSINE	DIFF	SINE	COSINE	DIFF
.001	.999 999 5	5	.051	.998 698 7	505	.101	.994 886 4	101 0
.002	.999 998 0	15	.052	.998 647 1	516	.102	.994 784 4	102 0
.003	.999 995 5	25	.053	.998 594 6	525	.103	.994 681 4	103 0
.004	.999 992 0	35	.054	.998 540 9	537	.104	.994 577 3	104 1
.005	.999 987 5	45	.055	.998 486 4	545	.105	.994 472 2	105 1
.006	.999 982 0	55	.056	.998 430 8	556	.106	.994 366 1	106 1
.007	.999 975 5	65	.057	.998 374 2	566	.107	.994 259 0	107 1
.008	.999 968 0	75	.058	.998 316 6	576	.108	.994 150 9	108 1
.009	.999 959 5	85	.059	.998 258 0	586	.109	.994 041 8	109 1
.010	.999 950 0	95	.060	.998 198 4	596	.110	.993 931 6	110 2
.011	.999 939 5	105	.061	.998 137 8	606	.111	.993 820 4	111 2
.012	.999 928 0	115	.062	.998 076 2	616	.112	.993 708 2	112 2
.013	.999 915 5	125	.063	.998 013 6	626	.113	.993 595 0	113 2
.014	.999 902 0	135	.064	.997 949 9	637	.114	.993 480 8	114 2
.015	.999 887 5	145	.065	.997 885 3	646	.115	.993 365 5	115 3
.016	.999 872 0	155	.066	.997 819 6	657	.116	.993 249 2	116 3
.017	.999 855 5	165	.067	.997 753 0	666	.117	.993 131 9	117 3
.018	.999 838 0	175	.068	.997 685 3	677	.118	.993 013 6	118 3
.019	.999 819 5	185	.069	.997 616 7	686	.119	.992 894 3	119 3
.020	.999 800 0	195	.070	.997 547 0	697	.120	.992 773 9	120 4
.021	.999 779 5	205	.071	.997 476 3	707	.121	.992 652 5	121 4
.022	.999 758 0	215	.072	.997 404 6	717	.122	.992 530 1	122 4
.023	.999 735 5	225	.073	.997 331 9	727	.123	.992 406 6	123 5
.024	.999 712 0	235	.074	.997 258 2	737	.124	.992 282 2	124 4
.025	.999 687 4	246	.075	.997 183 5	747	.125	.992 156 7	125 5
.026	.999 661 9	255	.076	.997 107 8	757	.126	.992 030 3	126 4
.027	.999 635 4	265	.077	.997 031 1	767	.127	.991 902 7	127 6
.028	.999 607 9	275	.078	.996 953 4	777	.128	.991 774 2	128 5
.029	.999 579 4	285	.079	.996 874 6	788	.129	.991 644 6	129 6
.030	.999 549 9	295	.080	.996 794 9	797	.130	.991 514 0	130 6
.031	.999 519 4	305	.081	.996 714 1	808	.131	.991 382 4	131 6
.032	.999 487 9	315	.082	.996 632 3	818	.132	.991 249 7	132 7
.033	.999 455 4	325	.083	.996 549 5	828	.133	.991 116 0	133 7
.034	.999 421 9	335	.084	.996 465 8	837	.134	.990 981 4	134 6
.035	.999 387 3	346	.085	.996 380 9	849	.135	.990 845 6	135 8
.036	.999 351 8	355	.086	.996 295 1	858	.136	.990 708 8	136 8
.037	.999 315 3	365	.087	.996 208 3	868	.137	.990 571 1	137 7
.038	.999 277 7	376	.088	.996 120 5	878	.138	.990 432 2	138 9
.039	.999 239 2	385	.089	.996 031 6	889	.139	.990 292 4	139 8
.040	.999 199 7	395	.090	.995 941 8	898	.140	.990 151 5	140 9
.041	.999 159 2	405	.091	.995 850 9	909	.141	.990 009 6	141 9
.042	.999 117 6	416	.092	.995 759 0	919	.142	.989 866 7	142 9
.043	.999 075 1	425	.093	.995 666 1	929	.143	.989 722 7	144 0
.044	.999 031 5	436	.094	.995 572 2	939	.144	.989 577 7	145 0
.045	.998 987 0	445	.095	.995 477 3	949	.145	.989 431 7	146 0
.046	.998 941 4	456	.096	.995 381 3	960	.146	.989 284 6	147 1
.047	.998 894 9	465	.097	.995 284 4	969	.147	.989 136 5	148 1
.048	.998 847 3	476	.098	.995 186 4	980	.148	.988 987 4	149 1
.049	.998 798 8	485	.099	.995 087 4	990	.149	.988 837 2	150 2
.050	.998 749 2	495	.100	.994 987 4	1000	.150	.988 686 0	151 2
		505			1010			152 2

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R. A. Davis' Table of Natural Sines and Cosines

SINE	COSINE	DIFF.	SINE	COSINE	DIFF.	SINE	COSINE	DIFF.
.151	.988 533 8	152 2	.201	.979 591 2	204 7	.251	.967 987 1	258 7
.152	.988 380 5	153 3	.202	.979 385 5	205 7	.252	.967 727 2	259 9
.153	.988 226 2	154 3	.203	.979 178 7	206 8	.253	.967 466 3	260 9
.154	.988 070 9	155 3	.204	.978 970 9	207 8	.254	.967 204 2	262 1
.155	.987 914 5	156 4	.205	.978 762 0	208 9	.255	.966 941 1	263 1
.156	.987 757 1	157 4	.206	.978 552 0	210 0	.256	.966 676 8	264 3
.157	.987 598 6	158 5	.207	.978 340 9	211 1	.257	.966 411 4	265 4
.158	.987 439 1	159 5	.208	.978 128 8	212 1	.258	.966 144 9	266 5
.159	.987 278 6	160 5	.209	.977 915 6	213 2	.259	.965 877 3	267 6
.160	.987 117 0	161 6	.210	.977 701 4	214 2	.260	.965 608 6	268 7
.161	.986 954 4	162 6	.211	.977 486 1	215 3	.261	.965 338 8	269 8
.162	.986 790 8	163 6	.212	.977 269 7	216 4	.262	.965 067 9	270 9
.163	.986 626 1	164 7	.213	.977 052 2	217 5	.263	.964 795 8	272 1
.164	.986 460 3	165 8	.214	.976 833 7	218 5	.264	.964 522 7	273 1
.165	.986 293 6	166 7	.215	.976 614 1	219 6	.265	.964 248 4	274 3
.166	.986 125 8	167 8	.216	.976 393 4	220 7	.266	.963 973 0	275 4
.167	.985 956 9	168 9	.217	.976 171 6	221 8	.267	.963 696 5	276 5
.168	.985 787 0	169 9	.218	.975 948 8	222 8	.268	.963 418 9	277 6
.169	.985 616 1	170 9	.219	.975 724 9	223 9	.269	.963 140 2	278 7
.170	.985 444 1	172 0	.220	.975 499 9	225 0	.270	.962 860 3	279 9
.171	.985 271 0	173 1	.221	.975 273 8	226 1	.271	.962 579 3	281 0
.172	.985 097 0	174 0	.222	.975 046 7	227 1	.272	.962 297 3	282 0
.173	.984 921 8	175 2	.223	.974 818 4	228 3	.273	.962 014 0	283 3
.174	.984 745 7	176 1	.224	.974 589 1	229 3	.274	.961 729 7	284 3
.175	.984 568 4	177 3	.225	.974 358 8	230 3	.275	.961 444 2	285 5
.176	.984 390 2	178 2	.226	.974 127 3	231 5	.276	.961 157 6	286 6
.177	.984 210 9	179 3	.227	.973 894 8	232 5	.277	.960 869 9	287 7
.178	.984 030 5	180 4	.228	.973 661 1	233 7	.278	.960 581 1	288 8
.179	.983 849 1	181 4	.229	.973 426 4	234 7	.279	.960 291 1	290 0
.180	.983 666 6	182 5	.230	.973 190 6	235 8	.280	.960 000 0	291 1
.181	.983 483 1	183 5	.231	.972 953 8	236 8	.281	.959 707 8	292 2
.182	.983 298 5	184 6	.232	.972 715 8	238 0	.282	.959 414 4	293 4
.183	.983 112 9	185 6	.233	.972 476 7	239 1	.283	.959 119 9	294 5
.184	.982 926 2	186 7	.234	.972 236 6	240 1	.284	.958 824 3	295 6
.185	.982 738 5	187 7	.235	.971 995 4	241 2	.285	.958 527 5	296 8
.186	.982 549 7	188 8	.236	.971 753 1	242 3	.286	.958 229 6	297 9
.187	.982 359 9	189 8	.237	.971 509 7	243 4	.287	.957 930 6	299 0
.188	.982 169 0	190 9	.238	.971 265 2	244 5	.288	.957 630 4	300 2
.189	.981 977 1	191 9	.239	.971 019 6	245 6	.289	.957 329 1	301 3
.190	.981 784 1	193 0	.240	.970 772 9	246 7	.290	.957 026 6	302 5
.191	.981 590 0	194 1	.241	.970 525 1	247 8	.291	.956 723 1	303 5
.192	.981 394 9	195 1	.242	.970 276 3	248 8	.292	.956 418 3	304 8
.193	.981 198 8	196 1	.243	.970 026 3	250 0	.293	.956 112 4	305 9
.194	.981 001 5	197 3	.244	.969 775 2	251 1	.294	.955 805 4	307 0
.195	.980 803 2	198 3	.245	.969 523 1	252 1	.295	.955 497 3	308 1
.196	.980 603 9	199 3	.246	.969 269 8	253 3	.296	.955 187 9	309 4
.197	.980 403 5	200 4	.247	.969 015 5	254 3	.297	.954 877 5	310 4
.198	.980 202 0	201 5	.248	.968 760 0	255 5	.298	.954 565 9	311 6
.199	.979 999 5	202 5	.249	.968 503 5	256 5	.299	.954 253 1	312 8
.200	.979 795 9	203 6	.250	.968 245 8	257 7	.300	.953 939 2	313 9
		204 7			258 7			315 1

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SINE	COSINE	DIFF.	SINE	COSINE	DIFF.	SINE	COSINE	DIFF
.301	.953 624 1	315 1	.351	.936 375 5	374 2	.401	.916 078 1	437 0
.302	.953 307 9	316 2	.352	.936 000 0	375 5	.402	.915 639 7	438 4
.303	.952 990 6	317 3	.353	.935 623 3	376 7	.403	.915 200 0	439 7
.304	.952 672 0	318 6	.354	.935 245 4	377 9	.404	.914 759 0	441 0
.305	.952 352 4	319 6	.355	.934 866 3	379 1	.405	.914 316 7	442 3
.306	.952 031 5	320 9	.356	.934 486 0	380 3	.406	.913 873 1	443 6
.307	.951 709 5	322 0	.357	.934 104 4	381 6	.407	.913 428 2	444 9
.308	.951 386 4	323 1	.358	.933 721 6	382 8	.408	.912 981 9	446 3
.309	.951 062 0	324 4	.359	.933 337 7	383 9	.409	.912 534 4	447 5
.310	.950 736 6	325 4	.360	.932 952 3	385 4	.410	.912 085 5	448 9
.311	.950 409 9	326 7	.361	.932 565 8	386 5	.411	.911 635 3	450 2
.312	.950 082 1	327 8	.362	.932 178 1	387 7	.412	.911 183 8	451 5
.313	.949 753 1	329 0	.363	.931 789 1	389 0	.413	.910 731 0	452 8
.314	.949 423 0	330 1	.364	.931 398 9	390 2	.414	.910 276 9	454 1
.315	.949 091 7	331 3	.365	.931 007 5	391 4	.415	.909 821 4	455 5
.316	.948 759 2	332 5	.366	.930 614 9	392 6	.416	.909 364 6	456 8
.317	.948 425 5	333 7	.367	.930 221 0	393 9	.417	.908 906 5	458 1
.318	.948 090 7	334 8	.368	.929 825 8	395 2	.418	.908 447 0	459 5
.319	.947 754 7	336 0	.369	.929 429 4	396 4	.419	.907 986 2	460 8
.320	.947 417 6	337 1	.370	.929 031 8	397 6	.420	.907 524 1	462 1
.321	.947 079 2	338 4	.371	.928 632 9	398 9	.421	.907 060 6	463 5
.322	.946 739 7	339 5	.372	.928 232 7	400 2	.422	.906 595 8	464 8
.323	.946 399 0	340 7	.373	.927 831 3	401 4	.423	.906 129 7	466 1
.324	.946 057 1	341 9	.374	.927 428 7	402 6	.424	.905 662 2	467 5
.325	.945 714 0	343 1	.375	.927 024 8	403 9	.425	.905 193 3	468 9
.326	.945 369 8	344 2	.376	.926 619 7	405 1	.426	.904 723 2	470 1
.327	.945 024 3	345 5	.377	.926 213 3	406 4	.427	.904 251 6	471 6
.328	.944 677 7	346 6	.378	.925 805 6	407 7	.428	.903 778 7	472 9
.329	.944 329 9	347 8	.379	.925 396 7	408 9	.429	.903 304 5	474 2
.330	.943 980 9	349 0	.380	.924 986 5	410 2	.430	.902 828 9	475 6
.331	.943 630 8	350 1	.381	.924 575 0	411 5	.431	.902 351 9	477 0
.332	.943 279 4	351 4	.382	.924 162 3	412 7	.432	.901 873 6	478 3
.333	.942 926 8	352 6	[illegible]	923 748 3	414 0	.433	.901 393 9	479 7
.334	.942 573 1	353 7	.384	.923 333 1	415 2	.434	900 912 9	481 0
.335	.942 218 1	355 0	.385	.922 916 6	416 5	.435	.900 430 5	482 4
.336	.941 862 0	356 1	.386	.922 498 8	417 8	.436	.899 946 7	483 8
.337	.941 504 6	357 4	.387	.922 079 7	419 1	.437	.899 461 5	485 2
.338	.941 146 1	358 5	.388	.921 659 4	420 3	.438	.898 975 0	486 5
.339	.940 786 4	359 7	.389	.921 237 8	421 6	.439	.898 487 1	487 9
.340	.940 425 4	361 0	.390	.920 814 9	422 9	.440	.897 997 8	489 3
.341	.940 063 3	362 1	.391	.910 390 7	424 2	.441	.897 507 1	490 7
.342	.939 700 0	363 3	.392	.919 965 2	425 5	.442	.897 015 1	492 0
.343	.939 335 4	364 6	.393	.919 538 5	426 7	.443	.896 521 6	493 5
.344	.938 969 6	365 8	.394	.919 110 4	428 1	.444	.896 026 8	494 8
.345	.938 602 7	366 9	.395	918 681 1	429 3	.445	.895 530 6	496 2
.346	.938 234 5	368 2	.396	718 250 5	430 6	.446	.895 033 0	497 6
.347	.937 865 1	369 4	.397	.917 818 6	431 9	.447	.894 534 0	499 0
.348	.937 494 5	370 6	.398	.917 385 4	433 2	.448	.894 033 6	500 4
.349	.937 122 7	371 8	.399	.916 950 9	434 5	.449	.893 531 8	501 8
.350	.936 749 7	373 0	.400	.916 515 1	435 8	.450	.893 028 6	503 2
		374 2			437 0			504 7

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SINE	COSINE	DIFF	SINE	COSINE	DIFF	SINE	COSINE	DIFF
.451	.892 523 9	504 7	.501	.865 449 3	578 1	.551	.834 505 3	659 4
.452	.892 017 9	506 0	.502	.864 867 6	579 7	.552	.833 844 1	661 2
.453	.891 510 5	507 4	.503	.864 286 4	581 2	.553	.833 181 3	662 8
.454	.891 001 7	508 8	.504	.863 703 6	582 8	.554	.832 516 7	664 6
.455	.890 491 4	510 3	.505	.863 119 3	584 3	.555	.831 850 4	.666 3
.456	.889 979 8	511 6	.506	.862 533 5	585 8	.556	.831 182 3	668 1
.457	.889 466 7	513 1	.507	.861 946 1	587 4	.557	.830 512 5	669 8
.458	.888 952 2	514 5	.508	.861 357 1	589 0	.558	.829 841 0	671 5
.459	.888 436 3	515 9	.509	.860 766 6	590 5	.559	.829 167 7	673 3
.460	.887 918 9	517 4	.510	.860 174 4	592 2	.560	.828 492 6	675 1
.461	.887 400 1	518 8	.511	.859 580 7	593 7	.561	.827 815 7	676 9
.462	.886 879 9	520 2	.512	.858 985 4	595 3	.562	.827 137 2	678 5
.463	.886 358 3	521 6	.513	.858 388 6	596 8	.563	.826 456 9	680 3
.464	.885 835 2	523 1	.514	.857 790 2	598 4	.564	.825 774 8	682 1
.465	.885 310 7	524 5	.515	.857 190 2	600 0	.565	.825 090 9	683 9
.466	.884 784 7	526 0	.516	.856 588 6	601 6	.566	.824 405 2	685 7
.467	.884 257 3	527 4	.517	.855 985 4	603 2	.567	.823 717 8	687 4
.468	.883 728 5	528 8	.518	.855 380 6	604 8	.568	.823 028 6	689 2
.469	.883 198 2	530 3	.519	.854 774 2	606 4	.569	.822 337 5	691 1
.470	.882 666 4	531 8	.520	.854 166 3	607 9	.570	.821 644 7	692 8
.471	.882 133 2	533 2	.521	.853 556 7	609 6	.571	.820 950 1	694 6
.472	.881 598 5	534 7	.522	.852 945 5	611 2	.572	.820 253 6	696 5
.473	.881 062 4	536 1	.523	.852 332 7	612 8	.573	.819 555 4	698 2
.474	.880 524 8	537 6	.524	.851 718 3	614 4	.574	.818 855 3	700 1
.475	.879 985 8	539 0	.525	.851 102 3	616 0	.575	.818 153 4	701 9
.476	.879 445 3	540 5	.526	.850 484 6	617 7	.576	.817 449 7	703 7
.477	.878 903 3	542 0	.527	.849 865 3	619 3	.577	.816 744 2	705 5
.478	.878 359 8	543 5	.528	.849 244 4	620 9	.578	.816 036 8	707 4
.479	.877 814 9	544 9	.529	.848 621 8	622 6	.579	.815 327 6	709 2
.480	.877 268 5	546 4	.530	.847 997 6	624 2	.580	.814 616 5	711 1
.481	.876 720 6	547 9	.531	.847 371 8	625 8	.581	.813 903 6	712 9
.482	.876 171 2	549 4	.532	.846 744 4	627 4	.582	.813 188 8	714 8
.483	.875 620 4	550 8	.533	.846 115 2	629 2	.583	.812 472 2	716 6
.484	.875 068 0	552 4	.534	.845 484 5	630 7	.584	.811 753 7	718 5
.485	.874 514 2	553 8	.535	.844 852 1	632 4	.585	.811 033 3	720 4
.486	.873 958 8	555 4	.536	.844 218 0	634 1	.586	.810 311 1	722 2
.487	.873 402 0	556 8	.537	.843 582 3	635 7	.587	.809 586 9	724 2
.488	.872 843 6	558 4	.538	.842 944 8	637 5	.588	.808 860 9	726 0
.489	.872 283 8	559 8	.539	.842 305 8	639 0	.589	.808 133 0	727 9
.490	.871 722 4	561 4	.540	.841 665 0	640 8	.590	.807 403 3	729 7
.491	.871 159 6	562 8	.541	.841 022 6	642 4	.591	.806 671 6	731 7
.492	.870 595 2	564 4	.542	.840 378 5	644 1	.592	.805 938 0	733 6
.493	.870 029 3	565 9	.543	.839 732 7	645 8	.593	.805 202 5	735 5
.494	.869 461 9	567 4	.544	.839 085 2	647 5	.594	.804 465 0	737 5
.495	.868 893 0	568 9	.545	.838 436 1	649 1	.595	.803 725 7	739 3
.496	.868 322 5	570 5	.546	.837 785 2	650 9	.596	.802 984 4	741 3
.497	.867 750 5	572 0	.547	.837 132 6	652 6	.597	.802 241 2	743 2
.498	.867 177 0	573 5	.548	.836 478 3	654 3	.598	.801 496 1	745 1
.499	.866 602 0	575 0	.549	.835 822 4	655 9	.599	.800 749 0	747 1
.500	.866 025 4	576 6	.550	.835 164 7	657 7	.600	.800 000 0	749 0
		578 1			659 4			751 0

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Sine	Cosine	Diff	Sine	Cosine	Diff	Sine	Cosine	Diff
.601	.799 249 0	751 0	.651	.759 077 7	856 5	.701	.713 161 3	981 5
.602	.798 496 1	752 9	.652	.758 219 0	858 7	.702	.712 177 0	984 3
.603	.797 741 2	754 9	.653	.757 357 9	861 1	.703	.711 189 8	987 2
.604	.796 984 3	756 9	.654	.756 494 5	863 4	.704	.710 200 0	989 8
.605	.796 225 5	758 8	.655	.755 628 9	865 6	.705	.709 207 3	992 7
.606	.795 464 7	760 8	.656	.754 760 9	868 0	.706	.708 211 8	995 5
.607	.794 701 8	762 9	.657	.753 890 6	870 3	.707	.707 213 5	998 3
.608	.793 937 0	764 8	.658	.753 017 9	872 7	.708	.706 212 4	1001 1
.609	.793 170 2	766 8	.659	.752 142 9	875 0	.709	.705 208 5	1003 9
.610	.792 401 4	768 8	.660	.751 265 6	877 3	.710	.704 201 7	1006 8
.611	.791 630 6	770 8	.661	.750 385 9	879 7	.711	.703 192 0	1009 7
.612	.790 857 8	772 8	.662	.749 503 8	882 1	.712	.702 179 5	1012 5
.613	.790 082 9	774 9	.663	.748 619 3	884 5	.713	.701 164 0	1015 5
.614	.789 306 0	776 9	.664	.747 732 6	886 7	.714	.700 145 7	1018 3
.615	.788 527 1	778 9	.665	.746 843 4	889 2	.715	.699 124 5	1021 2
.616	.787 746 2	780 9	.666	.745 951 7	891 7	.716	.698 100 3	1024 2
.617	.786 963 1	783 1	.667	.745 057 7	894 0	.717	.697 073 2	1027 1
.618	.786 178 1	785 0	.668	.744 161 3	896 4	.718	.696 043 1	1030 1
.619	.785 391 0	787 1	.669	.743 262 4	898 9	.719	.695 010 1	1033 0
.620	.784 601 8	789 2	.670	.742 361 1	901 3	.720	.693 974 1	1036 0
.621	.783 810 6	791 2	.671	.741 457 3	903 8	.721	.692 935 1	1039 0
.622	.783 017 2	793 4	.672	.740 551 1	906 2	.722	.691 893 1	1042 0
.623	.782 221 8	795 4	.673	.739 642 5	908 6	.723	.690 848 0	1045 1
.624	.781 424 3	797 5	.674	.738 731 3	911 2	.724	.689 800 0	1048 3
.625	.780 624 7	799 6	.675	.137 817 7	913 6	.725	.688 748 9	1051 1
.626	.779 823 0	801 7	.676	.736 901 6	916 1	.726	.687 694 7	1054 2
.627	.779 019 3	803 7	.677	.735 982 0	918 6	.727	.686 637 5	1057 2
.628	.778 213 3	806 0	.678	.735 061 9	921 1	.728	.685 577 1	1060 4
.629	.777 406 3	808 0	.679	.734 138 3	923 6	.729	.684 513 7	1063 4
.630	.776 595 1	810 2	.680	.733 212 1	926 2	.730	.683 447 1	1066 6
.631	.775 782 8	812 3	.681	.732 283 4	928 7	.731	.682 377 5	1069 6
.632	.774 968 3	814 5	.682	.731 352 2	931 2	.732	.681 304 6	1072 9
.633	.774 151 8	816 5	.683	.730 418 4	933 8	.733	.680 228 6	1076 0
.634	.773 333 0	818 8	.684	.729 482 0	936 4	.734	.679 149 5	1079 1
.635	.772 512 1	820 9	.685	.728 543 1	938 9	.735	.678 067 1	1082 4
.636	.771 689 1	823 0	.686	.727 601 5	941 6	.736	.676 981 5	1085 6
.637	.770 863 8	825 3	.687	.726 657 4	944 1	.737	.675 892 7	1088 8
.638	.770 036 4	827 4	.688	.725 710 7	946 7	.738	.674 800 7	1092 0
.639	.769 206 7	829 7	.689	.724 761 3	949 4	.739	.673 705 4	1095 3
.640	.768 374 9	831 8	.690	.723 809 4	951 9	.740	.672 606 9	1098 5
.641	.767 540 9	834 0	.691	.722 854 8	954 6	.741	.671 505 0	1101 9
.642	.766 704 6	836 3	.692	.721 897 5	957 3	.742	.670 399 9	1105 1
.643	.765 866 2	838 4	.693	.720 937 6	959 9	.743	.669 291 4	1108 5
.644	.765 025 5	840 7	.694	.719 975 0	962 6	.744	.668 179 6	1111 8
.645	.764 182 6	842 9	.695	.719 009 7	965 3	.745	.667 064 5	1115 1
.646	.763 337 4	845 2	.696	.718 041 8	967 9	.746	.665 945 9	1118 6
.647	.762 490 0	847 4	.697	.717 071 1	970 7	.747	.664 824 0	1121 9
.648	.761 640 3	849 7	.698	.716 097 8	973 3	.748	.663 698 7	1125 3
.649	.760 788 4	851 9	.699	.715 121 7	976 1	.749	.662 570 0	1128 7
.650	.759 934 2	854 2	.700	.714 142 8	978 9	.750	.661 437 8	1132 2
		856 5			981 5			1135 6

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SINE	COSINE	DIFF.	SINE	COSINE	DIFF.	SINE	COSINE	DIFF.
.751	.660 302 2	1135 6	.801	.598 664 3	1335 7	.851	.525 165 7	1617 0
.752	.659 163 1	1139 1	.802	.597 324 0	1340 3	.852	.523 541 8	1623 9
.753	.658 020 5	1142 6	.803	.595 979 0	1345 0	.853	.521 910 9	1630 9
.754	.656 874 4	1146 1	.804	.594 629 3	1349 7	.854	.520 273 0	1637 9
.755	.655 724 8	1149 6	.805	.593 274 8	1354 5	.855	.518 628 0	1645 0
.756	.654 571 6	1153 2	.806	.591 915 5	1359 3	.856	.516 975 8	1652 2
.757	.653 414 9	1156 7	.807	.590 551 4	1364 1	.857	.515 316 4	1659 4
.758	.652 254 6	1160 3	.808	.589 182 5	1368 9	.858	.513 649 7	1666 7
.759	.651 090 6	1164 0	.809	.587 808 6	1373 9	.859	.511 975 6	1674 1
.760	.649 923 1	1167 5	.810	.586 429 9	1378 7	.860	.510 294 0	1681 6
.761	.648 751 9	1171 2	.811	.585 046 1	1383 8	.861	.508 604 9	1689 1
.762	.647 577 0	1174 9	.812	.583 657 4	1388 7	.862	.506 908 3	1696 6
.763	.646 398 5	1178 5	.813	.582 263 7	1393 7	.863	.505 203 9	1704 4
.764	.645 216 2	1182 3	.814	.580 864 9	1398 8	.864	.503 491 8	1712 1
.765	.644 030 3	1185 9	.815	.579 461 0	1403 9	.865	.501 771 9	1719 9
.766	.642 840 6	1189 7	.816	.578 051 9	1409 1	.866	.500 044 0	1727 9
.767	.641 647 1	1193 5	.817	.576 637 7	1414 2	.867	.498 308 1	1735 9
.768	.640 449 8	1197 3	.818	.575 218 2	1419 5	.868	.496 564 2	1743 9
.769	.639 248 8	1201 0	.819	.573 793 5	1424 7	.869	.494 812 1	1752 1
.770	.638 043 9	1204 9	.820	.572 363 5	1430 0	.870	.493 051 7	1760 4
.771	.636 835 1	1208 8	.821	.570 928 2	1435 3	.871	.491 283 0	1768 7
.772	.635 622 5	1212 6	.822	.569 487 5	1440 7	.872	.489 505 9	1777 1
.773	.634 406 0	1216 5	.823	.568 041 4	1446 1	.873	.487 720 2	1785 7
.774	.633 185 6	1220 4	.824	.566 589 8	1451 6	.874	.485 925 9	1794 3
.775	.631 961 2	1224 4	.825	.565 132 7	1457 1	.875	.484 122 9	1803 0
.776	.630 732 9	1228 3	.826	.563 670 1	1462 6	.876	.482 311 1	1811 8
.777	.629 500 6	1232 3	.827	.562 201 9	1468 2	.877	.480 490 4	1820 7
.778	.628 264 3	1236 3	.828	.560 728 1	1473 8	.878	.478 660 6	1829 8
.779	.627 023 9	1240 4	.829	.559 248 6	1479 5	.879	.476 821 8	1838 8
.780	.625 779 5	1244 4	.830	.557 763 4	1485 2	.880	.474 973 7	1848 1
.781	.624 531 0	1248 5	.831	.556 272 4	1491 0	.881	.473 116 3	1857 4
.782	.623 278 4	1252 6	.832	.554 775 6	1496 8	.882	.471 249 4	1866 9
.783	.622 021 7	1256 7	.833	.553 273 0	1502 6	.883	.469 373 0	1876 4
.784	.620 760 8	1260 9	.834	.551 764 4	1508 6	.884	.467 486 9	1886 1
.785	.619 495 8	1265 0	.835	.550 249 9	1514 5	.885	.465 591 0	1895 9
.786	.618 226 5	1269 3	.836	.548 729 4	1520 5	.886	.463 685 2	1905 8
.787	.616 953 0	1273 5	.837	.547 202 9	1526 5	.887	.461 769 4	1915 8
.788	.615 675 2	1277 8	.838	.545 670 2	1532 7	.888	.459 843 4	1926 0
.789	.614 393 2	1282 0	.839	.544 131 4	1538 8	.889	.457 907 2	1936 2
.790	.613 106 8	1286 4	.840	.542 586 4	1545 0	.890	.455 960 5	1946 7
.791	.611 816 1	1290 7	.841	.541 035 1	1551 3	.891	.454 003 3	1957 2
.792	.610 521 1	1295 0	.842	.539 477 5	1557 6	.892	.452 035 4	1967 9
.793	.609 221 6	1299 5	.843	.537 913 6	1563 9	.893	.450 056 7	1978 7
.794	.607 917 8	1303 8	.844	.536 343 2	1570 4	.894	.448 067 0	1989 7
.795	.606 609 4	1308 4	.845	.534 766 3	1576 9	.895	.446 066 1	2000 9
.796	.605 296 6	1312 8	.846	.533 182 9	1583 4	.896	.444 054 1	2012 0
.797	.603 979 3	1317 3	.847	.531 592 9	1590 0	.897	.442 030 6	2023 5
.798	.602 657 4	1321 9	.848	.529 996 2	1596 7	.898	.439 995 4	2035 2
.799	.601 331 0	1326 4	.849	.528 392 8	1603 4	.899	.437 948 6	2046 8
.800	.600 000 0	1331 0	.850	.526 782 7	1610 1	.900	.435 889 9	2058 7
		1335 7			1617 0			2070 8

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SINE	COSINE	DIFF.	SINE	COSINE	DIFF.
.901	.433 819 1	2 0708	.951	.309 190 9	3 0590
.902	.431 736 0	2 0831	.952	.306 098 0	3 0929
.903	.429 640 5	2 0955	.953	.302 970 3	3 1277
.904	.427 532 5	2 1080	.954	.299 806 6	3 1637
.905	.425 411 6	2 1209	.955	.296 605 8	3 2008
.906	.423 277 7	2 1339	.956	.293 366 7	3 2391
.907	.421 130 6	2 1471	.957	.290 087 9	3 2788
.908	.418 970 2	2 1604	.958	.286 768 2	3 3197
.909	.416 796 1	2 1741	.959	.283 406 1	3 3621
.910	.414 608 3	2 1878	.960	.280 000 0	3 4061
.911	.412 406 4	2 2019	.961	.276 548 4	3 4516
.912	.410 190 2	2 2162	.962	.273 049 4	3 4990
.913	.407 959 6	2 2306	.963	.269 501 4	3 5480
.914	.405 714 2	2 2454	.964	.265 902 2	3 5992
.915	.403 453 8	2 2604	.965	.262 249 9	3 6523
.916	.401 178 3	2 2755	.966	.258 542 0	3 7079
.917	.398 887 2	2 2911	.967	.254 776 4	3 7656
.918	.396 580 4	2 3068	.968	.250 950 2	3 8262
.919	.394 257 5	2 3229	.969	.247 060 7	3 8895
.920	.391 918 4	2 3391	.970	.243 104 9	3 9558
.921	.389 562 6	2 3558	.971	.239 079 5	4 0254
.922	.387 189 9	2 3727	.972	.234 980 9	4 0986
.923	.384 799 9	2 3900	.973	.230 805 1	4 1758
.924	.382 392 5	2 4074	.974	.226 548 0	4 2571
.925	.379 967 1	2 4254	.975	.222 204 9	4 3431
.926	.377 523 5	2 4436	.976	.217 770 5	4 4344
.927	.375 061 3	2 4622	.977	.213 239 3	4 5312
.928	.372 580 2	2 4811	.978	.208 604 9	4 6344
.929	.370 079 7	2 5005	.979	.203 860 2	4 7447
.930	.367 559 5	2 5202	.980	.198 997 5	4 8627
.931	.365 019 2	2 5403	.981	.194 007 7	4 9898
.932	.362 458 3	2 5609	.982	.188 880 9	5 1268
.933	.359 876 4	2 5819	.983	.183 605 6	5 2753
.934	.357 273 0	2 6034	.984	.178 168 5	5 4371
.935	.354 647 7	2 6253	.985	.172 554 3	5 6142
.936	.352 000 0	2 6477	.986	.166 745 3	5 8090
.937	.349 329 4	2 6706	.987	.160 720 3	6 0250
.938	.346 635 3	2 6941	.988	.154 453 9	6 2664
.939	.343 917 1	2 7182	.989	.147 915 5	6 5384
.940	.341 174 4	2 7427	.990	.141 067 4	6 8481
.941	.338 406 6	2 7678	.991	.133 861 9	7 2055
.942	.335 612 9	2 7937	.992	.126 237 9	7 6240
.943	.332 792 7	2 8202	.993	.118 114 4	8 1235
.944	.329 945 5	2 8472	.994	.109 380 1	8 7343
.945	.327 070 3	2 8752	.995	.099 874 9	9 5052
.946	.324 166 6	2 9037	.996	.089 353 2	10 5217
.947	.321 233 6	2 9330	.997	.077 401 6	11 9516
.948	.318 270 3	2 9633	.998	.063 213 9	14 1877
.949	.315 276 1	2 9942	.999	.044 710 2	18 5037
.950	.312 249 9	3 0262	1.000	.000 000 0	44 7102
		3 0590			×

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R. A. Davis' Table of Natural Sines and Radians

By permission of the author, we are privileged to supply this table as a part of our series of Marchant Methods. The table shows lengths of arcs measured in terms of the radius for angles which have the tabulated sines; for example, if the radius of a circle is 10 ft., the length of arc that subtends an angle whose sine is .643 is .698409 x 10., or 6.98409 ft. It will be seen that the amounts in the columns headed "Arc-Rad" are measures of the angles expressed in Radians (an angle of 1 radian has its arc equal to its radius).

The range in which straight-line interpolation provides last-place accuracy (subject to rounding-off errors) is:

Sine	0 to .815	6 place accuracy of interpolated arc
	.815 to .960	5 " " " " "
	.96 to .990	4 " " " " "
	.99 to .998	3 " " " " "

In the higher ranges of the table, interpolation by Marchant Method MM-64 or MM-189 will provide six place accuracy.

DIRECT INTERPOLATION

A rapid method of straight-line interpolation is illustrated in the following example:

Find the length of arc (if radius is 1) of the angle whose sine is .738463.

Decimals: Upper Dial 3, Middle Dial 9, Keyboard Dial 6.

(1) Set up in Keyboard Dial the radians corresponding to sine .739 (.831585) and multiply by the ratio that the desired interpolate is "on the way" toward .739(.463).

(2) Change Keyboard Dial to read the radians corresponding to sine .738 (.830102) and, without clearing any dials, multiply by such amount as will cause Upper Dial to read 1.0000 (in this case, the multiplier is .537).
The desired arc (.830789) appears in Middle Dial.

INVERSE INTERPOLATION

Find the sine corresponding to length of arc (if radius is 1) which is .830789.

(1) Inspection of table shows that the desired sine lies between .738 and .739. Set up in Keyboard Dial the radians corresponding to the lower value of these sines (.830102) and, with carriage in 4th position, depress Add Bar.

(2) Set up in Keyboard Dial the difference (.001483). Clear Upper Dial and, by multiplication, build up the Upper Dial until the Middle Dial reads the length of arc (radians) to which it is desired to interpolate (.830789).
Upper Dial shows the three figures that are to be annexed to the lower value of the sine (.463). The desired sine is thus .738463.

On M models this process is most easily performed if the Non-Shift Key is depressed. Successive depressions of the No. 1 Key of the Single Row Keyboard with carriage in 3rd position will soon cause the Middle Dial to show an amount which is slightly greater than the desired amount. Next, reverse multiply by "1", shift to 2nd position and repeat, etc.

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SINE	ARC: RAD.	DIFF.	SINE	ARC: RAD.	DIFF	SINE	ARC: RAD.	DIFF.
.001	.001000	1000	.051	.051021	1001	.101	.101172	1005
.002	.002000	1000	.052	.052022	1001	.102	.102177	1005
.003	.003000	1000	.053	.053023	1001	.103	.103183	1006
.004	.004000	1000	.054	.054025	1002	.104	.104188	1005
.005	.005000	1000	.055	.055026	1001	.105	.105194	1006
.006	.006000	1000	.056	.056028	1002	.106	.106199	1005
.007	.007000	1000	.057	.057029	1001	.107	.107204	1005
.008	.008000	1000	.058	.058031	1002	.108	.108210	1005
.009	.009000	1000	.059	.059033	1002	.109	.109216	1006
.010	.010000	1000	.060	.060035	1002	.110	.110221	1005
.011	.011000	1000	.061	.061036	1001	.111	.111228	1007
.012	.012000	1000	.062	.062038	1002	.112	.112234	1006
.013	.013000	1000	.063	.063040	1002	.113	.113241	1007
.014	.014000	1000	.064	.064042	1002	.114	.114247	1006
.015	.015000	1000	.065	.065045	1003	.115	.115254	1007
.016	.016000	1000	.066	.066047	1002	.116	.116260	1006
.017	.017000	1000	.067	.067049	1002	.117	.117267	1007
.018	.018000	1000	.068	.068052	1003	.118	.118274	1007
.019	.019000	1000	.069	.069054	1002	.119	.119281	1007
.020	.020000	1000	.070	.070057	1003	.120	.120289	1008
.021	.021001	1001	.071	.071059	1002	.121	.121296	1007
.022	.022001	1000	.072	.072062	1003	.122	.122305	1009
.023	.023001	1000	.073	.073064	1002	.123	.123312	1007
.024	.024002	1001	.074	.074067	1003	.124	.124320	1008
.025	.025002	1000	.075	.075070	1003	.125	.125327	1007
.026	.026002	1000	.076	.076073	1003	.126	.126335	1008
.027	.027002	1000	.077	.077076	1003	.127	.127343	1008
.028	.028003	1001	.078	.078079	1003	.128	.128351	1008
.029	.029003	1000	.079	.079082	1003	.129	.129359	1008
.030	.030004	1001	.080	.080085	1003	.130	.130368	1009
.031	.031004	1000	.081	.081088	1003	.131	.131377	1009
.032	.032005	1001	.082	.082091	1003	.132	.132386	1009
.033	.033006	1001	.083	.083095	1004	.133	.133395	1009
.034	.034006	1000	.084	.084098	1003	.134	.134404	1009
.035	.035007	1001	.085	.085102	1004	.135	.135413	1009
.036	.036007	1000	.086	.086106	1004	.136	.136422	1009
.037	.037008	1001	.087	.087109	1003	.137	.137432	1010
.038	.038009	1001	.088	.088113	1004	.138	.138442	1010
.039	.039009	1000	.089	.089117	1004	.139	.139451	1009
.040	.040010	1001	.090	.090121	1004	.140	.140461	1010
.041	.041011	1001	.091	.091125	1004	.141	.141471	1010
.042	.042012	1001	.092	.092129	1004	.142	.142481	1010
.043	.043013	1001	.093	.093134	1005	.143	.143491	1010
.044	.044014	1001	.094	.094138	1004	.144	.144502	1011
.045	.045015	1001	.095	.095142	1004	.145	.145512	1010
.046	.046016	1001	.096	.096147	1005	.146	.146523	1011
.047	.047017	1001	.097	.097152	1005	.147	.147534	1011
.048	.048018	1001	.098	.098157	1005	.148	.148545	1011
.049	.049019	1001	.099	.099161	1004	.149	.149556	1011
.050	.050020	1001	.100	.100167	1006	.150	.150568	1012
					1005			1011

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SINE	ARC : RAD.	DIFF.	SINE	ARC : RAD.	DIFF.	SINE	ARC : RAD.	DIFF.
.151	.151 579	1011	.201	.202 378	1021	.251	.253 713	1033
.152	.152 591	1012	.202	.203.399	1021	.252	.254 746	1033
.153	.153 602	1011	.203	.204 420	1021	.253	.255 779	1033
.154	.154 614	1012	.204	.205 441	1021	.254	.256 814	1035
.155	.155 627	1013	.205	.206 463	1022	.255	.257 848	1034
.156	.156 639	1012	.206	.207 485	1022	.256	.258 882	1034
.157	.157 652	1013	.207	.208 507	1022	.257	.259 917	1035
.158	.158 664	1012	.208	.209 529	1022	.258	.260 951	1034
.159	.159 677	1013	.209	.210 551	1022	.259	.261 986	1035
.160	.160 690	1013	.210	.211 574	1023	.260	.263 021	1035
.161	.161 703	1013	.211	.212 597	1023	.261	.264 057	1036
.162	.162 716	1013	.212	.213 620	1023	.262	.265 093	1036
.163	.163 729	1013	.213	.214 643	1023	.263	.266 130	1037
.164	.164 743	1014	.214	.215 667	1024	.264	.267 166	1036
.165	.165 757	1014	.215	.216 691	1024	.265	.268 203	1037
.166	.166 771	1014	.216	.217 715	1024	.266	.269 241	1038
.167	.167 785	1014	.217	.218 739	1024	.267	.270 278	1037
.168	.168 799	1014	.218	.219 764	1025	.268	.271 316	1038
.169	.169 814	1015	.219	.220 788	1024	.269	.272 354	1038
.170	.170 829	1015	.220	.221 813	1025	.270	.273 393	1039
.171	.171 843	1014	.221	.222 839	1026	.271	.274 431	1038
.172	.172 859	1016	.222	.223 864	1025	.272	.275 470	1039
.173	.173 874	1015	.223	.224 890	1026	.273	.276 509	1039
.174	.174 889	1015	.224	.225 916	1026	.274	.277 549	1040
.175	.175 905	1016	.225	.226 942	1026	.275	.278 589	1040
.176	.176 920	1015	.226	.227 969	1027	.276	.279 629	1040
.177	.177 936	1016	.227	.228 995	1026	.277	.280 670	1041
.178	.178 952	1016	.228	.230 022	1027	.278	.281 711	1041
.179	.179 968	1016	.229	.231 049	1027	.279	.282 752	1041
.180	.180 985	1017	.230	.232 077	1028	.280	.283 793	1041
.181	.182 001	1016	.231	.233 104	1027	.281	.284 835	1042
.182	.183 018	1017	.232	.234 132	1028	.282	.285 877	1042
.183	.184 036	1018	.233	.235 161	1029	.283	.286 920	1043
.184	.185 053	1017	.234	.236 189	1028	.284	.287 963	1043
.185	.186 070	1017	.235	.237 218	1029	.285	.289 006	1043
.186	.187 088	1018	.236	.238 246	1028	.286	.290 049	1043
.187	.188 106	1018	.237	.239 276	1030	.287	.291 093	1044
.188	.189 124	1018	.238	.240 305	1029	.288	.292 137	1044
.189	.190 142	1018	.239	.241 335	1030	.289	.293 181	1044
.190	.191 161	1019	.240	.242 365	1030	.290	.294 226	1045
.191	.192 180	1019	.241	.243 395	1030	.291	.295 271	1045
.192	.193 199	1019	.242	.244 426	1031	.292	.296 316	1045
.193	.194 218	1019	.243	.245 457	1031	.293	.297 363	1047
.194	.195 237	1019	.244	.246 488	1031	.294	.298 409	1046
.195	.196 257	1020	.245	.247 519	1031	.295	.299 455	1046
.196	.197 276	1019	.246	.248 551	1032	.296	.300 502	1047
.197	.198 296	1020	.247	.249 583	1032	.297	.301 549	1047
.198	.199 316	1020	.248	.250 615	1032	.298	.302 597	1048
.199	.200 337	1021	.249	.251 647	1032	.299	.303 645	1048
.200	.201 357	1020	.250	.252 680	1033	.300	.304 692	1047
		1021			1033			1049

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SINE	ARC: RAD.	DIFF.	SINE	ARC: RAD.	DIFF.	SINE	ARC: RAD.	DIFF.
.301	.305741	1049	.351	.358638	1068	.401	.412607	1091
.302	.306790	1049	.352	.359706	1068	.402	.413699	1092
.303	.307839	1049	.353	.360774	1068	.403	.414791	1092
.304	.308888	1050	.354	.361843	1069	.404	.415884	1093
.305	.309938	1050	.355	.362913	1070	.405	.416978	1094
.306	.310988	1051	.356	.363983	1070	.406	.418072	1094
.307	.312039	1051	.357	.365053	1070	.407	.419167	1095
.308	.313090	1051	.358	.366124	1071	.408	.420262	1095
.309	.314141	1052	.359	.367196	1072	.409	.421358	1096
.310	.315193	1052	.360	.368268	1072	.410	.422454	1096
.311	.316245	1052	.361	.369339	1071	.411	.423550	1096
.312	.317297	1053	.362	.370412	1073	.412	.424647	1097
.313	.318350	1053	.363	.371485	1073	.413	.425745	1098
.314	.319403	1053	.364	.372559	1074	.414	.426844	1099
.315	.320456	1054	.365	.373633	1074	.415	.427943	1099
.316	.321510	1054	.366	.374707	1074	.416	.429043	1100
.317	.322564	1054	.367	.375782	1075	.417	.430142	1099
.318	.323618	1055	.368	.376857	1076	.418	.431243	1101
.319	.324673	1056	.369	.377933	1076	.419	.432344	1101
.320	.325729	1056	.370	.379009	1076	.420	.433445	1101
.321	.326785	1056	.371	.380085	1076	.421	.434548	1103
.322	.327841	1056	.372	.381162	1077	.422	.435650	1102
.323	.328897	1057	.373	.382240	1078	.423	.436753	1103
.324	.329954	1058	.374	.383318	1078	.424	.437857	1104
.325	.331012	1057	.375	.384396	1078	.425	.438961	1104
.326	.332069	1058	.376	.385475	1079	.426	.440066	1105
.327	.333127	1058	.377	.386554	1079	.427	.441172	1106
.328	.334185	1059	.378	.387635	1081	.428	.442278	1106
.329	.335244	1059	.379	.388715	1080	.429	.443385	1107
.330	.336303	1060	.380	.389796	1081	.430	.444492	1107
.331	.337363	1060	.381	.390878	1082	.431	.445600	1108
.332	.338423	1060	.382	.391959	1081	.432	.446709	1109
.333	.339483	1061	.383	.393041	1082	.433	.447817	1108
.334	.340544	1061	.384	.394124	1083	.434	.448927	1110
.335	.341605	1061	.385	.395207	1083	.435	.450038	1111
.336	.342666	1062	.386	.396291	1084	.436	.451148	1110
.337	.343728	1063	.387	.397375	1084	.437	.452260	1112
.338	.344791	1062	.388	.398460	1085	.438	.453372	1112
.339	.345853	1063	.389	.399545	1085	.439	.454485	1113
.340	.346916	1064	.390	.400631	1086	.440	.455598	1113
.341	.347980	1064	.391	.401717	1086	.441	.456711	1113
.342	.349044	1064	.392	.402804	1087	.442	.457826	1115
.343	.350108	1065	.393	.403891	1087	.443	.458941	1115
.344	.351173	1065	.394	.404979	1088	.444	.460057	1115
.345	.352238	1065	.395	.406067	1088	.445	.461173	1116
.346	.353303	1067	.396	.407155	1088	.446	.462290	1117
.347	.354370	1067	.397	.408245	1090	.447	.463408	1118
.348	.355437	1066	.398	.409335	1090	.448	.464527	1119
.349	.356503	1067	.399	.410425	1090	.449	.465645	1118
.350	.357570	1068	.400	.411516	1091	.450	.466765	1120

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SINE	ARC : RAD.	DIFF.	SINE	ARC : RAD.	DIFF.	SINE	ARC : RAD	DIFF
.451	.467885+		.501	.524754-	1155	.551	.583562-	1198
.452	.469006+	1121	.502	.525910-	1156	.552	.584761-	1199
.453	.470127+	1121	.503	.527066+	1156	.553	.585961-	1200
.454	.471249+	1122	.504	.528224-	1158	.554	.587161+	1200
.455	.472372+	1123	.505	.529382-	1158	.555	.588363	1202
.456	.473495+	1123	.506	.530541-	1159	.556	.589566-	1203
.457	.474619+	1124	.507	.531701-	1160	.557	.590769+	1203
.458	.475744	1125	.508	.532861+	1160	.558	.591974-	1205
.459	.476869+	1125	.509	.534023-	1162	.559	.593179+	1205
.460	.477995+	1126	.510	.535185-	1162	.560	.594386-	1207
.461	.479122-	1127	.511	.536348-	1163	.561	.595593+	1207
.462	.480249	1127	.512	.537511+	1163	.562	.596802-	1209
.463	.481377-	1128	.513	.538676	1165	.563	.598011+	1209
.464	.482505+	1128	.514	.539841+	1165	.564	.599222-	1211
.465	.483635-	1130	.515	.541008-	1167-	.565	.600433+	1211
.466	.484765-	1130	.516	.542175-	1167	.566	.601646-	1213
.467	.485895	1130	.517	.543342+	1167+	.567	.602859+	1213
.468	.487026+	1131	.518	.544511+	1169	.568	.604074-	1215
.469	.488158+	1132	.519	.545681-	1170	.569	.605289+	1215
.470	.489291-	1133	.520	.546851-	1170	.570	.606506-	1217
.471	.490424	1133	.521	.548022+	1171	.571	.607723+	1217
.472	.491558-	1134	.522	.549194+	1172	.572	.608942	1219
.473	.492693-	1135	.523	.550367-	1173	.573	.610162-	1220
.474	.493828	1135	.524	.551540+	1173	.574	.611382+	1220
.475	.494964	1136	.525	.552715	1175	.575	.612604+	1222
.476	.496101-	1137	.526	.553890+	1175	.576	.613827-	1223
.477	.497238+	1137	.527	.555067-	1177	.577	.615051-	1224
.478	.498376+	1138	.528	.556244-	1177	.578	.616276-	1225
.479	.499515+	1139	.529	.557422-	1178	.579	.617502-	1226
.480	.500655-	1140	.530	.558601-	1179	.580	.618729-	1227
.481	.501795-	1140	.531	.559780+	1179	.581	.619957-	1228
.482	.502936	1141	.532	.560961-	1181	.582	.621186	1229
.483	.504078-	1142	.533	.562142+	1181	.583	.622416+	1230
.484	.505220	1142	.534	.563324+	1182	.584	.623648-	1232
.485	.506363+	1143	.535	.564508-	1184	.585	.624880	1232
.486	.507507	1144	.536	.565692-	1184	.586	.626114-	1234
.487	.508652-	1145	.537	.566877-	1185	.587	.627348+	1234
.488	.509797-	1145	.538	.568063-	1186	.588	.628584	1236
.489	.510943-	1146	.539	.569249+	1186+	.589	.629821-	1237
.490	.512090-	1147	.540	.570437+	1188	.590	.631059	1238
.491	.513237+	1147	.541	.571626-	1189	.591	.632298-	1239
.492	.514385+	1148	.542	.572815+	1189	.592	.633538+	1240
.493	.515534+	1149+	.543	.574006-	1191	.593	.634780-	1242
.494	.516684+	1150+	.544	.575197-	1191	.594	.636022	1242
.495	.517835-	1151	.545	.576389+	1192	.595	.637266-	1244
.496	.518986+	1151	.546	.577582+	1193	.596	.638511-	1245
.497	.520138	1152	.547	.578776+	1194	.597	.639756+	1245
.498	.521291-	1153	.548	.579971+	1195	.598	.641003+	1247
.499	.522444+	1153	.549	.581167+	1196	.599	.642252-	1249
.500	.523599-	1155	.550	.582364+	1197	.600	.643501	1249
					1198			1251

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SINE	ARC: RAD.	DIFF.	SINE	ARC: RAD.	DIFF.	SINE	ARC: RAD.	DIFF.
.601	.644752⁻	1251	.651	.708901	1317	.701	.776799⁻	1401
.602	.646003⁺	1251	.652	.710219⁺	1318	.702	.778202⁻	1403
.603	.647256⁺	1253	.653	.711539⁻	1320	.703	.779607	1405
.604	.648511⁻	1255	.654	.712860	1321	.704	.781014⁺	1407
.605	.649766⁻	1255	.655	.714183⁻	1323	.705	.782423⁺	1409
.606	.651022⁺	1256	.656	.715507⁻	1324	.706	.783834⁺	1411
.607	.652280	1258	.657	.716832⁺	1325	.707	.785247⁺	1413
.608	.653539	1259	.658	.718160⁻	1328	.708	.786662⁺	1415
.609	.654799⁺	1260	.659	.719489⁻	1329	.709	.788079⁺	1417
.610	.656061⁻	1262	.660	.720819⁻	1330	.710	.789498⁺	1419
.611	.657323⁺	1262	.661	.722151⁻	1332	.711	.790919⁺	1421
.612	.658587	1264	.662	.723484	1333	.712	.792342⁺	1423
.613	.659852	1265	.663	.724819	1335	.713	.793768⁻	1426
.614	.661118⁺	1266	.664	.726156⁻	1337	.714	.795195⁻	1427
.615	.662386	1268	.665	.727494⁻	1338	.715	.796624⁺	1429
.616	.663655⁻	1269	.666	.728834⁻	1340	.716	.798056⁻	1432
.617	.664925⁻	1270	.667	.730175	1341	.717	.799489	1433
.618	.666196⁺	1271	.668	.731518	1343	.718	.800925⁻	1436
.619	.667469⁻	1273	.669	.732863⁻	1345	.719	.802363⁻	1438
.620	.668743⁻	1274	.670	.734209⁻	1346	.720	.803802⁺	1439
.621	.670018⁻	1275	.671	.735557⁻	1348	.721	.805244⁺	1442
.622	.671294⁺	1276	.672	.736906⁺	1349	.722	.806689⁻	1445
.623	.672572⁺	1278	.673	.738257⁺	1351	.723	.808135⁺	1446
.624	.673851⁺	1279	.674	.739610⁺	1353	.724	.809584⁻	1449
.625	.675132⁻	1281	.675	.740965⁻	1355	.725	.811034⁺	1450
.626	.676413⁺	1281	.676	.742321	1356	.726	.812487⁺	1453
.627	.677696⁺	1283	.677	.743679⁻	1358	.727	.813943⁻	1456
.628	.678980⁺	1284	.678	.745038⁺	1359	.728	.815400⁺	1457
.629	.680266⁺	1286	.679	.746400⁻	1362	.729	.816860	1460
.630	.681553⁺	1287	.680	.747763⁻	1363	.730	.818322	1462
.631	.682842⁻	1289	.681	.749127⁺	1364	.731	.819786⁺	1464
.632	.684131⁺	1289	.682	.750494⁻	1367	.732	.821253⁺	1467
.633	.685422⁺	1291	.683	.751862	1368	.733	.822722⁻	1469
.634	.686715⁻	1293	.684	.753232	1370	.734	.824193	1471
.635	.688008⁺	1293	.685	.754604⁻	1372	.735	.825667⁻	1474
.636	.689304⁻	1296	.686	.755977⁺	1373	.736	.827143⁻	1476
.637	.690600⁺	1296	.687	.757353⁻	1376	.737	.828621⁺	1478
.638	.691898⁺	1298	.688	.758730⁻	1377	.738	.830102⁻	1481
.639	.693197⁺	1299	.689	.760108⁺	1378	.739	.831585⁻	1483
.640	.694498⁺	1301	.690	.761489⁺	1381	.740	.833070⁺	1485
.641	.695800⁺	1302	.691	.762872⁻	1383	.741	.834558⁺	1488
.642	.697104	1304	.692	.764256⁺	1384	.742	.836049⁻	1491
.643	.698409	1305	.693	.765642	1386	.743	.837542⁻	1493
.644	.699715⁺	1306	.694	.767030	1388	.744	.839037	1495
.645	.701023⁺	1308	.695	.768420	1390	.745	.840535⁻	1498
.646	.702333⁻	1310	.696	.769812⁻	1392	.746	.842035⁺	1500
.647	.703643⁺	1310	.697	.771205⁺	1393	.747	.843538	1503
.648	.704956⁻	1313	.698	.772601⁻	1396	.748	.845043⁺	1505
.649	.706269⁺	1313	.699	.773998⁺	1397	.749	.846552⁻	1509
.650	.707584⁺	1315	.700	.775398⁻	1400	.750	.848062	1510
		1317			1401			1513

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SINE	ARC: RAD.	DIFF.	SINE	ARC: RAD.	DIFF.	SINE	ARC: RAD.	DIFF.
.751	.849575+	1513	.801	.928964-	1669	.851	1.017887-	1902
.752	.851091-	1516	.802	.930636	1672	.852	1.019794-	1907
.753	.852609+	1518	.803	.932312	1676	.853	1.021707-	1913
.754	.854130+	1521	.804	.933992-	1680	.854	1.023626-	1919
.755	.855654-	1524	.805	.935675+	1683	.855	1.025551-	1925
.756	.857180+	1526	.806	.937363	1688	.856	1.027482+	1931
.757	.858709+	1529	.807	.939054+	1691	.857	1.029420-	1938
.758	.860241+	1532	.808	.940750-	1696	.858	1.031363+	1943
.759	.861776-	1535	.809	.942449-	1699	.859	1.033313+	1950
.760	.863313-	1537	.810	.944152+	1703	.860	1.035270-	1957
.761	.864853+	1540	.811	.945859+	1707	.861	1.037233-	1963
.762	.866396	1543	.812	.947571-	1712	.862	1.039202+	1969
.763	.867942-	1546	.813	.949286	1715	.863	1.041178+	1976
.764	.869490	1548	.814	.951006-	1720	.864	1.043161-	1983
.765	.871041+	1551	.815	.952729+	1723	.865	1.045150+	1989
.766	.872596-	1555	.816	.954457	1728	.866	1.047147-	1997
.767	.874153-	1557	.817	.956189+	1732	.867	1.049150+	2003
.768	.875712+	1559	.818	.957925+	1736	.868	1.051160+	2010
.769	.877275+	1563	.819	.959666	1741	.869	1.053178-	2018-
.770	.878841-	1566	.820	.961411	1745	.870	1.055202+	2024+
.771	.880410-	1569	.821	.963160+	1749	.871	1.057234-	2032
.772	.881982-	1572	.822	.964914-	1754	.872	1.059273+	2039
.773	.883556+	1574	.823	.966672+	1758	.873	1.061320-	2047
.774	.885134+	1578	.824	.968435	1763	.874	1.063374-	2054
.775	.886715	1581	.825	.970202+	1767	.875	1.065436-	2062
.776	.888299	1584	.826	.971974	1772	.876	1.067505+	2069
.777	.889886	1587	.827	.973750+	1776+	.877	1.069583-	2078
.778	.891476+	1590	.828	.975532-	1782-	.878	1.071668-	2085
.779	.893069+	1593	.829	.977317+	1785+	.879	1.073761-	2093
.780	.894666-	1597	.830	.979108-	1791	.880	1.075862+	2101
.781	.896265+	1599	.831	.980903	1795	.881	1.077972-	2110-
.782	.897868+	1603	.832	.982703-	1800	.882	1.080089+	2117+
.783	.899474+	1606	.833	.984508	1805	.883	1.082216-	2127-
.784	.901084-	1610	.834	.986318	1810	.884	1.084351-	2135
.785	.902696+	1612	.835	.988133-	1815	.885	1.086494+	2143
.786	.904312	1616	.836	.989953-	1820	.886	1.088646+	2152
.787	.905931+	1619	.837	.991778-	1825	.887	1.090807+	2161
.788	.907554-	1623	.838	.993608-	1830	.888	1.092977+	2170
.789	.909180-	1626	.839	.995443-	1835	.889	1.095157-	2180
.790	.910809	1629	.840	.997283+	1840	.890	1.097345+	2188
.791	.912442-	1633	.841	.999129	1846	.891	1.099543+	2198
.792	.914078	1636	.842	1.000980-	1851	.892	1.101750+	2207
.793	.915718-	1640	.843	1.002836+	1856	.893	1.103968-	2218-
.794	.917361-	1643	.844	1.004698-	1862	.894	1.106194	2226
.795	.919008-	1647	.845	1.006565+	1867	.895	1.108432-	2238
.796	.920658-	1650	.846	1.008438	1873	.896	1.110678+	2246
.797	.922312-	1654	.847	1.010316+	1878	.897	1.112935+	2257
.798	.923969+	1657	.848	1.012200+	1884	.898	1.115203-	2268
.799	.925630+	1661	.849	1.014090-	1890	.899	1.117481-	2278
.800	.927295+	1665	.850	1.015985+	1895	.900	1.119770	2289
		1669			1902			2299

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SINE	ARC : RAD.	DIFF.	SINE	ARC : RAD.	DIFF.
.901	1.122 069	2 299	.951	1.256 454	3218
.902	1.124 380	2 311	.952	1.259 705	3 251
.903	1.126 702	2 322	.953	1.262 989	3 284
.904	1.129 035	2 333	.954	1.266 306	3 317
.905	1.131 380	2 345	.955	1.269 660	3 354
.906	1.133 736	2 356	.956	1.273 050	3 390
.907	1.136 104	2 368	.957	1.276 478	3 428
.908	1.138 485	2 381	.958	1.279 945	3 467
.909	1.140 878	2 393	.959	1.283 453	3 508
.910	1.143 284	2 406	.960	1.287 003	3 550
.912	1.145 703	2 419	.961	1.290 596	3 593
.913	1.148 134	2 431	.962	1.294 235	3 639
.913	1.150 578	2 444	.963	1.297 921	3 686
.914	1.153 036	2 458	.964	1.301 657	3 736
.915	1.155 508	2 472	.965	1.305 444	3 787
.916	1.157 994	2 486	.966	1.309 284	3 840
.917	1.160 494	2 500	.967	1.313 180	3 896
.918	1.163 008	2 514	.968	1.317 135	3 955
.919	1.165 537	2 529	.969	1.321 151	4 016
.920	1.168 081	2 544	.970	1.325 231	4 080
.921	1.170 640	2 559	.971	1.329 379	4 148
.922	1.173 215	2 575	.972	1.333 598	4 219
.923	1.175 806	2 591	.973	1.337 892	4 294
.924	1.178 413	2 607	.974	1.342 265	4 373
.925	1.181 036	2 623	.975	1.346 722	4 457
.926	1.183 676	2 640	.976	1.351 267	4 545
.927	1.186 334	2 658	.977	1.355 907	4 640
.928	1.189 009	2 675	.978	1.360 649	4 742
.929	1.191 702	2 693	.979	1.365 498	4 849
.930	1.194 413	2 711	.980	1.370 462	4 964
.931	1.197 143	2 730	.981	1.375 551	5 089
.932	1.199 892	2 749	.982	1.380 774	5 223
.933	1.202 661	2 769	.983	1.386 144	5 370
.934	1.205 450	2 789	.984	1.391 673	5 529
.935	1.208 259	2 809	.985	1.397 375	5 702
.936	1.211 090	2 831	.986	1.403 269	5 894
.937	1.213 942	2 852	.987	1.409 377	6 108
.938	1.216 816	2 874	.988	1.415 722	6 345
.939	1.219 712	2 896	.989	1.422 336	6 614
.940	1.222.631	2 919	.990	1.429 257	6 921
.941	1.225 574	2 943	.991	1.436 532	7 275
.942	1.228 541	2 967	.992	1.444 221	7 689
.943	1.231 533	2 992	.993	1.452 407	8 186
.944	1.234 551	3 018	.994	1.461 198	8 791
.945	1.237 595	3 044	.995	1.470 755	9 557
.946	1.240 666	3 071	.996	1.481 324	10 569
.947	1.243 765	3 099	.997	1.493 317	11 993
.948	1.246 892	3 127	.998	1.507 542	14 225
.949	1.250 049	3 157	.999	1.526 071	18 529
.950	1.253 236	3 187	1.000	1.570 796+	44 725
		3 218			×

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A SHORT METHOD FOR EVALUATING DETERMINANTS AND SOLVING SYSTEMS OF LINEAR EQUATIONS WITH REAL OR COMPLEX COEFFICIENTS

by
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FOREWORD
by
MARCHANT CALCULATING MACHINE COMPANY

By permission of the author and publishers, we are privileged to issue this highly significant and valuable paper as a part of our Marchant Methods series.

In addition to the applications mentioned in the author's introduction, examples similar to No. 4 on page 5 are frequently found in the field of Statistical Method.

It will be noted that each element of the calculation is obtained as a result of one continuous calculator operation comprising a series of positive or negative summations of products, with or without final division. This being the case, it is obvious that calculators selected for this work should preferably conform to the following specifications:

(1) Multiplication and division, either positive or negative, with accumulation when desired.
(2) Complete capacity carry-over in both product and counting registers.
(3) Straight-line figure proof of entry of multipliers, multiplicands, dividends and divisors; and particularly that both multiplier and multiplicand shall be visible for permitting check of correctness of entry of the multiplication factors at each step of the continuous process.
(4) Positive decimal control, assuring no uncertainty in the entry of amounts with reference to decimal or in the pointing off of the intermediate and final products or quotients.

****

The within paper, as published, included a Mathematical Appendix containing the mathematical foundation of the method, and also consideration of complex equations whose symmetrical elements are conjugate. This appendix is not reprinted herein.

A SHORT METHOD FOR EVALUATING DETERMINANTS AND SOLVING SYSTEMS OF LINEAR EQUATIONS WITH REAL OR COMPLEX COEFFICIENTS

Prescott D. Crout

1. Introduction. The purpose of this paper is to describe without proof a short method for solving arbitrary systems of linear algebraic equations, and evaluating determinants, the quantities involved being either real or complex. The cases considered are

1. Arbitrary systems with real coefficients, which occur in obtaining stresses in structures, in solving systems of linear differential equations with constant coefficients (transient problems), etc.

2. Symmetrical systems with real coefficients, which occur with direct current networks, undamped vibration, deflections in structures, least square processes, Ritz' method, etc.

3. Symmetrical systems with complex coefficients, which occur with alternating current networks, and forced vibration with dissipation.

4. Arbitrary systems with complex coefficients, which occur in certain vibration problems involving gyroscopic action.

5. Systems involving two sets of variables, which occur when the currents in a network are to be found for a variety of impressed voltages, also in the approximate solution of integral equations arising in electric field problems.

The work of solving a system of equations (or evaluating a determinant) is largely concentrated in the determination of an "auxiliary matrix", and is roughly half that required by a matrix multiplication. The process is particularly adapted for use with a computing machine, for each element is determined by one continuous machine operation (sum of products with or without a final division). The setting down of this matrix and of the final solution is the only writing required by the process. The work involved is cut almost in half if the given equations (or determinant) is symmetrical, as very often happens. A "check column" can be carried along if desired.

The amount of work required to obtain a solution is considerably less than that required by Gauss' method[1], even when there is symmetry and the coefficients are real, in which case Gauss' method has been considerably refined by Doolittle[2]. (Gauss' method is much shorter than

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[1] "Numerisches Rechnen", by Runge, p. 65.

[2] "Practical Least Squares", by Leland, p. 40.

a solution by determinants.)

The method as given is applicable to m equations in n unknowns, there being no restriction on the rank of the matrix of the coefficients.

2. Description of the Method. Let the given system of equations be specified by its given matrix,[3] thus

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ 12.1719 & 27.3941 & 1.9827 & 7.3757 & 6.6355 \\ 8.1163 & 23.3385 & 9.8397 & 4.9474 & 6.1304 \\ 3.0706 & 13.5434 & 15.5973 & 7.5172 & 4.6921 \\ 3.0581 & 3.1510 & 6.9841 & 13.1984 & 2.5393 \end{array} \quad (1)$$

the first equation being $12.1719x_1 + 27.3941x_2 + 1.9827x_3 + 7.3757x_4 = 6.6355$. The solution requires the formation of one matrix and a set of final results; thus we have an auxiliary matrix

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ 12.1719 & 2.2506 & .16289 & .60596 & .54515 \\ 8.1163 & 5.0720 & 1.6793 & .0057629 & .33632 \\ 3.0706 & 6.6327 & 3.9585 & 1.4193 & .19891 \\ 3.0581 & -3.7316 & 12.7526 & -6.7332 & .060806 \end{array} \quad (2)$$

and a final matrix

$$\begin{aligned} x_1 &= .15942 \\ x_2 &= .14687 \\ x_3 &= .11261 \\ x_4 &= .060806 . \end{aligned} \quad (3)$$

---

[3] A matrix is a rectangular array of numbers, or elements. Those elements which have the same row and column index form the principal diagonal, which slopes down to the right starting with the element in the upper left corner. The diagonal element of any element to the right of the principal diagonal is that element of this diagonal which lies in the same row as the given element. The diagonal element of any element below the principal diagonal is that element of this diagonal which lies in the same column as the given element.

Since a modern computing machine gives in one continuous operation a sum or difference of products with or without a final division, we see that each element of the auxiliary matrix is given by a single machine operation.[5]

The procedure for obtaining the one-columned final matrix from the auxiliary matrix is contained in the following rules.

1. The elements are determined in the following order: last, next to last, second from last, third from last, etc.

2. The last element is equal to the corresponding element in the last column of the auxiliary matrix.

3. Each element is equal to the corresponding element of the last column of the auxiliary matrix minus the sum of those products of elements in its row in the auxiliary matrix and corresponding elements in its column in the final matrix which involve only previously computed elements.

We see that in forming products only those elements of the auxiliary matrix are used which lie to the right of the principal diagonal and to the left of the last column. The calculations made in obtaining (3) are

R3C1 $.11261 = .19891 - 1.4193 \cdot .060806$

R2C1 $.14687 = .33632 - 1.6793 \cdot .11261 - .0057629 \cdot .060806$

R1C1 $.15942 = .54515 - 2.2506 \cdot .14687 - .16289 \cdot .11261 - .60596 \cdot .060806.$

It may be noted that each element of the final matrix is given by a single machine operation.

It is not necessary but is strongly recommended that the values of the unknowns, which compose the final matrix, be substituted in each of the given equations, the result being a number of checks equal to the number of equations. Since the satisfaction of these checks guarantees the correctness of the solution, it is not necessary to check the calculations which gave the auxiliary matrix and the final matrix[6]. The first of the four checks obtained from (1) and (3) is

$$12.1719 \cdot .15942 + 27.3941 \cdot .14687 + 1.9827 \cdot .11261 + 7.3757 \cdot .060806 = 6.6355.$$

Evidently each check requires but one machine operation.

The above method is applicable as described to n equations in n unknowns. The only writing involved is that required in recording the auxiliary matrix and the final matrix.

---

[5] In carrying out such an operation it is not necessary to foresee whether the result is positive or negative.

[6] Since in solving a given set of equations only a limited number of

3. Systems Having Symmetrical Coefficients. If there is symmetry, the work of computing the auxiliary matrix is cut almost in half by the fact that if the coefficients of the unknowns (or the elements of the given matrix) are symmetrical about the principal diagonal; each element of the auxiliary matrix below the principal diagonal gives, if divided by its diagonal element, the symmetrically opposite element above this diagonal.[3] Elements above the principal diagonal of the auxiliary matrix are thus obtained as by-products of calculations made in determining elements below this diagonal.

As an example the symmetrical set of equations

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ 245.95 & 768.49 & 233.08 & 261.13 & 40.384 \\ 768.49 & 2665.85 & 880.54 & 915.95 & 90.887 \\ 233.08 & 880.54 & 688.91 & 458.17 & 6.5783 \\ 261.13 & 915.95 & 458.17 & 652.85 & 24.471 \end{array} \qquad (4)$$

has the auxiliary matrix

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ 245.95 & 3.1246 & .94769 & 1.0617 & .16420 \\ 768.49 & 264.63 & .57536 & .37800 & -.13338 \\ 233.08 & 152.25 & 380.42 & .40259 & -.029928 \\ 261.13 & 100.03 & 153.15 & 276.13 & -.0017367 \end{array} \qquad (5)$$

and the final matrix

$$\begin{aligned} x_1 &= .55591 \\ x_2 &= -.11591 \\ x_3 &= -.029229 \\ x_4 &= -.0017367 \ . \end{aligned} \qquad (6)$$

---

decimal places can be carried, the question arises as to how much the error accumulated during the course of the calculations affects the final results. An answer to this question cannot be given, since it depends upon whether or not the determinant of the system is on the point of vanishing (or, more generally, upon whether the matrix is on the point of shifting rank). Nevertheless it will be mentioned that in solving each of six systems of four equations in four unknowns using a ten bank computing machine, the final checks showed the given equations to be satisfied except for errors numerically less than three units in the ninth digit. Also it is known (from a separate investigation) that the values (10) of the unknowns of the complex system (8) computed with a ten bank machine contain errors each of which is numerically smaller than four units in the tenth digit. Although no general statement can be made, it is believed that in the usual case the accuracy tends to remain high during the course of the calculations.

In the auxiliary matrix the element in row 3 and column 4 is

$$\frac{(458.17 - 233.08 \cdot 1.0617 - 152.25 \cdot .37800)}{380.42} = \frac{153.15}{380.42} = .40259 ,$$

the numerator 153.15 being recorded in the symmetrically opposite position before the final division by the diagonal element 380.42 is carried out.[4] The final matrix is obtained in the usual manner.

4. Evaluation of Determinants. It is true that the value of a determinant is equal to the product of the elements which form the principal diagonal of the auxiliary matrix. The auxiliary matrix is computed as described in Section 2, the elements of the determinant being treated in the same manner as the coefficients of the x's before. The auxiliary matrix is square, since there is no column corresponding to the last column in (2); also, there is no final matrix.

As an example, the determinant composed of the first four columns of (1) has an auxiliary matrix composed of the first four columns of (2); hence the value of the determinant is (12.1719)(5.0720)(3.9585)(-6.7332) = -1645.4.

If the determinant is symmetrical about the principal diagonal, the work of evaluating it is cut almost in half because of the smaller amount of work required in computing the auxiliary matrix. (See Section 3.) For example, the determinant composed of the first four columns of (4) has an auxiliary matrix composed of the first four columns of (5). The value of the determinant is hence (245.95)(264.63)(380.42)(276.13) $= 6.8368 \cdot 10^9$.[4]

5. Continuous Check on Calculations. If desired, a "check column" may be written at the right of the given matrix, each element of this column being the sum of the elements of the corresponding row in the matrix. This column is now treated in exactly the same manner as the last column of the given matrix, the calculations being carried along with those for the other columns, and the result being the addition of corresponding "check columns" to the auxiliary matrix and the final matrix. The check columns thus obtained for (1), (2), and (3) are, respectively,[4]

$$\begin{matrix} 55.560 \\ 52.372 \\ 44.421 \\ 28.931 \end{matrix} , \quad \begin{matrix} 4.5646 \\ 3.0214 \\ 2.6182 \\ 1.0608 \end{matrix} , \quad \text{and} \quad \begin{matrix} 1.1594 \\ 1.1469 \\ 1.1126 \\ 1.0608 \end{matrix} \qquad (7)$$

These columns provide checks at all stages of the computation, because

1. In the auxiliary matrix any element in the check column is equal to one plus the sum of the other elements in its row which lie to the right of the principal diagonal.

2. In the final matrix any element in the check column is equal to one plus the sum of the other elements in its row.

For example noting (2), (3), and (7), two of the checks are

$$1 + 1.6793 + .0057629 + .33632 = 3.0214$$

$$1 + .11261 = 1.1126 .$$

The above statements are true and the procedure the same for any number of equations and unknowns; also, for the evaluation of determinants, in which, however, only Statement 1 is applicable.

6. Improvement in Accuracy. Since the number of decimal places in the computations is limited, the values obtained for the unknowns are in general not exact. However, if they are placed in the given equations and the differences between the two sides are obtained, and if these differences are then inserted in place of the right hand sides of the given equations, the resulting equations have as their solution the corrections to the values first obtained. Noting that the above differences are obtained in applying the final checks (see Section 2), and that the auxiliary matrix for the modified equations is the same as that for the original equations except for the last column, it follows that if the column of the differences[7] obtained in applying the final checks be annexed to the given matrix and then treated in the same manner as the last column, the corresponding column obtained in the final matrix is composed of the required corrections.

Since the problem of solving the modified equations is similar to the original problem, the above process may be repeated; thus the final checks on the corrections give data for another column in the given matrix, which leads to a column in the final matrix composed of corrections to the first corrections, etc. In the usual case each application of this process increases the number of significant figures in the results by approximately the same number obtained with the original solution, the data in the given equations being considered exact.

7. Systems of Equations Having Complex Coefficients. Since the proofs which establish the above method do not require the quantities involved to be real, the method is applicable to complex equations. The only question is that of whether the required calculations can be easily performed on a computing machine.

Since $(a+jb)(c+jd) = (ac-bd) + j(ad+bc)$, we see that the real and imaginary parts of a sum of products of complex numbers are each a sum of products of real numbers, given by one machine operation. It follows that each element of the final matrix, and each element of the auxiliary matrix which lies on or below the principal diagonal can be obtained by two machine operations one of which gives the real part, the other, the imaginary part. It also follows that each check run on the final matrix consists of two machine operations.

Each element of the auxiliary matrix which lies to the right of the principal diagonal is given by a sum of products with a final division.

---

[7] In recording the computed values of these differences, as many non-vanishing digits should be kept as can be handled in the subsequent machine calculations.

The sum is first computed and recorded, after which the final division by the diagonal element $A = a + jb$ is carried out as a separate multiplication by $1/A = a/(a^2 + b^2) - jb/(a^2 + b^2)$. Since the best form for recording the supplementary data depends upon how much of it there is, we shall consider separately the following two cases.

1. Symmetrical Equations. Noting Section 3 we see that if the coefficients in the given equations are symmetrical about the principal diagonal, the sum required for computing any element of the auxiliary matrix above the principal diagonal is already recorded as the symmetrically opposite element below this diagonal. It follows that only the last column of the auxiliary matrix requires separately recorded sums, which may be written in a right supplementary column. Data on the final multipliers 1/A may be placed in two left supplementary columns. (See (9).) In determining the auxiliary matrix the elements in any row which lie in the supplementary columns are determined just before those in that row to the right of the principal diagonal.

As an example the given set of four equations

x_1	x_2	x_3	x_4	=
84.276 71.184j	-91.342 10.741j	32.463 -12.142j	-45.417 -32.984j	6558.50 -185.00j
-91.342 10.741j	45.374 52.741j	-21.212 74.342j	80.414 -20.732j	-8780.75 670.75j
32.463 -12.142j	-21.212 74.342j	72.713 -20.221j	37.642 -77.814j	-3463.25 -3535.41j
-45.417 -32.984j	80.414 -20.732j	37.642 -77.814j	-20.114 -29.741j	-4268.87 1449.04j ,

(8)

where the imaginary part of each element is written below the real part for compactness, has the auxiliary matrix

$a^2 + b^2$	1/A	x_1	x_2	x_3	x_4	=	RS
12170.	.0069251 -.0058493j	84.276 71.184j	-.56973 .60867j	.15379 -.27397j	-.50745 .037241j	44.336 -39.644j	
13101.	-.000009789 -.0087368j	-91.342 10.741j	- .1282 114.458j	.41654 .087841j	-.10413 -.30098j	-29.887 45.088j	-5156.8 -3426.6j
7119.4	.0111588 .0039928j	32.463 -12.142j	-10.107 47.665j	79.444 -28.426j	.75944 -.77633j	-29.404 - 8.3789j	-2574.2 170.19j
3284.5	-.00050088 -.0174415j	-45.417 -32.984j	34.462 -11.880j	38.265 -83.263j	- 1.6452 57.287j	-51.713 -22.421j	1369.5 -2925.6j ,

(9)

where the supplementary columns are partitioned off by vertical lines, and the final matrix[6]

$$
\begin{aligned}
x_1 &= \begin{matrix} 20.471 \\ 10.782j \end{matrix} \\
x_2 &= \begin{matrix} -42.652 \\ 37.913j \end{matrix} \\
x_3 &= \begin{matrix} 27.275 \\ -31.498j \end{matrix} \\
x_4 &= \begin{matrix} -51.713 \\ -22.421j \end{matrix} \quad .
\end{aligned}
\tag{10}
$$

The following are typical calculations made in obtaining the above solution. In each case the part computed is specified by "real" or "imag.", after which the letters R, C, LS, and RS are used in place of "row", "column", "left supplementary", and "right supplementary", respectively. In the auxiliary matrix[4]

Real, R3C3 $\quad 79.444 = 72.713 - 32.463 \cdot .15379 + 12.142 \cdot .27397 + 10.107 \cdot .41654 + 47.665 \cdot .087841$

Imag., R4C3 $\quad -83.263 = -77.814 - 45.417 \cdot .27397 + 32.984 \cdot .15379 - 34.462 \cdot .087841 + 11.880 \cdot .41654$

Real, R3CRS $\quad -2574.2 = -3463.25 - 32.463 \cdot 44.336 + 12.142 \cdot 39.644 - 10.107 \cdot 29.887 + 47.665 \cdot 45.088$

R3CLS1 $\quad 7119.4 = 79.444 \cdot 79.444 + 28.426 \cdot 28.426$

Imag., R3CLS2 $\quad .0039928 = 28.426 \div 7119.4$

Real, R3C4 $\quad .75944 = .0111588 \cdot 38.265 + .0039928 \cdot 83.263$

Imag., R3C5 $\quad -8.3789 = .0111588 \cdot 170.19 - .0039928 \cdot 2574.2$.

In the final matrix

Real R2C1 $\quad -42.652 = -29.887 - .41654 \cdot 27.275 - .087841 \cdot 31.498 - .10413 \cdot 51.713 + .30098 \cdot 22.421$.

2. Non-symmetrical Equations. In the general case where there is no symmetry, and a sum must be recorded for every element of the auxiliary matrix to the right of the principal diagonal, the supplementary data may be recorded in a supplementary matrix having the same number of rows and columns as the auxiliary matrix. Each element to the right of the principal diagonal is the sum required in computing the corresponding element of the auxiliary matrix; the diagonal elements are the final multipliers 1/A; and values of $a^2 + b^2$ may be placed in a left supplementary column.

Since to obtain any element of the auxiliary matrix or the final matrix a real and imaginary part must be obtained, and since the sums involved have twice as many terms as they would have in the case of real

coefficients, we see that the work involved in solving a set of equations with complex coefficients is a little more than four times that required in solving a similar set with real coefficients.

The procedures just described under Case 1 and Case 2 can be used to obtain the square auxiliary matrix required in evaluating a determinant with complex elements. (See Section 4.)

A check column can be carried along if desired, the entries being complex. (See Section 5.)

8. Equations Involving Two Sets of Variables. If two sets of variables are involved, the above method may be used to obtain one set in terms of the other. For example the given set of equations[4]

$$
\begin{array}{cccccccc}
x_1 & x_2 & x_3 & x_4 = & y_1 & y_2 & y_3 & y_4 \\
12.172 & 27.394 & 1.983 & 7.376 & 6.636 & 0 & 0 & 0 \\
8.116 & 23.339 & 9.840 & 4.947 & 0 & 6.636 & 0 & 0 \\
3.071 & 13.543 & 15.597 & 7.517 & 0 & 0 & 6.636 & 0 \\
3.058 & 3.151 & 6.984 & 13.198 & 0 & 0 & 0 & 6.636 ,
\end{array}
\tag{11}
$$

of which the first is $12.172x_1 + 27.394x_2 + 1.983x_3 + 7.376x_4 = 6.636y_1$, has the auxiliary matrix

$$
\begin{array}{cccccccc}
x_1 & x_2 & x_3 & x_4 = & y_1 & y_2 & y_3 & y_4 \\
12.172 & 2.251 & .1629 & .6060 & .5452 & 0 & 0 & 0 \\
8.116 & 5.072 & 1.679 & .00576 & -.8724 & 1.308 & 0 & 0 \\
3.071 & 6.633 & 3.958 & 1.419 & 1.039 & -2.192 & 1.676 & 0 \\
3.058 & -3.732 & 12.753 & -6.733 & 2.699 & -4.877 & 3.175 & -.9855 ,
\end{array}
\tag{12}
$$

and the final matrix

$$
\begin{array}{cccccc}
 & & y_1 & y_2 & y_3 & y_4 \\
x_1 & = & -9.187 & 17.053 & -12.117 & 5.643 \\
x_2 & = & 3.800 & -6.607 & 4.734 & -2.343 \\
x_3 & = & -2.791 & 4.730 & -2.830 & 1.399 \\
x_4 & = & 2.699 & -4.877 & 3.175 & -.9855 ,
\end{array}
\tag{13}
$$

from which we see that $x_1 = -9.187y_1 + 17.053y_2 - 12.117y_3 + 5.643y_4$, etc. Each y column of (11) is treated in the same manner as the last column of (1), the result being a corresponding column in the final matrix. Four final checks on each of these is obtained by equating the corresponding y to 1 and the other y's to 0, and substituting (13) in (11). If a check column is not used, it is desirable to finish one y column and apply the final checks before filling in the other y columns. (The calculations for each y column are independent of those for the others.) If a check column is

desired, the procedure and rules given in Section 5 are worded so as to apply to the present case, the y columns being included with the others. There is no restriction on the number of equations or the number of y's.

9. m Equations in n Unknowns. Unusual Cases. Let there be given m equations in n unknowns. These may be arbitrarily (in any order) labelled 1,2,..., m, and 1,2,..., n, respectively, and the above procedure followed as though the equations and unknowns were actually written in the orders indicated by the labels. We shall consider the rows and columns both numbered in the orders in which they are completed in the auxiliary matrix. In general these orders are those in which the equations and unknowns are actually written, so that the labels are superfluous; however, in determining the auxiliary matrix two unusual cases may arise, as follows.[3]

1. If in completing a column the diagonal element is found to vanish, the (unlabelled) row chosen to be completed and labelled is one which has a non-vanishing element in this column.

2. If in completing a column all of the newly computed elements are found to vanish, that column is left unlabelled; but is (with its zeros) in its final form, and requires no further calculations.

The calculation of the auxiliary matrix is continued until either the rows or the unfinished x columns are exhausted, the total number of labelled rows and columns then being R. (R is the rank of the matrix of the coefficients in the given equations.) The m-R unlabelled rows and n-R unlabelled x columns will be called excess. Either, both, or neither of the quantities m-R and n-R may vanish, depending upon the given equations.

The one column (or more if there are several y columns as in Section 8)arising from the right hand sides of the equations is now completed, the procedure being identical with that used in completing the last labelled column. If in so doing the computed elements (which lie in excess rows) all vanish, the given equations are compatible; otherwise they are incompatible.

If the equations are found to be compatible, we next omit the excess rows (the equations corresponding to which are superfluous), and determine the final matrix from the remaining rows. Each excess column is treated like the last column, and contributes a column to the final matrix, in which it is labelled -x with the appropriate subscript. We thus obtain R unknowns as linear combinations of the others.

As an example, the given equations[4]

---

[3] These cases may also arise in evaluating a determinant. Should Case 1 occur, the sign of the product of the diagonal elements must be reversed if the number of inversions of the row labels is odd; otherwise, left unaltered. Should Case 2 occur, the value of the determinant is zero.

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ 47.126 & -74.150 & 14.222 & -64.372 & 27.304 \\ -29.312 & 54.332 & -37.998 & 70.028 & -80.476 \\ 32.470 & -46.984 & -4.7770 & -29.358 & -12.934 \\ -5.7490 & 17.257 & -30.887 & 37.842 & -66.824 \\ 8.9070 & -9.9090 & -11.888 & 2.8280 & -26.586 \end{array} \qquad (14)$$

have the auxiliary matrix

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ 47.126 & -1.5734 & .30179 & -1.3660 & .57938 \\ -29.312 & 8.2113 & -3.5502 & 3.6522 & -7.7324 \\ 32.470 & 4.1056 & 0 & 0 & 0 \\ -5.7490 & 8.2113 & 0 & 0 & 0 \\ 8.9070 & 4.1056 & 0 & 0 & 0 \end{array} \qquad (15)$$

and the final matrix

$$\begin{array}{ccccc} & & -x_3 & -x_4 & \\ x_1 & = & -5.2843 & 4.3805 & -11.5871 \\ x_2 & = & -3.5502 & 3.6522 & -7.7324 \end{array} \qquad (16)$$

The first unknown is $x_1 = 5.2843x_3 - 4.3805x_4 - 11.5871$. Since the two labelled rows and columns of the auxiliary matrix are in consecutive order, the labels are omitted, the excess rows and columns being clearly indicated by the block of zeros. Here $R = 2$.

m final checks can be obtained on each column of the final matrix by substituting the results in the given equations, the other columns in the final matrix and the corresponding columns in the given equations being omitted.

If a check column is desired, the procedure and rules given in Section 5 are worded so as to apply to the present case. In addition it is true that in the auxiliary matrix those elements of the check column which lie in excess rows must vanish if the given equations are compatible.

If the given equations are homogeneous (right hand sides all zero), all columns of the final matrix arise from excess columns of the auxiliary matrix. In those systems ordinarily met in stability and vibration problems $m = n = R + 1$.

NOTES ON USE OF THE MARCHANT CALCULATOR FOR SOLUTION OF SIMULTANEOUS EQUATIONS BY THE METHOD OF PRESCOTT D. CROUT AS DESCRIBED IN MARCHANT METHOD MM-434B1

Those who frequently use the Crout Method soon memorize the procedure for selecting the proper amounts for calculator entries, in accordance with the precise generalized instructions given in Dr. Crout's paper. However, when the process is to be used infrequently, it is believed that a guide will be useful for showing the manner of progressing with the calculation, as well as for selection of the amounts at each stage of the work.

Such a guide for third and fourth order real matrices is shown on pages 3 and 4 respectively. Except where the amounts are the same in each matrix, the elements of the auxiliary matrix are designated by "row" and "column," with lower case letters, whereas those for the given matrix are capitalized.

DECIMALS: For the computed elements to be accurate to as great a number of significant figures as possible, it is advisable to place the Middle Dial Decimal as far to the left as will barely accomodate the largest whole number (at left of its decimal) that exists in the given matrix. The Upper Dial (or Keyboard Dial) Decimal should, likewise, be as far to the left as will barely accomodate the largest whole number (at left of its decimal) in the factor which is to be entered in the Upper Dial (or Keyboard Dial), and the decimal for the Keyboard Dial (or Upper Dial) factor is then set in accordance with the usual rule. There are a few instances, such as when terminal division is by amounts less than 1, in which this rule will need modification.

This plan requires that under most circumstances the Upper and Keyboard Dial Decimals must be re-set for each pair of amounts which are to be multiplied and simultaneously subtracted.

In applying this rule for either the auxiliary or final matrices, it is possible that at times there may not be sufficient capacity to accomodate the accumulation of products of negative expressions which, when subtracted, actually increase the amount of the first entry. In such cases, the eye should be kept upon the dial to be sure that the number that would normally appear in the dial that would be next at the left will not be lost.

Where extreme accuracy is not required, a single decimal setting often can be used which will accomodate all amounts that are likely to be entered in computing the auxiliary (or final) matrix. Such a setting would provide for accomodating the longest whole number likely to be entered in the Upper and Keyboard Dials respectively. The Middle Dial Decimal is then set to conform to these decimals.

CALCULATOR ENTRIES: In the guide shown on pages 3 and 4, the factors are selected from the matrix in the following order; the left-hand of any pair of factors to be subtractively multiplied is the one in the same row as the element being calculated, and the right-hand of the pair is the one in the same column. It is suggested that this same arrangement be used in calculating; viz, by entering the "row" factor in Keyboard Dial, and entering the "column" factor in Single-Row Keyboard so it appears in the Upper Dial.

In computing the right-hand column of the auxiliary matrix, it is sometimes helpful to imagine that the elements of the corresponding column of the given matrix are multiplied or divided by some power of 10.

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(over)

EVALUATING NEGATIVE AMOUNTS: When the Middle Dial shows an amount in its complementary or "negative" form; i.e., preceded by a succession of 9's, it may be converted to its "positive" form by duplicating the amount in the Keyboard Dial directly below its Middle Dial position, preceding the Keyboard Dial set up with a few 9's. Upon reverse multiplication by 2, the amount will show in its positive form in the Middle Dial at right of as many ciphers as there are preceding 9's in the Keyboard Dial.

DIVIDING A NEGATIVE NUMBER: In the evaluation of elements at right of principal diagonal, a terminal division is necessary. If the Middle Dial amount prior to such division is in its negative form, a positive quotient is obtained by dividing the negative number as follows:

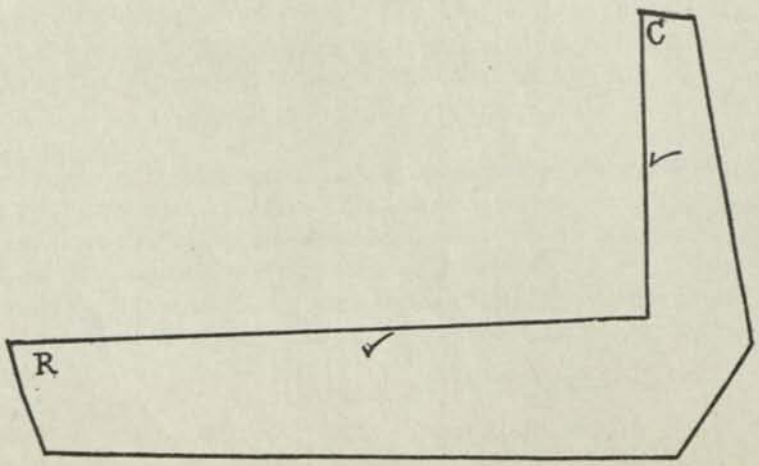
(1) Set up the divisor in Keyboard Dial around pre-set decimal and shift carriage until its left-hand figure is directly below the Middle Dial figure that is directly at the right of the succession of 9's.

(2) Multiply by any amount that will clear the 9's from left of Middle Dial. (It is not necessary to multiply by the exact amount that will do so. An "over-multiplication" will do no harm.)

(3) Without clearing Upper Dial, move Manual Counter Control toward the operator, and divide so that Upper Dial does not clear: Hold down the Stop Key while touching Division Key by a rolling motion of two fingers.

The quotient in positive form will appear in the Upper Dial.

READING ELEMENTS DIRECTLY FROM MATRIX: One of the important benefits of the Crout Method is that intermediate copying to work sheet of all items except those of the Auxiliary Matrix is eliminated. This may be aided by using the device illustrated below, which if made from an easily erasable celluloid material permits pencil check marks to be placed upon the edges so that the particular element being entered is positively identified. In using this device, the vertical edge "C" is placed at right of the column and the horizontal edge "R" is placed directly below the row from which elements are to be read.



THIRD ORDER SIMULTANEOUS EQUATIONS

Guide to Solution by Prescott D. Crout Method

ORIGINAL MATRIX:

X_1	X_2	X_3	S
R1C1	R1C2	R1C3	R1C4
R2C1	R2C2	R2C3	R2C4
R3C1	R3C2	R3C3	R3C4

AUXILIARY MATRIX:

R1C1	r1c2	r1c3	r1c4
R2C1	r2c2	r2c3	r2c4
R3C1	r3c2	r3c3	r3c4

r1c2 = R1C2 ÷ R1C1
r1c3 = R1C3 ÷ R1C1
r1c4 = R1C4 ÷ R1C1
r2c2 = R2C2 − R2C1 · r1c2
*r3c2 = R3C2 − R3C1 · r1c2
**r2c3 = (R2C3 − R2C1 · r1c3) ÷ r2c2
r2c4 = (R2C4 − R2C1 · r1c4) ÷ r2c2
r3c3 = R3C3 − R3C1 · r1c3 − r3c2 · r2c3
r3c4 = (R3C4 − R3C1 · r1c4 − r3c2 · r2c4) ÷ r_3 c_3

The final matrix is:

X_3 = r3c4

X_2 = r2c4 − X_3 · r2c3

X_1 = r1c4 − X_3 · r1c3 − X_2 · r1c2

(*) If matrix is symmetrical, after recording this element, set up the corresponding diagonal element as a divisor, thus producing the symmetrically opposite element above the diagonal.

(**) If matrix is symmetrical, this element is produced by dividing its symmetrically opposite element by the corresponding diagonal element.

(over)

FOURTH ORDER SIMULTANEOUS EQUATIONS

Guide to Solution by Prescott D. Crout Method - Marchant Method MM-434B1

ORIGINAL MATRIX:

X_1	X_2	X_3	X_4	S
R1C1	R1C2	R1C3	R1C4	R1C5
R2C1	R2C2	R2C3	R2C4	R2C5
R3C1	R3C2	R3C3	R3C4	R3C5
R4C1	R4C2	R4C3	R4C4	R4C5

AUXILIARY MATRIX:

R1C1	r1c2	r1c3	r1c4	r1c5
R2C1	r2c2	r2c3	r2c4	r2c5
R3C1	r3c2	r3c3	r3c4	r3c5
R4C1	r4c2	r4c3	r4c4	r4c5

r1c2 = R1C2 ÷ R1C1
r1c3 = R1C3 ÷ R1C1
r1c4 = R1C4 ÷ R1C1
r1c5 = R1C5 ÷ R1C1
r2c2 = R2C2 − R2C1 • r1c2
*r3c2 = R3C2 − R3C1 • r1c2
*r4c2 = R4C2 − R4C1 • r1c2
**r2c3 = (R2C3 − R2C1 • r1c3) ÷ r2c2
**r2c4 = (R2C4 − R2C1 • r1c4) ÷ r2c2
r2c5 = (R2C5 − R2C1 • r1c5) ÷ r2c2
r3c3 = R3C3 − R3C1 • r1c3 − r3c2 • r2c3
*r4c3 = R4C3 − R4C1 • r1c3 − r4c2 • r2c3
**r3c4 = (R3C4 − R3C1 • r1c4 − r3c2 • r2c4) ÷ r3c3
r3c5 = (R3C5 − R3C1 • r1c5 − r3c2 • r2c5) ÷ r3c3
r4c4 = R4C4 − R4C1 • r1c4 − r4c2 • r2c4 − r4c3 • r3c4
r4c5 = (R4C5 − R4C1 • r1c5 − r4c2 • r2c5 − r4c3 • r3c5) ÷ r4c4

The final matrix is:

X_4 = r4c5

X_3 = r3c5 − X_4 • r3c4

X_2 = r2c5 − X_4 • r2c4 − X_3 • r2c3

X_1 = r1c5 − X_4 • r1c4 − X_3 • r1c3 − X_2 • r1c2

(*) If matrix is symmetrical, after recording this element, set up the corresponding diagonal element as a divisor, thus producing the symmetrically opposite element above the diagonal.

(**) If matrix is symmetrical, this element is produced by dividing its symmetrically opposite element by the corresponding diagonal element.

1 026 643 6 8

EVALUATION OF A POLYNOMIAL

Remarks: Workers in science and mathematics often must evaluate polynomials. The method to be described is that of synthetic substitution. A frequent use for this type of calculation is in the evaluation of a polynomial and its derivative (which is also a polynomial) when solving equations by iteration methods such as the Newton-Raphson, etc.

Outline: A polynomial is an algebraic expression in the form

$$Y = A + BX + CX^2 + DX^3 + EX^4 \ldots.$$

in which X may have any numerical value, and A, B, . . . are constant numerical coefficients. The polynomial may be evaluated by merely substituting the desired value of X, and performing the multiplications and sums indicated. However, instead of forming a table of powers of X, and substituting their values in the expression, it is simpler to develop the function completely without need of transferring any amounts to work-sheet, or pre-computing a table of powers.

Example: Evaluate $5.146 + .312\,X - 2.495\,X^2 + 6.155\,X^3 + .876\,X^4$ when $X = 1.417$

Operations: Decimals: Upper Dial 3, Middle Dial 8, Keyboard Dial 5, No. 4 Tab Key down.

(1) Set up coefficient of highest power (.876) in Keyboard Dial, and multiply by X (1.417), producing EX (1.24129) in Middle Dial.

(2) Clear-Tab Upper and Keyboard Dials, which shifts carriage to 4th position, and set up coefficient of next highest power (6.155) in Keyboard Dial and add, producing $D + EX$ (7.39629) in Middle Dial.

(3) Transfer Middle-Dial amount (7.39629) to Keyboard Dial, clear Upper and Middle Dials and multiply by X (1.417), producing $DX + EX^2$ (10.48054) in Middle Dial.

(4) Clear-Tab Upper and Keyboard Dials, set up coefficient of next highest power (2.495) in Keyboard Dial and subtract, producing $-C + DX + EX^2$ (7.98554) in Middle Dial.

(5) Transfer Middle-Dial amount (7.98554) to Keyboard Dial, clear Upper and Middle Dials and multiply by X (1.417), producing $-CX + DX^2 + EX^3$ (11.31551) in Middle Dial.

(6) Clear-Tab Upper and Keyboard Dials, set up coefficient of next highest power (.312) and add, producing $B - CX + DX^2 + EX^3$ (11.62751) in Middle Dial.

(7) Transfer Middle-Dial amount (11.62751) to Keyboard Dial, clear Upper and Middle Dials and multiply by X (1.417), producing $BX - CX^2 + DX^3 + EX^4$ (16.47618) in Middle Dial.

(8) Clear-Tab Upper and Keyboard Dials, set up Constant term (5.146) and add, producing the desired value of the polynomial (21.622) in Middle Dial.

NOTE A: If any power of X is missing, such as would be the case in the above if the term "$6.155\,X^3$" were omitted, proceed as above except that the coefficient of X^3 is regarded as zero. In such a case, omit step 2, which adds the coefficient of X^3. Step 3 in such a case produces only EX^2, etc.

NOTE B: If a negative amount is transferred from Middle Dial to Keyboard Dial (an amount preceded by 9's), the product of the subsequent multiplication will also be negative. In such a case, after the product is formed, sufficient extra 9's should be entered to fill Middle Dial with 9's substantially to its leftmost dial.

THE BIRGE-VIETA METHOD
of
FINDING REAL ROOTS OF RATIONAL INTEGRAL FUNCTION

PREFACE: Few realize the extent that classical mathematical methods have evolved under the control of the "parameter" (to use a mathematician's word) that pencil-and-paper shall be used in the calculations required by such methods. If the modern calculating machine had been available to the mathematicians of the Renaissance, it is possible that even such a familiar tool as the Briggs Logarithm might not have been developed. Certainly the art would have progressed along far different lines if from the start there had been available a machine that could multiply or divide as rapidly as one could enter amounts in a keyboard.

The disclosure herein is an interesting example of how an early method, which was discarded because it involved so much numerical computation that was "unfit for a Christian," to quote from a writer of that day, has now been found to possess decided advantages when compared with methods that displaced it. This is because present-day calculating machines remove the drudgery element which caused the method to be relegated to the shelf over 200 years ago.

The method to which we refer was originally proposed by Francis Vieta (1540-1603). Raymond T. Birge, Ph.D., Professor of Physics and Chairman of the Department, University of California, is responsible for re-establishing it as a modern computing tool. Dr. Birge has noted that it possessed many advantages over the methods that have been developed to take its place (merely because of the excessive amount of pencil-and-paper work that it entailed).

In applying the Vieta method to the modern calculating machine, Dr. Birge has reduced it to simple systematic procedure that permits speedy determination of the root under conditions of controlled accuracy.

USES OF THE BIRGE-VIETA METHOD: The method is ideal for finding a real root of the usual algebraic equation when rough approximation of the root is known, particularly if the equation is of higher degree than the second. It is also excellent for solving transcendental equations (those that involve logarithmic or trigonometric functions in combination with analytic functions), particularly when the equations are in such form that substitutions of odd amounts in the equations or in their first derivatives are difficult. Inasmuch as the usual problem of inverse curvilinear interpolation is one of finding the root when the value of the function is a given amount, it will be seen that the Birge-Vieta method is adapted to such work, assuming of course that the tabular values are first expressed as an Interpolation Polynomial of degree "n" that fits n + 1 equidistant values of such tabulated function (See Marchant Method 434-F).

In the case of solving equations involving transcendental functions, tabular values are, likewise, obtained. An Interpolation Polynomial is then fitted to the values and then solved for the desired root. If, however, the equation has a simple first derivative and substitution of amounts in the original equation or its first derivative is not too difficult, the Newton-Raphson Method of obtaining the root is to be preferred.

(over)

OUTLINE: It is assumed that the reader is familiar with the usual Horner Synthetic Division process which is described in most College Algebra texts. However. a Note is appended which describes this procedure in a way that will enable it to be understood by a computer who is not familiar with it. (See top of Page 4).

An algebraic statement of the sample computation is given. This is followed by detailed instructions for performing the work on a Marchant Silent-Speed calculator. An Appendix then states the particular advantages of the Birge-Vieta Method, as compared with methods that are ordinarily used for such work.

The symbolism of the Horner Method is employed insofar as possible.

EXAMPLE: Find correctly to nine figures the real root nearest to $x = 1.0$ of the following equation:

$$y = g(x) = x^5 - x - 0.2 \text{ (true value is } 1.044\ 761\ 700_{07})$$

Assume $x = +1 = p_1$ as first approximation of the root.

I Transfer from $g(x)$ to $g'(x - p_1) = g'(x - 1) = g'(u)$
Transfer factor, $p_1 = +1$. Apply Horner Shift for A_0 and A_1 (See Note A, Page 4).

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	1	0	0	0	-1	-0.2
		1	1	1	1	0
	1	1	1	1	0	$-0.2 = A_0$
		1	2	3	4	
	1	2	3	4	$4 = A_1$	

Therefore $u = -\frac{A_0}{A_1} = -\frac{-0.2}{4} = +0.05 = x - p_1$

or $x = p_1 - \frac{A_0}{A_1} = 1.0 + 0.05 = 1.05 = p_2$, as second approximation.

It will be noted that the above represents the first steps of an ordinary Horner synthetic division. Only A_0 and A_1 need be found.

II Transfer from $g(x) = g''(x - p_2) = g''(x - 1.05) = g''(v)$.

It is a characteristic of this method that the calculations need be carried <u>only</u> to the reliability that the ratio of the next coefficients (in this case, B_0 and B_1) is likely to have. A practical rule is to carry <u>twice</u> as many <u>decimal</u> places in all sums and products used in obtaining B_0 as there are decimal places in the transfer factor. Hence, since 1.05 is the transfer factor, carry B_0 calculations to <u>four</u> decimal places. We find, in this problem, three significant figures for B_0, and hence carry all calculations for B_1 to at least <u>three</u> significant figures (it is really simpler to carry four and round off to three).

We now return to the <u>original</u> coefficients, an essential of the method, and one of its best features from the viewpoint of accuracy control.

$p_2 = 1.05$ transfer factor.

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	1	0	0	0	- 1	- 0.2
		1.05	1.1025	1.1576_2	$+ 1.2155_{01}$	$+ 0.2262_{76}$
	1	1.05	1.1025	1.1576_2	$+ 0.2155_{01}$	$+ 0.0263 = B_0$
		1.05	2.2050	3.472	4.860	
	1	2.10	3.307	4.629	$5.075 = B_1$	

By inspection $v = -B_0/B_1$ will have two ciphers. Therefore, by rule given, the ratio should be correct to two significant figures.

Therefore $$v = -\frac{B_0}{B_1} = -\frac{+0.0263}{5.07} = -0.005187$$

rounded to -0.0052

It will be noted that <u>four</u> decimal places carried in the B_0 calculations were sufficient to give B_0 to three significant figures, as is desired in order to be sure that B_0/B_1 is correctly calculated to <u>two</u> significant figures (i.e., in addition to the two ciphers with which it starts).

Therefore $$x = p_2 - \frac{B_0}{B_1} = 1.05 - 0.0052 = 1.0448 = p_3,$$ as second approximation.

III Transfer from $g(x)$ to $g'''(x - p_3) = g'''(x - 1.0448) = g'''(w)$

$p_3 = 1.0448$ transfer factor

As before, since there are four decimal places in $v = -B_0/B_1$ or in p_3, the next transfer factor, we carry eight decimal places in getting C_0, i.e., close to full capacity of a ten-key calculator, so for simplicity the full ten-key capacity is utilized. Then from the C_0 result, carry only six significant figures in computing C_1.

x^5	x^4	x^3	x^2	x^1	x^0
1	0	0	0	-1	-0.2
	1.0448	1.091 607 04	1.140 511 035$_4$	$+1.191\ 605\ 929_4$	$+0.200\ 189\ 874_6$
1	1.0448	1.091 607 04	1.140 511 035$_4$	$+0.191\ 605\ 929_4$	$+0.000\ 189\ 875 = C_0$
	1.0448	2.183 21$_4$	3.421 53$_2$	$+4.766\ 41_9$	
1	2.0896	3.274 82	4.562 04	4.958 02	$= C_1$

There will be four ciphers in C_0/C_1, therefore carry five or six significant figures.

Therefore $$w = -\frac{C_0}{C_1} = -\frac{+0.000\ 189\ 875}{4.958\ 02} = -0.000\ 038\ 296\ 5_4$$

This ratio should be satisfactory to four significant figures. However, we retain five as this is to be the final approximation.

Therefore $$x = p_3 - \frac{C_0}{C_1} = 1.0448 - 0.000\ 038\ 296_{54} = 1.044\ 761\ 703_{46} = p_4$$

This root should be accurate to nine digits. It is seen that the error is 0.3 in the 9th digit.

A continuation of this process with transfer factor 1.044 761 703 gives $D_0 = +0.000\ 000\ 014_3$ and $D_1 = C_1$ (closely enough) $= 4.958$

Therefore $$-D_0/D_1 = -0.000\ 000\ 002_9 \text{ or } p_5 = p_4 - D_0/D_1 = 1.044\ 761\ 700_1$$

which is correct to ten figures.

An alternate continuation process is to use $p_4 = 1.044\ 761\ 7$ as transfer factor and by double multiplication (see Marchant Method 421A) carry all products to full 20-

(over)

digit capacity of the calculator, thus producing p_5 correct to 18 or 19 digits (1.044 761 700 075 552 795).

Note that the actual error in p_1 is -0.045, in p_2 is +0.0052, in p_3 is +0.000 038, and in p_4 is +0.000 000 003. Thus, each approximation is correct to about double the number of digits of its predecessor. This is a characteristic feature of the present method. For this reason, p_5 should be correct to about 18 digits.

NOTE A - THE HORNER SHIFT

For those not familiar with the Horner Shift, the procedure is easily understood by reference to the calculation for B_0/B_1 on Page 2, with factors manipulated as below:

						Transfer factor p
	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients of x^n	a	b	c	d	e	f
		pa	pm	pn	po	pq
	a	m	n	o	q	$r = B_0$

in which $m = b + pa$, $n = c + pm$, $o = d + pn$, etc. and similarly for the next row that produces B_1.

MATHEMATICAL BASIS OF METHOD

The Birge-Vieta process obtains the value of the function and of its first derivative when the approximate roots (the transfer factors) are substituted for "x." That part of the process which obtains A_0, B_0, C_0, etc., obtains successively more accurate values of the function, and A_0, B_0, and C_0, etc. are these successive values. The step that obtains A_1, B_1, C_1, etc., similarly obtains successively more accurate values of the first derivative when the transfer factors are substituted for "x." This is done, however, not by duplicating the first step with respect to the equation of the first derivative of the function but by taking advantage of partial products and sums developed during the first step. This makes it unnecessary to set up the equation of the derived function.

The successive transfer factors may have the same or different signs. Under some conditions they may alternate in sign.

COMBINING SUBSTITUTION METHODS WITH THE BIRGE-VIETA PROCESS

Inasmuch as A_0, B_0, C_0, etc are the values of the function when the transfer factor is substituted for "x," and A_1, B_1, C_1, etc., are the first derivatives of the function with respect to "x" when the transfer factor is likewise substituted for "x," there will be cases in which the first two steps of the computation may be more easily done by taking advantage of these facts, using a Table of Powers for direct computation of these amounts. This plan reserves the Birge-Vieta process for cases in which direct substitution is not easy and where the first derivative also is not easy to compute, which by the premise at bottom of Page 1 is its indicated use, anyway. These conditions usually are met when the transfer factor exceeds three figures if a Table of Powers of three-figure amounts is available. It is met with two-digit transfer factors if a Table of Powers is not available, (assuming, of course, that the usual small powers of integers 1 to 9 are known).

A readily available table of "First Ten Powers of the Integers from 1 to 1000" is that of Works Project for Computation of Mathematical Tables, Table MT-1, Information Section, National Bureau of Standards, Washington, D. C.; price 50 cents.

It happens that the example used to illustrate this method is in such form that with the aid of a Table of Powers of three-figure amounts the results of the second section may be obtained somewhat faster by substitution. (The work of the first section is obviously merely a matter of inspection.)

As an example of this straight substitution, let us apply it to this second section. We first note that the powers of 1.05, to four decimals, are $x^5 = 1.2763$ and $x^4 = 1.2155$, (these are the only powers needed for substituting in the equation or in its first derivative).

From this, we have $1.05^5 - 1.05 - 0.2 = 0.0263 = B_0$

and its first derivative

$$5 \times 1.05^4 - 1 = 5.0775 = B_1$$

APPLICATION OF THE BIRGE-VIETA METHOD TO THE MARCHANT CALCULATOR

The skilled computer who prefers to add or subtract mentally, or who wishes to use auxiliary means for such addition or subtraction doubtless would prefer to set up the transfer factor as a constant in the Keyboard Dial and multiply by the various factors as needed. The amounts are then entered on a work sheet exactly as shown in the above analysis.

Others will wish to perform all additions and subtractions on the Marchant. The detailed Marchant operations for this procedure, when applied to the calculation of p_4, are as follows:

OPERATIONS: Decimals: Upper Dial 9, Middle Dial 18, Keyboard Dial 9. Use any 10 column Model M Marchant with Upper Green Shift Key down.

Inasmuch as the coefficient of x^5 is 1.0, the calculator computation is started for development of the x^3 column; thus,

(1) Set up 1.0448 and multiply by transfer factor (1.0448).

Copy 1.091 607 04 from Middle Dial to x^3 column.

(2) As there is no amount to add to this, the normal adding step is skipped. Shift to 10th position, clear Upper and Keyboard Dials, and copy Middle Dial amount (1.091 607 04) into Keyboard Dial, clear Middle Dial and multiply by transfer factor (1.0448).

Copy $1.140\ 511\ 035_4$ from Middle Dial to x^2 column.

(3) Repeat Step (2) with Keyboard Dial setup of 1.140 511 035.

(4) Clear Keyboard Dial, shift to 10th position, set up 1.0, and subtract.

Copy $0.191\ 605\ 929_4$ from Middle Dial to x^1 column.

(5) Repeat Step (2) with Keyboard Dial setup of 0.191 605 929.

(6) Clear Keyboard Dial, shift to 10th position, set up 0.2, and subtract.

Copy C_0 (0.000 189 875) to x^0 column.

(7) Clear all dials, set up 1.0448, and multiply by 2.0.

Copy 2.0896 from Middle Dial to x^4 column.

(8) Shift to 10th position, clear Keyboard Dial and transfer Middle Dial amount (2.0896) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

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(9) Shift to 10th position, clear Keyboard Dial, set up 1.091 607 04, and add.

Copy 3.274 82 from Middle Dial to x^3 column.

(10) Transfer Middle Dial amount (3.274 82) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(11) Shift to 10th position, clear Keyboard Dial, set up 1.140 511 035, and add.

(12) Transfer Middle Dial amount (4.562 04) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(13) Shift to 10th position, clear Keyboard Dial, set up 0.191 605 929, and add.

Copy 4.958 03 from Middle Dial as C_1.

(14) Clear dials, set up C_0 (0.000 189 875) and add.

(15) Set up C_1 (4.958 03) and divide.

$-w = 0.000\ 038\ 296$ appears in Upper Dial.

(16) Clear Middle and Keyboard Dials, shift to 10th position, set up 1.0448, and add.

(17) Set up 1.0 and reverse multiply by Upper Dial amount that is at right of decimal (.000 038 296), reducing it to ciphers.

Root (1.044 761 704) appears in Middle Dial.

That the error is "4" in the 10th significant figure, whereas the analysis on Page 3 shows it to be "3," comes about because the Marchant does not drop off right-hand figures in producing $4.766\ 41_9$ of Step (12). Slight variations of this sort from the analysis are to be expected. The root, however, is still accurate to 9 figures, which is all that this stage of the computation is expected to obtain.

The continuation of the process with transfer factor 1.044 761 704, if it is desired to go so far, may be done in the same manner as above.

Reference is made in the analysis to "double multiplication" with carrying all products to 20 digits. This is assisted by the means mentioned in Marchant Method 421A, "Multiplication of Large Factors."

APPENDIX -- ADVANTAGES OF THE BIRGE-VIETA METHOD

Dr. Birge gives the following reasons why the Vieta process, when adapted to a calculator, is to be preferred, as compared with the more commonly used Ruffini-Horner Method. These advantages are in addition to the extra speed of the Vieta process because of there being fewer steps.

(1) One always deals with the <u>same</u> original coefficients (which often contain relatively few significant figures), instead of with constantly new sets of coefficients, which inevitably get more complex, as in the R-H method.

(2) Any <u>error</u> in the calculation affects <u>only</u> the particular transfer being made, and can never affect the final result. The same thing is true for the Newton iteration method, and constitutes the greatest advantage of that method. Thus, due to an error, a certain approximation may be poorer than the preceding approximation, but this fact immediately shows up in the <u>next</u> approximation. In other words, p_1 p_2 p_3 should constitute a series of numbers that rapidly settles down to a <u>constant</u> value, just as x_1 x_2 x_3 etc. in Newton's iteration method (for square roots, etc.) rapidly become constant.

But in the R-H method, <u>any</u> error makes the new function incorrect, and since we <u>then</u> proceed to get the root o the <u>new</u> function, the final result is necessarily incorrect. In other words, any such error carries through to the end. This advantage of the Vieta method over the R-H method can scarcely be overemphasized, and should be alone sufficient to make the R-H method completely obsolete.

(3) In the Vieta method the transfer factors p_1 p_2 etc., are all approximately the same size, and since the original coefficients are always used (advantage (1), <u>all</u> corresponding products and sums appearing in successive Horner shifts are approximately the same. Hence we do not need to figure the position of the decimal point, after the first Horner shift has been made. This fact is of great advantage in avoiding errors, and it results in much time saved.

(4) As already stated, one needs to calculate <u>only</u> the first two coefficients of each new function, whereas all coefficients must be calculated in the R-H method.

(5) In calculating these first two coefficients, we do not need, at first, to get the various sums and products to the <u>final</u> desired accuracy (as <u>is</u> necessary in the R-H method).

APPROXIMATING POLYNOMIAL FROM DIFFERENCE ARRAY (STIRLING METHOD)

REMARKS: *It is often desired to obtain an algebraic expression for a function that is determined by the relation that a series of tabulated amounts bears to corresponding values of the independent variable. When values of the latter are taken at equidistant points so that an array of differences may be set up, an equation in the form of an Approximating Polynomial may be readily obtained. If the nth difference of the array is constant, the Approximating Polynomial will represent the function correctly provided differences up to, and including those of the nth order are taken into account. If there are differences in the array which are of higher order than those taken into account, the Approximating Polynomial will approximate the function insofar as it can be done by a polynomial of degree "n".*

Obtaining an Approximating Polynomial by means outlined herein provides the most rapid method of fitting an equation to non-periodic tabulated data of scientific and statistical computations. It is assumed that the data are "smoothed"; that is to say, the obvious errors of observation are eliminated as is the case when the tabulated values are taken from a curve or determined by least-squares methods. If functions appear in periodic form, the Approximating Polynomial found by the method herein is generally suitable only for showing one quarter-period (approximately) of the periodic function. Fourier Series analysis is generally employed for obtaining equations of periodic functions.

OUTLINE: The Approximating Polynomial described herein has the form

(1) $y = a_0 + a_1u + a_2u^2 + a_3u^3 \ldots\ldots + a_nu^n$

in which the "a" values are coefficients to be determined, and "u" represents the independent variable reduced to the initial condition that $u = 0$ when $y = a_0$ and that the difference between tabulated values of the independent variable, in terms of "u", is "1". For example, in the table showing the relation between x and y, below, the values of "u" are shown in the middle column assuming that the 0 point of "u" is to be at $x = 0.3$. It is obvious that if an Approximating Polynomial in the form of (1) is obtained, it is easy to convert it to one that shows y as a function of x. This simple transformation is not discussed herein.

EXAMPLE:

Function			Differences				
x	u	y	1st	2nd	3rd	4th	5th
0.	-3	1.00000					
			-4865				
0.1	-2	0.95135		1547			
			-3318		-299		
0.2	-1	0.91817		1248		100	
			$-2070\,(d'_{-\frac{1}{2}})$		$-199\,(d'''_{-\frac{1}{2}})$		$-32\,(d^{v}_{-\frac{1}{2}})$
0.3	0	0.89747		$1049\,(d''_{0})$		$68\,(d'''_{0})$	
			$-1021\,(d'_{\frac{1}{2}})$		$-131\,(d'''_{\frac{1}{2}})$		$-32\,(d^{v}_{\frac{1}{2}})$
0.4	+1	0.88726		918		36	
			- 103		- 95		
0.5	+2	0.88623		823			
			720				
0.6	+3	0.89343					

The method exemplified herein will be applied to the tabulated values listed on the previous page which are shown with differences. An Approximating Polynomial is to be obtained in such form that it will show optimum accuracy in the vicinity of x = 0.3. This value is then chosen as the base point for obtaining the "a" coefficients, so "u" is set at 0 when x = 0.3.

The formula used is that of Stirling and is chosen because it is the easiest to apply. The Bessel formula* gives somewhat more accurate results in the region that is half-way between the equidistant tabular values. This difference, however, is exceedingly slight so that rarely will it be advisable to go to this refinement. The Newton formula* is useful for obtaining an Approximating Polynomial when only the values at the top of a table are obtainable. However, even in this case the Stirling Method may be used if it is satisfactory to extrapolate probable differences upward from the known differences.

In the Approximating Polynomial (1), Page 1, the Stirling formulas for coefficients, up to consideration of 5th differences, are below:

(2) $a_1 = \frac{1}{2}\,(d'_{-\frac{1}{2}} + d'_{\frac{1}{2}}) - 1/12\,(d'''_{-\frac{1}{2}} + d'''_{\frac{1}{2}}) + 1/60\,(d^{v}_{-\frac{1}{2}} + d^{v}_{\frac{1}{2}})$

(3) $a_2 = \frac{1}{2}\,d''_0 - 1/24\,d''''_0$

(4) $a_3 = 1/12\,(d'''_{-\frac{1}{2}} + d'''_{\frac{1}{2}}) - 1/48\,(d^{v}_{-\frac{1}{2}} + d^{v}_{\frac{1}{2}})$

(5) $a_4 = 1/24\,d''''_0$

(6) $a_5 = 1/240\,(d^{v}_{-\frac{1}{2}} + d^{v}_{\frac{1}{2}})$

The terms up to 4th differences appear in Scarborough: Numerical Mathematical Analysis, 1930 edition, Page 80. Those for 5th differences were supplied by courtesy of Dr. Raymond T. Birge, Professor of Physics, University of California, to whom we are also indebted for other helpful data in connection with this process.

The nomenclature of equations (2) to (6), inclusive, applies to the preceding difference array and is further explained in Marchant Method MM-189a. It will be noted that certain factors are repeated or bear simple ratios to others.

For ordinary computing, any terms that do not affect the final result in one place at the right of the one that is to be retained would be omitted. If values of the polynomial are desired close to the centering point, it is often possible to shorten the work if advantage is taken of this principle. In this case, it is not possible to do this if 5-place accuracy is desired without uncertainty within the range u = -1 to u = +1, because the maximum effect of the 5th difference is noted in coefficient a_5 as 0.000 013 3 (see below) so it would affect 6th place by 13.

If accuracy to the number of places of the tabulated values is desired up to the limits of the values from which differences are taken; i.e., the extreme range of the tabulated values, the higher-order coefficients must be taken to more places than those of lower order. For example, at the extreme range of the table, u = ± 3. As the coefficient a_5 multiplies u^5, or 243, it is evident that a_5 must be carried to a sufficient number of places so that the error of its right-hand digit when multiplied by 243 will not affect 6th place.

(*) Scarborough: Numerical Mathematical Analysis, 1930 edition, the Johns Hopkins Press, Pages 80-81, gives coefficients for Newton, Bessel, and Stirling formulas up to and including 4th differences. Values including those due to 5th differences will be supplied upon application.

Using these principles, the coefficients are obtained as follows:

$a_1 = -0.015\ 455 - (-0.000\ 275) + (-0.000\ 011) = -0.015\ 191$

$a_2 = +0.005\ 245 - 0.000\ 028\ 3 = +0.005\ 216\ 7$

$a_3 = -0.000\ 275 - (-0.000\ 013\ 33) = -0.000\ 261\ 67$

$a_4 = +0.000\ 028\ 33$

$a_5 = -0.000\ 002\ 667$

The Approximating Polynomial, accordingly, is

$$(7)\ y = 0.89747 - 0.015191u + 0.0052167u^2 - 0.00026167u^3 + 0.000028333u^4 - 0.000002667u^5$$

To show how closely this approximates the tabulated function, when u varies from -3 to +3, its values are computed to six places.

x	u	y Computed to 6 places	Tabulated y	6th place error
0	-3	1.000 000	1.000 00	0
0.1	-2	0.951 351	.951 35	1
0.2	-1	0.918 170	.918 17	0
0.3	0			
0.4	+1	0.887 260	.887 26	0
0.5	+2	0.886 229	.886 23	1
0.6	+3	0.893 429	.893 43	1

MARCHANT CALCULATOR APPLICATION

No exemplification of the details of Marchant application to this work is given because it embodies the simplest of calculator manipulation. Because work of this sort is usually infrequently done and because some of the factors of equations (2) to (6) inclusive are repetitions or bear simple numerical ratios to others, it is usually advisable to evaluate each factor individually, copying the amounts to work sheet and summing them afterwards. For these reasons, the accumulation of partial products is not recommended, though this procedure should undoubtedly be followed if there is a great volume of the work to be done.

In nearly all cases, except where it is desired to obtain an empirical formula (see below), it is usually satisfactory to use the function in terms of "u"; thus, for direct interpolation and related work the "x" is converted to the corresponding "u" before applying the formula, and in cases of inverse interpolation and the like, the "x" is obtained after the "u" has been found.

APPROXIMATION OF FUNCTIONS BY POLYNOMIALS

Polynomials of the type considered herein for the representation of a tabulated function have not been given the consideration in mathematical literature that their importance warrants. It is believed that this is due to the usual comparatively laborious process of setting them up by solving systems of linear equations, which has long been the conventional method of converting n tabulated values into a power series of degree n. Now that it is recognized that they are much more easily obtained from their difference arrays, more and more uses are certain to be found for them.

One principal use of these polynomials is to provide means of handling complicated analytical or transcendental functions in which substitution is difficult owing to the complexity of the terms. Equidistant values are established, sufficient to determine the Approximation Polynomial. A few intermediate values are obtained for a later check of the error. The Polynomial is then used in

(over)

place of the function for which it is a substitution. When given the "u" value, the "y" is obtainable by direct substitution in the polynomial. When given the "y" value. the "u" is easily obtained by the Birge-Vieta Method (see Marchant Method MM-434D).

The above-described procedure is particularly helpful in cases where the differential or integral values of a complicated function are desired. Many of these cannot be integrated directly and differentiation is often difficult. If the expression is approximated as a polynomial, however, it is a simple matter to obtain successive differential or integral forms and without the discontinuities which use of the original expression might entail. A characteristic of the Approximation Polynomial is that its graph has minimum curvature.

The use of these polynomials in cases of large volume of interpolation, such as in table preparation, is obvious, though in such instances the procedure of Marchant Method MM-439E1 should be compared. Inverse interpolation is easily handled by using the Birge-Vieta Method for solving for "y". (Compare also Marchant Methods MM-439J2, 439H, and 439J1.

The polynomials readily lend themselves to extrapolation provided it is understood that the uncertainty of the result increases (sometimes rapidly) as one leaves the region contained between the extreme values from which the differences are tabulated. This effect becomes increasingly serious as the degree of the polynomial increases.

The polynomials also provide a way of exploring the effect of the powers of the independent variable in cases of experimental tabulated data, thus leading directly to an empirical formula to express the relationships. Obviously, if the coefficient of, say, the third power of x (not of u) is large and those of other powers are negligible, the experimenter will be on the lookout for influences that vary according to the cube of the independent variable. Care must be taken not to accept too literally the significance of the polynomial as a working formula, however, because an empirical formula should, if possible, have some physical meaning or reasonable basis for being in the form used.

If the polynomial shows comparatively large coefficients of x (not of u) for certain powers, an empirical formula, however, may generally be set up using those coefficients and powers only. The values of y corresponding to the tabulated x's may then be computed from this new polynomial and compared with the original tabulated values. The residuals then may be considered as n values of another new polynomial containing only the powers that are to be retained in the proposed empirical formula. By solving these as a system of linear equations, applying least squares methods, a modification is obtained of the coefficients of the powers that are to be retained in the empirical formula. This modified formula then becomes the improved empirical formula.*

The above-described procedure is the usual one of taking advantage of an approximating polynomial (power series) as a base for an empirical formula. Another case in which such a polynomial may be converted into a simplified empirical formula is that in which the successive coefficients follow a definite law, indicating a convergent series, which represents some other function such as an exponential, trigonometric, etc.

(*) Steinmetz: Engineering Mathematics. 3rd Edition, McGraw-Hill Publ. Co., Pages 215-16. See also Marchant Method MM-434B2 and Marchant Method MM-434B1 (Page 7, Section 6). These relate to the Crout Method for solving such systems of equations.

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Note: The following article is reproduced for inclusion in the Marchant-Method series because it describes the proper computing procedure for obtaining the Probable Error or Standard Deviation of a function when the Probable Errors or Standard Deviations of its elements are known. Functions that differ from the ordinary formula-type, as shown herein, are handled according to the principles used in this example, but care must be taken in applying them.

HOW MANY FIGURES FOR THE ANSWER?

Business figures usually involve money and consequently must be exact to "nearest cent." With scientific figures, on the other hand, it is sometimes not so simple to express the degree of accuracy of the result.

The object of this article is to outline means of caring for a most frequently encountered case of this sort; viz., how should the answer be written when the Probable Error of each of its factors is known?

The solution for such cases is somewhat time-consuming. However, because of the fact that most calculating follows a "pattern" in which the factors of the successive problems do not vary greatly from each other, it is usually advisable to study a few cases from any series in order to know the error trend. Many instances require the computation of each case because without it the results are of doubtful value.

For example, suppose that all factors of the following problem, except the Constant (pi = 3.14159) represent values obtained from observations or experiment, each having the Probable Error of Col. (b) as tabulated below. What is the P. E. of the answer, and how may the answer be expressed?

$$\frac{3.14159 \times 4.962^3 \times 0.0343}{1.623 \times 3.050^2} = ?$$

A ten-place answer, obviously ridiculously "correct" is 0.8719582822.
Though there are short-cuts to the P. E. in cases like this, the principle is shown below:

	(a) Factor	(b) Probable Error of (a)	(c) Relative P. E. (b) ÷ (a)	(d) Square Col. (c)	(e) (d) x Square of Exponents
(1)	4.962	.018	.00363	.000013	.000117 (mult. by 3^2)
(2)	.0343	.0012	.03499	.001224	.001224
(3)	1.623	.024	.01479	.000219	.000219
(4)	3.050	.011	.00361	.000013	.000052 (mult. by 2^2)
	All constants are omitted.			Total of Col. (e)	.001612

Relative P. E. of answer is square root of 0.001612, or 0.040.
Actual P. E. of answer is 0.040 x 0.8719 . . ., or 0.035.
Answer is, therefore, written as 0.872 ± 0.035 P. E.

It follows from the definition of Probable Error that it is as even a chance that the true answer lies within the limits of 0.837 and 0.907 as that it lies outside of these limits.

To avoid intrusion of errors due to rounding, it is important that the amounts which have Probable Errors be written to the same number of decimals as shown in the P. E. For a similar reason, it is important that any Constants be expressed to a sufficient number of places so that rounding of their right-hand digits will not introduce an error approaching the magnitude of the right-hand digit of the P. E. to the number of figures expressed. Rounding errors do not follow the normal Probability Curve. In this instance, it is obvious that the loss of the 6th decimal in the Constant, which could not exceed 5 and is actually 3, creates a relative error of the Constant of such a small amount that it could not affect the P. E. of the answer.

The same computing plan applies if the original errors are given in terms of Standard Deviations. In such a case, the final error figure is the Standard Deviation of the answer. As both Probable Error and Standard Deviation are based upon the same Probability Curve, they have a constant relationship. The Probable Error is 0.6745 x Standard Deviation, so when one is known the other is easily found.

NOGRADY METHOD FOR SOLUTION OF CUBIC EQUATIONS

REMARKS: *The application of the Birge-Vieta Method (See MM-434D) to the solution of a cubic (third degree) equation gives the real root that is nearest to the first approximation. The work must then be repeated for other real roots. No imaginary roots are found. Special study has been given by Henry A. Nogrady* to the problem of obtaining all roots of such equations, both real and imaginary. Complete exposition of the method is given in his monograph, "A New Method for the Solution of Cubic Equations."* By aid of a table included in this book, the work is greatly simplified.*

The description herein exemplifies the use of the Marchant calculator when applied to the general cubic equation having three real roots, or having one real root and two conjugate complex roots. Modification to fit cases of two real roots, one real and two non-conjugate complex roots, and three complex roots, as well as tests for recognizing in what classification any equation comes, is fully covered in the Nogrady monograph, which is assumed to be in possession of the reader.

OUTLINE: The general cubic equation

(1) $ax^3 + bx^2 + cx + d = 0$

where a, b, c, and d are any numbers, is transformed into

(2) $y^3 + py + q = 0$

by substituting

(3) $\dfrac{3ac - b^2}{3a^2} = p$ and $\dfrac{2b^3}{27a^3} - \dfrac{bc}{3a^2} + \dfrac{d}{a} = q$

(2) then becomes

(4) $z^3 + nz + n = 0$

by substituting

(5) $\dfrac{p^3}{q^2} = n$

If n is real, eq. (4) has at least one real root. Its value is tabulated in the Nogrady monograph as z_1. By substitutions not outlined herein, the other roots of (4) are

(6) $z_2 = \dfrac{z_1}{2}\left(-1 + \sqrt{\dfrac{z_1 - 3}{z_1 + 1}}\right)$

(7) and $z_3 = \dfrac{z_1}{2}\left(-1 - \sqrt{\dfrac{z_1 - 3}{z_1 + 1}}\right)$

(*) *"A New Method for the Solution of Cubic Equations" by Henry A. Nogrady, ~~29 - 18 Boylne Ave.~~ Detroit, Michigan. For sale by the author, price $1.25 postpaid.*

When z_1, z_2, and z_3 are found, the corresponding y's are found by multiplying the z's by the ratio q/p.

The corresponding x's are found by subtracting b/3a from the y's.

The computation is expedited if the following terms are evaluated in the order named:

$3a$, $3ac$, $3a^2$, $27a^3$, p, bc, q, $b/3a$, q^2, n, q/p, $(z_1 - 3)/(z_1 + 1)$. Extract root of previous amount, and then evaluate the y's and x's. This listing of elements of the computation does not comprise the bettering of the table value of z_1 (see Eq. 6).

EXAMPLE I

Find roots to 5 places of $x^3 + 2x^2 + 10x - 3 = 0$

By substitutions outlined above

$$y^3 + 8.66667y - 9.07407 = 0$$

and

$$z^3 + 7.90592z + 7.90592 = 0$$

From Nogrady Table, Page XXIV, the nearest $n = 7.911462$ for which the corresponding root z_1 is -0.906.

This value is improved to six figures by the following process:

(8) Six-figure value of $z_1 = \dfrac{2z_1^{\,3} - n}{3z_1^{\,2} + n}$ *NOTE: A four-figure value requires only linear interpolation except at certain extremes of table.*

in which $z_1 = -0.906$ and $n = 7.90592$; or Six-figure $z_1 = -0.905950$

from which $z_2 = .45298 + 2.91919\ i$ and $z_3 = .45298 - 2.91919\ i$

Multiplying these z's by q/p, we have

$$y_1 = 0.94854;\ y_2 = -0.47427 + 3.05642\ i;\ y_3 = -0.47427 - 3.05642\ i$$

Subtracting b/3a, we have

$$x_1 = 0.28187;\ x_2 = 1.14094 + 3.05642\ i;\ x_3 = 1.14094 - 3.05642\ i$$

The latter two roots, because of symmetry, are termed Conjugate Complex Roots. The symbol "i" indicates $\sqrt{-1}$.

OPERATIONS: Decimals; Upper Dial 6, Middle Dial 11, Keyboard Dial 5. Use any Marchant 8 or 10 column model.

NOTE: Because the coefficients are simple integers, certain operations listed below normally would be omitted. For sake of completeness, however, they are listed. Whether a multiplication or division is positive or negative depends upon the sign of the factors and whether their product is to be added or subtracted. The procedure given below requires this obvious modification in the case of examples that have different signs from the equation considered herein.

(1) Set up in Keyboard Dial "a" (1.00000) and multiply by 3.
Copy "3a" (3.00000) from Middle Dial to Work Sheet.

(2) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "c" (10.00000).

Copy "3ac" (30.00000) from Middle Dial to Work Sheet.

(3) Clear Upper and Middle Dials, and multiply by "a" (1.00000).

Copy "$3a^2$" (3.00000) from Middle Dial to Work Sheet.

(4) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "a" (1.00000).

(5) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by 9.

Copy "$27a^3$" (27.00000) from Middle Dial to Work Sheet.

(6) Clear all dials, set up in Keyboard Dial "3ac" (30.00000), shift to 7th position, and depress Add Bar. Then depress Subtract Bar, set up "b" (2.00000) in Keyboard Dial, and reverse multiply by "b" (2.00000).

(7) Change Keyboard Dial to read "$3a^2$" (3.00000), and divide.

Copy "p" (8.66667) from Upper Dial to Work Sheet.

(8) Clear all dials, set up in Keyboard Dial "b" (2.00000), and multiply by "c" (10.00000).
Copy "bc" (20.00000) from Middle Dial to Work Sheet.

(9) Clear Upper and Middle Dials and multiply by "b" (2.00000).

(10) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "2b" (4.00000).

"$2b^3$" (16.00000) appears in Middle Dial, but it need not be copied to Work Sheet.

(11) Change Keyboard Dial to read "$27a^3$" (27.00000) and divide.

(12) Clear Keyboard and Middle Dials, set up in Keyboard Dial "bc" (20.00000), shift to 7th position, depress Add Bar and then depress Subtract Bar, change Keyboard Dial to read "$3a^2$" (3.00000), moye Manual Counter Control toward the operator, and depress Division Key in the manner that will not cause Upper Dial to clear.

(13) Clear Keyboard and Middle Dials, set up in Keyboard Dial "d" (3.00000), shift to 7th position, depress Add Bar, and then depress Subtract Bar, change Keyboard Dial to read "a" (1.00000), and inasmuch as "d" is negative the Manual Counter Control will be left as it was in Step 12; i.e., toward the operator. Depress Division Key in the manner that will not cause Upper Dial to clear. Move Manual Counter Control away from operator.

NOTE: It will now be observed that Upper Dial shows a negative amount. This is evaluated as a positive amount and copied to Work Sheet as "q" (-9.07407).

(14) Clear all dials, set up "b" (2.00000), and, with carriage in 7th position, depress Add Bar.

(over)

(15) Change Keyboard Dial to read "3a" (3.00000) and divide.

Copy "b/3a" (0.66667) from Upper Dial to Work Sheet.

(16) Clear all dials, set up "q" (9.07407) and multiply by "q" (9.07407).

Copy "q^2" (82.33875) from Middle Dial to Work Sheet.

(17) Clear all dials, set up "p" (8.66667) and multiply by "p" (8.66667).

(18) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "p" (8.66667).

(19) Change Keyboard Dial to read "q^2" (82.33875) and divide.
Copy "n" (7.90592) from Upper Dial to Work Sheet.

(20) From Table of Nogrady Roots, Page XXIV, the nearest "n" is 7.911462 for which corresponding root "z_1" is -0.906.

NOTE: The computation for improving this root to 0.905950 by formula 8 is obvious. It is taken to 5 places as -0.90595.

(21) Clear all dials, set up in Keyboard Dial "q" (9.07407) and, with carriage in 7th position, depress Add Bar. Change Keyboard Dial to read "0" (8.66667) and divide.

Copy "q/p" (1.04701) from Upper Dial to Work Sheet.

(22) Clear all dials, set up in Keyboard Dial "z_1-3" (3.90595) and with carriage in 7th position, depress Add Bar. Change Keyboard Dial to read "$z_1 + 1$" (0.09405) and divide.

Copy $(z_1-3)/(z_1+1)$ or (-41.53057) from Upper Dial to Work Sheet.

(23) Extract Square Root of -41.53057 by a Marchant Table, producing a five-figure root of 6.4444 which is expressed as 6.4444 i, indicating that it is the square root of a negative number.

NOTE: This square root may be improved, if desired, to 6.44442 i.

(24) Clear all dials, set up in Keyboard Dial "$z_1/2$" (0.45298) and multiply by square root from Step 23 (6.44442).

Copy coefficient of i (2.91919) from Middle Dial to Work Sheet, thus completing all figures from z_2 and z_3.

(25) Clear all dials, set up in Keyboard Dial "q/p" (1.04701) and multiply by z_1 (0.90595) and the real and imaginary parts of z_2 and z_3 (0.45298) and (2.91919) producing

y_1 (0.94854); y_2 (-0.47427 + 3.05642 i);
and y_3 (-0.47427 - 3.05642 i).

(26) Clear all dials. With carriage in 7th position, set up y_1 (0.94854), and add. Set up "b/3a" (0.66667) and, with Non-Shift Key down, reverse multiply by 1.

x_1 (0.28187) appears in Middle Dial.

(27) Clear Middle Dial and touch Add Bar. Set up the real part of y_2 and y_3 (0.47427) and add, thus completing values for

x_2 (1.14094 + 3.05642 i)
x_3 (1.14094 - 3.05642 i)

EXAMPLE II

Find roots to 5 places of $x^3 - 7x + 6 = 0$

This is in the form of $y^3 + py + q = 0$, so the operations following Step No. 15 need only be done with certain obvious deletions. The outline is below:

$n = p^3/q^2 = -343/36 = -9.52778.$

From Table, nearest "n" is -9.516913 for which z_1 is -1.169.

This value is improved by *(8)* to

$$z_1 = \frac{2\ (-1.169^3) + 9.52778}{3\ (-1.169^2) - 9.52778} = \frac{6.33276}{-5.42810} = -1.16667$$

$q/p = 6/-7 = -0.85714$

$\sqrt{(z_1 - 3)/(z_1 + 1)} = \sqrt{25} = 5$

$z_2 = -0.58333 \cdot 4 = -2.33333$

$z_3 = -0.58333 \cdot -6 = 3.50000$

$x_1 = y_1 = -0.85714 \cdot -1.16667 = 1.$

$x_2 = y_2 = -0.85714 \cdot -2.33333 = 2.$

$x_3 = y_3 = -0.85714 \cdot 3.50000 = -3$

The Marchant operations are similar to most of those following Step 16 of Example I.

DIRECT INTERPOLATION AND SUB-TABULATION

(IF FOURTH DIFFERENCES DO NOT EXCEED 1000)

Those familiar with elementary mathematics of interpolation will have no difficulty in following the method outlined herein. For the benefit of others an Appendix (Marchant-Method MM-439E2) is provided which describes the notations and conventions of this process.

REMARKS: *This method obtains intermediate values on a smooth curve connecting observed or calculated values. If the curve is that of a continuous function with a continuous derivative of the highest order of the differences that are taken into account in the process, the method also obtains intermediate values of the function between calculated values within the limits of accuracy of the method. The process used in this method comprises the correction of the straight-line (direct proportion) interpolate to reflect the influence of differences higher than the first. If there is a very large volume of sub-tabulation to be done, it is advisable to consider an alternative process according to which the interpolates are integrated from the higher order differences of the sub-divided intervals, using special modifications of certain accounting machines for the successive integration.*

The methods herein accomodate all cases in which fourth differences do not exceed 1000, it being substantially necessary, however, only to use first and second differences for the computation. A limited amount of differencing to third and fourth orders is necessary at infrequent intervals merely to obtain their general range in order that a proper selection of method may be made from the choices herein, and to provide the "4th difference correction" of Example IV if 4th differences exceed 20. (See Appendix Marchant Method MM-439E3).

Marchant Method MM-419 provides a rapid way of obtaining differences, though if a great deal of such work is to be done, it is suggested that the possibilities of certain accounting machines be investigated for this purpose.

OUTLINE: This method describes:

I "Straight-Line" or "Direct Proportion" Interpolation.
If accuracy is desired to the last place of the tabular values, this should not be used when second differences exceed 4.

II "Second Difference" Interpolation.
If accuracy is desired to the last place of the tabular values, this should not be used if third differences exceed 60 or fourth differences exceed 20.

III "Third Difference" Interpolation.
If accuracy is desired to the last place of the tabular values, this should not be used if fourth differences exceed 20.

IV "Fourth Difference" Interpolation, using the Comrie "Fourth Difference Throwback."
For use when fourth differences do not exceed 1000.

ACCURACY: If the chart of coefficients (Fig.1) is used for obtaining coefficients and chart values are read to the nearest cross-section line or mid-point thereof the maximum error of the reading of any coefficient will be .00025. In "Second Difference" Interpolation (see above), the sum of two adjacent second differences is multiplied by the coefficient. Under this condition, second difference should not exceed 1000 if the chart is to be used, provided accuracy is desired to last figure of table values. In "Third Difference" and in "Fourth Difference" Interpolation (see above), each difference is multiplied by the coefficient, so the differences may be as large as 2000 without the error exceeding 5 in the digit at the right of the table values. This assures "last place accuracy" as the term is generally understood.

If tables of coefficients are available showing five significant figures with interpolation to six figures, second differences may be as high as 1,000,000. Where extreme accuracy is required and tables of coefficients are not available, they may be computed algebraically, the formulas being on the Chart (Fig. 1).

The other limitations to largest difference that may be used as referred to in the preceding section, are inherent in the process, and should be observed regardless of whether coefficients are taken from the chart or from tables.

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EXAMPLE I: STRAIGHT-LINE INTERPOLATION
(Use when second difference does not exceed 4)

Argument	Function	Differences First	Differences Second
3351	3.057368 (f_{-1})		
		27312 ($d'_{-\frac{1}{2}}$)	
3352	3.084680 (f_0)		-4 (d''_0)
		27308 ($d'_{\frac{1}{2}}$)	
3353	3.111988 (f_1)		

Find value for 3352.47 (n = .47)

OPERATIONS: Decimals; Upper Dial 2, Middle Dial 8. Keyboard Dial 6. Non-Shift Key should be down.

(1) Set up in Keyboard Dial f (3.084680) and add with carriage in 3rd position. Clear Upper Dial.

(2) Set up in Keyboard Dial $d'_{\frac{1}{2}}$ (.027308) and as function is ascending, multiply by 1.0 and depress Upper Green Shift Key.

f_1 (3.111988) appears in Middle Dial showing the function was properly differenced.

(3) Without entries or clearances, reverse multiply by 1, and then multiply by n (.47): f_n (3.097515) appears in Middle Dial.

"n" (.47) appears in Upper Dial.

NOTE A: If this were a case of sub-tabulation in which it was desired to obtain a large number of intermediate interpolates, the Upper Dial would be built up to the desired "n's" and upon completion would be increased to read 1.0- Upper Dial is then cleared so a new series of "n's" may be started. "f_1" (3.111988) is then in Middle Dial where it becomes "f_0" for sub-tabulation in the next interval 3353-3354. Under these conditions, Step (1) of the above method would be omitted.

If function is descending; that is to say, becomes smaller as the argument increases, move Manual Counter Control toward the operator and keep it so for the entire series of operations, proceeding as follows for Steps (2) and (3):

(2a) Set up $d'_{\frac{1}{2}}$ and reverse multiply by 1.

(3a) Multiply by 1 and then reverse multiply by n.

NOTE B: The Marchant Method series includes many special applications of Straight-Line Interpolation, all of which are based upon the above process. Mathematicians may be interested in MM-439B, which describes double interpolation between four corner values of row and column which surround the interpolate desired.

INTERPOLATION OR SUB-TABULATION TAKING INTO ACCOUNT HIGHER DIFFERENCES

REMARKS: The general process is to calculate corrections to the "straight-line" or "direct proportion" interpolate ($f_0 + nd'_{\frac{1}{2}}$). Whether this correction is positive or negative depends upon the signs of the differences; that is to say, the shape of the curve of the function. As a check of this, the computer should note the trend of the curve of the function as originally tabulated. If intermediate values should be below a straight-line connecting the tabulated points, the corrections are obviously negative, and vice-versa.

In each of the cases considered below, an example will be given of sub-tabulation; that is to say, of obtaining a number of equally spaced intermediate amounts such as would be the case when preparing a table or in drawing a more accurate curve of a function. An example will also be given of finding a single fractional interpolate. The processes are similar, but there are slight alterations of method which warrant their being separately considered.

It must be remembered that it is possible to interpolate and sub-tabulate without differences by using formulas of LaGrange type (See Marchant Method MM-439D). Whichever method should be used in any case will depend upon circumstances. In general, it may be said that whenever the differences have been computed as a means of locating possible errors of the table values, the methods based upon differences as herein may be used to advantage. Methods based upon differences are also to be preferred when obtaining an interpolate to an odd fractional interval. The choice of method in other cases will depend upon a number of conditions, discussion of which is beyond the scope of this work.

(over)

EXAMPLE II: SECOND-DIFFERENCE SUB-TABULATION
(Use when 3rd difference does not exceed 60 and 4th does not exceed 20)

Argument	Function	Differences First	Second	Third	Fourth
10	28.7694 (f_{-2})		689 (d''_{-2})		
		8372 ($d'_{-1\frac{1}{2}}$)			
11	29.6066 (f_{-1})		714 (d''_{-1})		
		9086 ($d'_{-\frac{1}{2}}$)		30	
12	30.5152 (f_0)		744 (d''_0)		6
		9830 ($d'_{+\frac{1}{2}}$)		36	
13	31.4982 (f_{+1})		780 (d''_{+1})		
		10610 ($d'_{+1\frac{1}{2}}$)			
14	32.5592 (f_{+2})		823 (d''_{+2})		

Sub-tabulate the above to tenths between arguments 12 and 13.

NOTE: Only the first and second differences are required for this and succeeding examples. However, enough differencing to third and fourth orders is done to establish the fact that the example comes within the limitations of the method. Third and fourth differences are also taken near the upper and lower parts of any table so that second and first differences for the end intervals may be obtained by building up from fourth to third differences, a process of extrapolation.

OPERATIONS: Decimals; Upper Dial 9 and 4, Middle Dial 14 and 9, Keyboard Dial 5.

Use any 10 column model Marchant. (The Keyboard Constant Device is desirable in the work of Part B.)

PART A: **OBTAINING CORRECTIONS TO STRAIGHT-LINE INTERPOLATES FOR THE INTERVAL 12 TO 13.**

(1) With carriage in 5th position, set up in Keyboard Dial d''_0 (.0744) and add.

(2) Similarly set up d''_1 (.0780) and add.
$d''_0 + d''_1$ (.1524) appears in Middle Dial.

(3) Transfer Middle Dial amount (.1524) to Keyboard Dial, clear Upper and Middle Dials, and multiply in succession at 4th Upper Dial decimal by the Chart (Fig. 1) coefficient B''_{0+1}, taken from the middle curve for n = 0.1, 0.2, ... 0.9, and tabulate results as follows:

n	B''_{0+1} (from Chart)	Correction to 1st Difference
0.1	-.0225	-.00343
0.2	.04	.00610
0.3	.0525	.00800
0.4	.06	.00914
0.5	.0625	.00953
0.6	.06	.00914
0.7	.0525	.00800
0.8	.04	.00610
0.9	.0225	.00343

It is noted that calculation need be made only to n = .5.

PART B: OBTAINING INTERPOLATES FOR THE SUB-TABULATION (Have Non-Shift Key depressed during this part of the work).

As the above corrections are negative, the corrections shown are negative and are to be deducted from the "straight-line" interpolates.

(4) Set up in Keyboard Dial at 5th decimal f_0 (30.5152) and with carriage in 10th position, add into Middle Dial at 14th decimal.

(5) Similarly set up $d'_{\frac{1}{2}}$ (.9830) and multiply by 1.
f_1 (31.4982) appears in Middle Dial, proving correctness of differencing.

(6) Reverse multiply by 1, which restores f_0 in Middle Dial.

(7) Clear Upper Dial, shift to 9th position, and multiply by first n(0.1).
Straight-line Interpolate for n = 0.1 appears in Middle Dial (30.6135) but it need not be separately noted.

(8) Shift to 3rd position,* set up at extreme left of Keyboard Dial correction for n = 0.1 (343) taken from table obtained in Step 3. It now appears as .00343 with reference to 12th (phantom) Keyboard Dial decimal. Reverse multiply by 1.
Interpolate for 12.1 (30.61007) appears in Middle Dial.

(9) Multiply by 1 to restore Straight-line Interpolate (30.6135) in Middle Dial.

(10) Clear correction (.00343) from left if Keyboard Dial,** shift to 9th position, and multiply by 0.1.
Straight-line interpolate for n = 0.2 (30.7118) appears in Middle Dial, but it need not be separately noted.
"n" (0.2) appears in Upper Dial as control count.

(11) Shift to 3rd position,* set up correction for n = 0.2 from Step 3 (610) at extreme left of Keyboard Dial and reverse multiply by 1.
Interpolate for 12.2 (30.70570) appears in Middle Dial.

(12) Multiply by 1 to restore Straight-line Interpolate (30.7118).

(13) Repeat Steps 10, 11, and 12 for all succeeding values, developing the interpolates below:

12.1	30.61007	rounded up as	30.6101
12.2	30.70570		30.7057
12.3	30.80210		30.8021
12.4	30.89926		30.8993
12.5	30.99717		30.9972
12.6	31.09586		31.0959
12.7	31.19530		31.1953
12.8	31.29550		31.2955
12.9	31.39647		31.3965

(14) Upon completion of the work, shift to 9th position, clear the correction for n = 0.9 (.00343) from Middle Dial, ** and multiply by 0.1.

(over)

Upper Dial will show 1.0.
Middle Dial will show f_1(31.4982) which checks the work, provided the corrections were properly entered.

(15) Change Keyboard Dial to read $d'_{1\frac{1}{2}}$(10610) and continue as before to sub-tabulate in the 2nd interval.

(*) A decimal marker may be set at Upper Dial 2 to act as a guide for this position.

(**) Models with Keyboard Constant Device permit clearance by Keyboard Clear Key of amounts at left of Keyboard Dial, without disturbing amounts set up in the five right-hand columns of Keyboard Dial.

EXAMPLE IIa: SECOND DIFFERENCE INTERPOLATION
(Use when 3rd difference does not exceed 60
and 4th difference does not exceed 20)

In the table of Example II, interpolate for 12.626.

Using same decimal setting as for Example II, proceed as follows:

(1) With carriage in 5th position, set up in Keyboard Dial f_0(30.5152) and depress Add Bar.

(2) Set up in Keyboard Dial $d'_{\frac{1}{2}}$(.9830) and multiply* by n(.626).

(3) Clear Upper and Keyboard Dials, set up in Keyboard Dial the sum of d''_0 and d''_1 (.1524) and reverse multiply** by B'_{0+1} (.0585) taken from Chart (Fig. 1) that corresponds to n = .626.
$f_{.626}$(31.1216) appears in Middle Dial.
Complement of B'(...9.9415) appears in Upper Dial. If it is desired to show this in true figures, move Manual Counter Control toward the operator.

(*) If first difference is negative, reverse multiply at this point.

(**) If second difference is positive reverse multiply, as shown. If it is negative, omit reverse.

EXAMPLE III: THIRD DIFFERENCE SUB-TABULATION
(Use when third difference exceeds 60,
and fourth difference does not exceed 20)

Argument	Function	Differences First	Second	Third	Fourth
1.5	1.484362(f_{-1})		134(d''_{-1})		
		3782($d'_{-\frac{1}{2}}$)		97	
1.6	1.488144(f_0)		231(d''_0)		7
		4013($d'_{\frac{1}{2}}$)		104	
1.7	1.492157(f_1)		335(d''_1)		
		4348($d'_{1\frac{1}{2}}$)			
1.8	1.496505(f_2)		447(d''_2)		

Sub-tabulate interval 1.6 to 1.7 to 20th's.

OPERATIONS: Decimals; Upper Dial 11 (phantom) and 4, Middle Dial 17, Keyboard Dial 13 (phantom) and 6. A decimal marker should also be placed between Middle Dials 11 and 10 as a guide for retention of desired figures of the answer. (The Keyboard Constant Device is desirable for Part B.)

PART A: OBTAINING CORRECTION TO STRAIGHT-LINE INTERPOLATES FOR THE INTERVAL:

NOTE: The process used for this part is a marked improvement over that which was included in the first edition of this method. Our attention was called to the short-cut possibilities of this process by William H. Watkins, Senior Naval Architect and Supervisor of the Scientific and Test Groups, Design Section, Puget Sound Navy Yard.

It is a characteristic of E''_0 and E''_1 (see Chart Fig. 1) that each is an invert of the other; viz., E''_0 for $n = .15$, equals E''_1 for $n = .85$. When second differences are less than 1000, as in this case, it is possible to use double multiplication on 10 column Marchants as exemplified herein, thus reducing calculating time materially. When second differences exceed 1000, the multiplications to produce the corrections corresponding to E''_0 and E''_1 must be separately made.

(1) Set up d''_1 (.000335) at extreme left of Keyboard Dial, with reference to phantom Keyboard Dial decimal at 13, and d''_0 (.000231) with reference to 6th Keyboard Dial decimal, and lock in place as constant. Keyboard Dial reads 3350.000231.

(2) Prepare Columns 1, 2, and 3 of work sheet, as below. Column 2 is obtained by reference to Chart (Fig. 1), using upper curve for values of n to 0.5 and lower curve for values 0.5 to 0.95. Column 3 is the invert of Column 2.

(1)	(2) E''_0 (from Chart)	(3) E''_1 (invert of Col. 2)	(4) $E''_0 d''_0 + E''_1 d''_1$
.05	-.0153	-.0082	-.0000063
.10	.0285	.0165	121
.15	.0393	.0245	173
.20	.048	.032	218
.25	.0547	.0391	257
.30	.0595	.0455	290
.35	.0625	.0512	316
.40	.064	.056	335
.45	.064	.0598	348
.50	.0625	.0625	354
.55	.0598	.064	353
.60	.056	.064	344
.65	.0512	.0625	328
.70	.0455	.0595	304
.75	.0391	.0547	274
.80	.032	.048	235
.85	.0245	.0393	188
.90	.0165	.0285	134
.95	.0082	.0153	70

(3) Multiply Keyboard Dial set-up (see Step 1) by first E''_0 (.0153) at 11th Upper Dial decimal and by first E''_1 (.0082) at 4th Upper Dial decimal. Upper Dial now reads 153000.0082. Middle Dial reads 51.2550062.8130018942. Of these, only the central group (62813) is noted, it being read with reference to 17th decimal as .0000063, equaling $E''_0 d''_0 + E''_1 d''_1$. Copy in Column 4.

(over)

(4) Clear Upper and Middle Dials and repeat Step 3 for remaining intervals, thus completing Col. 4.

NOTE: In some cases the values of E''_0 and E''_1 in the above, being read from Chart, differ slightly from the actual coefficients, but the results obtained are well within the error limit assumed.

PART B: OBTAINING INTERPOLATES FOR THE SUB-TABULATION (Have Non-Shift Key down during this part of the work:)

Alter decimal setting to the following: Upper Dial 9 and 2, Middle Dial 15, Keyboard Dial 13 (phantom) and 6;

(1) With carriage in 10th position, set up in Keyboard Dial at 6th decimal f_0 (1.488144) and multiply by "1".

(2) Change Keyboard Dial to read $d'_{1/2}$ (.004013) and multiply by "1".

f_1 (1.492157) appears in Middle Dial as proof that first difference is correct.

(3) Reverse multiply by 1 to restore f_0 (1.488144) in Middle Dial.

(4) Clear Upper Dial, shift to 8th position, and multiply by .05.

Straight-line interpolate ($f_0 + nd'_{1/2}$) for n = 0.05 appears in Middle Dial but it need not be separately noted (1.48834465).

(5) Shift to 3rd position, set up in Keyboard Dial at 13th (phantom) decimal, the correction from Col. 4 of work sheet, corresponding to n = 0.05 (.0000063) and reverse multiply by 1.

NOTE: The 6th Keyboard Dial decimal acts as a guide for entering this and subsequent similar amounts. The "3" of .0000063 is directly at left of Keyboard Dial decimal.

Interpolate for 1.605 (1.488338) appears in Middle Dial.

(6) Without shifting carriage, multiply by 1 to restore straight-line interpolate for n = 0.05 in Middle Dial (1.48834465).

(7) Clear correction factor (.0000063) from Keyboard Dial at left, shift to 8th position, and multiply by 0.05.

Straight-line interpolate for n = 0.10 appears in Middle Dial, but it need not be separately noted.

(8) Repeat Steps 5, 6, and 7 for all succeeding values of n to 0.95, developing the interpolates, the first ten values of which are shown on the following page.

Argument		Interpolate	
	1.605		1.488338
	1.61		1.488533
	1.615		1.488729
	1.62		1.488925
	1.625		1.489122
	1.63		1.489319
	1.635		1.489517
	1.64		1.489716
	1.645		1.489915
	1.65		1.490115

9 Clear correction factor for n = 0.95 (.0000070) from Middle Dial at left, shift to 8th position, and multiply by .05.

f_2(1.492157) appears in Middle Dial. This proves the work, provided correction factors were properly entered from the work sheet.

EXAMPLE IIIa: THIRD DIFFERENCE INTERPOLATION
(Use when 3rd difference does not exceed 60
and 4th difference does not exceed 20)

In the table of Example III, interpolate for 1.63357 using the same decimal setting as for Part B of Example II.

(1) With carriage in 10th position, set up in Keyboard Dial f_0(1.488144) and depress Add Bar.

(2) Set up in Keyboard Dial $d'_{\frac{1}{2}}$(.004013) and multiply* at 9th Upper Dial decimal by n(.3357).

(3) Clear Upper and Keyboard Dials, set up in Keyboard Dial d''_0(.000231) and reverse multiply** at 9th Upper Dial decimal by E''_0 taken from Chart (Fig. 1) that corresponds to n = .3357 (.0619).

(4) Clear Upper and Keyboard Dials, set up in Keyboard Dial d''_1(.000335) and reverse multiply** at 9th Upper Dial decimal by E''_1 taken from Chart (Fig. 1) that corresponds to n = .3357 (.050).

Interpolate for 1.63357 (1.489460) appears in Middle Dial.

(*) *If first difference is negative, reverse multiply at this point.*

(**) *If second difference is positive, reverse multiply as shown; if it is negative, omit reverse.*

EXAMPLE IV: FOURTH DIFFERENCE SUB-TABULATION
(Fourth difference exceeding 20, but less than 1000)

The correction for straight-line interpolate is made without the necessity of computing 3rd or 4th differences, except to see if the example comes within the classification desired. Inasmuch as the coefficient for fourth differences varies little for values of n between 0 and 1.0, the fourth difference, if exceeding 20 and less than 1000, may be "thrown-back" and taken into account by merely modifying the 2nd difference. If second differences used in the calculating are reduced by 0.184 X d''''_0 the fourth difference is accounted for if within the limits named. This fact was first brought to attention by Dr. L. J. Comrie. It is, therefore, designated as the Comrie Throw-Back.

(over)

Argument	Function	Differences First	Second	Third	Fourth
-2	16.9 (f_{-4})				
		-150 ($d'_{-3\frac{1}{2}}$)			
-1	1.0 (f_{-3})		140 (d''_{-3})		
		-10 ($d'_{-2\frac{1}{2}}$)			
0	0 (f_{-2})		20 (d''_{-2})		
		10 ($d'_{-1\frac{1}{2}}$)			
1	1.0 (f_{-1})		140 (d''_{-1})		
		150 ($d'_{-\frac{1}{2}}$)		360	
2	16.0 (f_0)		500 (d''_0)		240
		650 ($d'_{\frac{1}{2}}$)		600	
3	81.0 (f_1)		1100 (d''_1)		
		1750 ($d'_{1\frac{1}{2}}$)			
4	256.0 (f_2)				

In the above tabulation of X^4, it is seen that 4th differences are +240. (They are, in fact, constant at this amount). We should, therefore, be able to interpolate any value of X^4 correctly to tenths of units, whereas the value would not necessarily be correct to hundredths of units, as this would make d''''_0 equal to 2400.

Sub-tabulate interval 2.0 to 3.0 to tenths.

OPERATIONS: Decimals: Upper Dial 10 &4, Middle Dial 11. Keyboard Dial 7 & 1. Use any 10-column Marchant. (The Keyboard Constant Device is very desirable in the work of Part B.

PART A: <u>OBTAINING CORRECTION TO STRAIGHT-LINE INTERPOLATES FOR THE INTERVAL</u>:

(1) Compute "modified" differences, as follows: (Be sure 4th difference has proper sign.)

$M''_0 = 50.0 - 0.184 \times 24 = 45.6$
$M''_1 = 110.0 - 0.184 \times 24 = 105.6$

(2) Set up M''_1 (105.6) at 8th Keyboard Dial decimal and M''_0 (45.6) at 1st Keyboard Dial decimal, and lock as constant.

(3) Prepare Columns 1, 2, and 3 of work sheet as below, as described in Step 2 of Example III:

(1) n	(2) E''_0 from Chart	(3) E''_1 (invert of Col. 2)	(4) $E''_0M''_0 + E''_1M''_1$
.10	.0285	.0165	3.04
.20	.048	.032	5.57
.30	.0595	.0455	7.52
.40	.064	.056	8.83
.50	.0625	.0625	9.45
.60	.056	.064	9.31
.70	.0455	.0595	8.36
.80	.032	.048	6.53
.90	.0165	.0285	3.76

(4) Proceed as in Example III and multiply the Keyboard Dial set-up by E''_0 of Col. (2) at left and by E''_1 of Col. 3 at right, thus completing Col. 4.

PART B: <u>OBTAINING INTERPOLATES FOR THE SUB-TABULATION</u> (Have Non-Shift Key down during this part of the work).

Alter decimal setting to the following: Upper Dial 9 and 3, Middle Dial 11, Keyboard Dial 8 and 2.

(1) With carriage in 10th position, set up in Keyboard Dial at 2nd decimal f_0 (16.0) and multiply by 1.

(2) Change Keyboard Dial to read $d'_{½}$ (65.0) and multiply by 1.

f_1 (81.0) appears in Middle Dial as proof that first difference is correct.

(3) Reverse multiply by 1 to restore f_0 (16.0) in Middle Dial.

(4) Clear Upper Dial, shift to 9th position, and multiply by 0.1.

Straight-line interpolate ($f_0 + nd'_{½}$) for n=0.1 (22.5) appears in Middle Dial, but it need not be separately noted.

(5) Shift to 4th position, set up in Keyboard Dial at 8th decimal the correction from Col. 4 of work sheet corresponding to n=0.1 (3.04) and reverse multiply by 1.

Interpolate for 2.1 (19.46) appears in Middle Dial.

(6) Without shifting carriage, multiply by 1 to restore straight-line interpolate for n=0.1 (22.5).

(7) Clear correction factor (3.04) from Keyboard Dial at left, shift to 9th position, and multiply by 0.1.

Straight-line interpolate for n=0.2 appears in Middle Dial, but it need not be separately noted.

(8) Repeat Steps 4, 5, and 6 for all succeeding values of n to 0.9, developing the interpolates as below:

	(1)		(2)		(3)	(4)
Argument	2.1	Interpolate	19.46	Rounded to	19.5	X^4 = 19.45
	2.2		23.43		23.4	23.43
	2.3		27.99		28.0	27.98
	2.4		33.17		33.2	33.18
	2.5		39.05		39.1	39.06
	2.6		45.69		45.7	45.70
	2.7		53.15		53.2	53.14
	2.8		61.47		61.5	61.47
	2.9		70.74		70.7	70.73

NOTE: As this is an algebraic function which is easily calculated, the actual calculated amounts are shown in right-hand column for comparison.

(over)

(9) Clear correction factor for n = 0.9 (3.76) from Middle Dial at left, shift to 9th position, and multiply by 0.1.

f_1(81.0) appears in Middle Dial as proof (provided correction factors were properly entered from the work sheet).

NOTE: The method produces results accurate to the same number of decimals as used in the tabulated functions: in this case to 0.05. It will be observed, however, from comparison of Col. (2) and Col. (4) of Step 8 that in this case the accuracy was within .01. Inasmuch as the limit of accuracy is approached as the fourth difference approaches 1000 and the fourth difference in this case is 240 it is seen that the example supports the claim as to accuracy, because if fourth difference were 1000, the error would not exceed .04.

EXAMPLE IVa: FOURTH DIFFERENCE INTERPOLATION
(Fourth difference exceeding 20, but less than 1000)

In the table of Example IV, interpolate for 2.375 using the same decimal setting as for Part A of Example IV.

(1) With carriage in 5th position, set up at 2nd Keyboard Dial decimal f_0(16.) and depress Add Bar.

(2) Set up in Keyboard Dial $d'_{\frac{1}{2}}$(65.0) and multiply* by n(.375).

(3) Clear Upper and Keyboard Dials and set up in Keyboard Dial the modified second difference M''_0(45.6) and reverse multiply** by E''_0 taken from Chart (Fig. 1) that corresponds to n = .375 (.0635).

(4) Clear Upper and Keyboard Dials and set up in Keyboard Dial the modified second difference M''_1(105.6) and reverse multiply** by E''_1 taken from Chart (Fig. 1) that corresponds to n = .375 (.0538).

Interpolate for 2.375 (31.798) appears in Middle Dial and is written as 31.8.

(*) *If first difference is negative, reverse multiply at this point.*

(**) *If modified second difference is positive reverse multiply as shown: if it is negative, omit reverse.*

NOTE: If E''_0 and E''_1 had been calculated or taken from tables at more exact values of .06347 and .05371 respectively, the interpolate would be 31.8090. The exact interpolate is 31.81665. The slight difference is due to the approximation occasioned by use of the Throw-Back but the result is well within the limit of accuracy of the method.

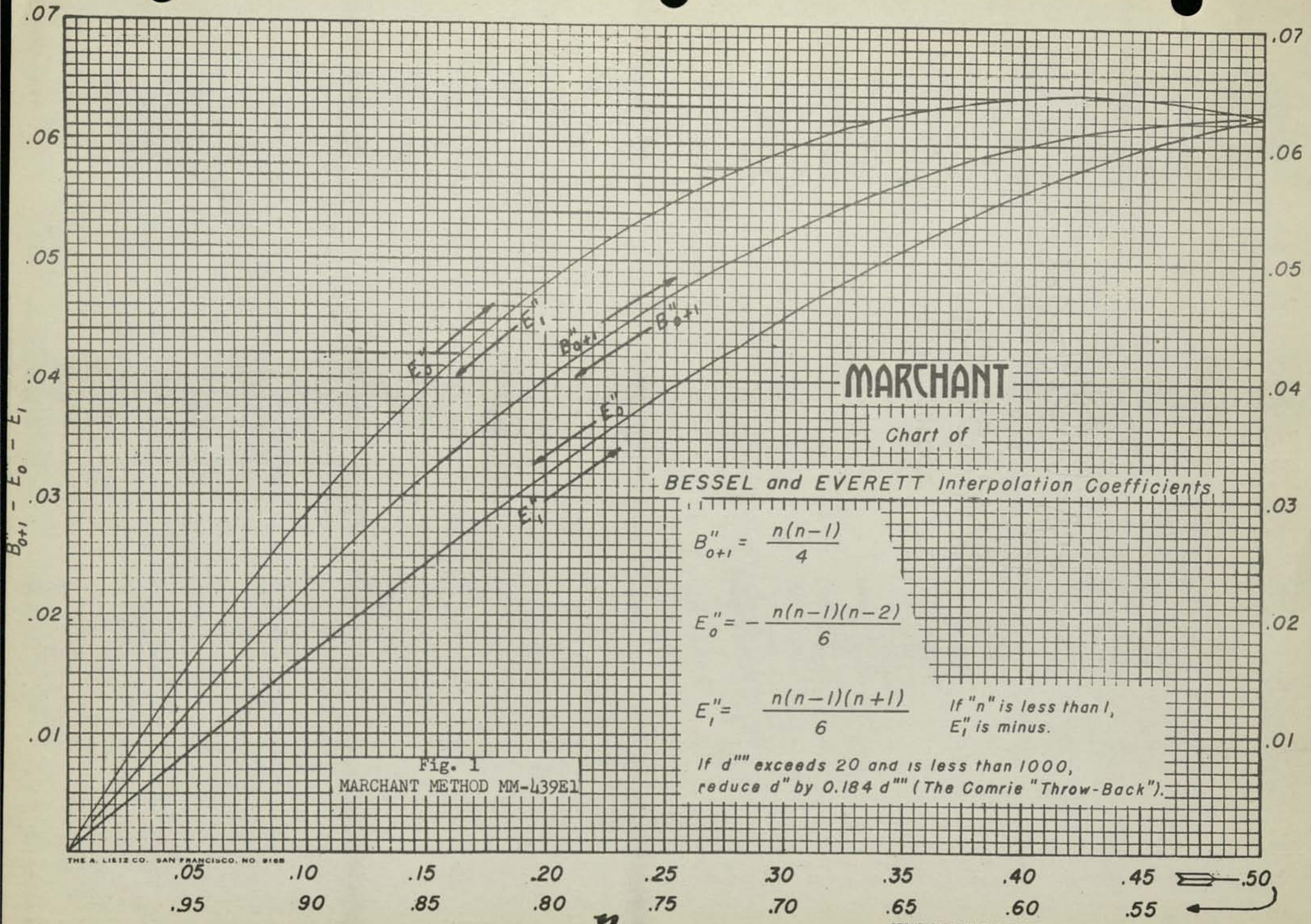

Fig. 1
MARCHANT METHOD MM-439E1

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DIRECT INTERPOLATION AND SUB-TABULATION (IF FOURTH DIFFERENCES DO NOT EXCEED 1000) EXPLANATORY APPENDIX TO MARCHANT-METHOD MM-439E1

As mentioned in MM-439E1, this Appendix is provided for those who require more information concerning the basic principles of the methods exemplified in MM-439E1.

NOMENCLATURE

The methods outlined in the examples are based upon what is known as Central Difference formulas. The nomenclature is:

a_{-1}	f_{-1}		d''_{-1}		d'''_{-1}
		$d'_{-\frac{1}{2}}$		$d'''_{-\frac{1}{2}}$	
a_0	f_0		d''_0		d'''_0
		$d'_{\frac{1}{2}}$		$d'''_{\frac{1}{2}}$	
a_1	f_1		d''_1		d'''_1
		$d'_{1\frac{1}{2}}$		$d'''_{1\frac{1}{2}}$	
a_2	f_2		d''_2		d'''_2

In which the column of "a" represents the table argument; i.e., the independent variable; the column of "f" represents the functions of "a": i.e., the dependent variable. The successive differences are then shown in the other columns; i.e., d' first difference, d", second difference, etc.

This nomenclature differs from that used with descending or ascending difference formulas which have certain disadvantages for practical computing.

Any pair of tabular values, argument and corresponding function, may be selected as a_0 and f_0 All other values, as well as the pattern of differences, then assume the designation of the above nomenclature. If, for example, it is desired to sub-tabulate intermediate values in one interval, the upper tabular value becomes f_0 and the lower value becomes f_1 If it is then desired to sub-tabulate in the interval next below, the former f_1 becomes the f_0 for the second interval, etc.

SIGNS

The work will be somewhat expedited if the functions are listed as increasing values as they progress from the top of the column toward the bottom; i.e., first differences are positive. Sometimes this is not convenient, in which case the instructions herein require obvious modification to obtain the correct "first difference" interpolate.

Care should be taken to see that the differences have the correct sign. In this nomenclature the columns are read downwards. If a <u>positive</u> amount added to the upper of a pair of tabular values, or differences, produces the lower of such pair, the resulting difference is positive, and vice-versa.

EXAMPLES:

(a) $-8\ (f_{-2})$
$-1\ (f_{-1})$ $d'_{\frac{1}{2}}$ is $+7$, because $-8 + 7 = -1$

(b) $\begin{matrix} -1 \ (d''_{-\frac{1}{2}}) \\ 1 \ (d''_{\frac{1}{2}}) \end{matrix}$ d''_0 is $+2$, because $-1+2=+1$

(c) $\begin{matrix} -5 \ (d''_{-\frac{1}{2}}) \\ -9 \ (d''_{\frac{1}{2}}) \end{matrix}$ d''_0 is -4 because $-5+(-4)=-9$

In making spot computations of 3rd and 4th differences to determine which of the various methods herein should be used, it is just as important to determine the sign of the fourth difference as it is to determine its amount, as obviously a negative fourth difference <u>increases</u> the second difference when producing the Modified Second Difference by using the Comrie Throw-Back (as in Examples IV and IVa).

ABSOLUTE VALUES OF DIFFERENCES

The explanation of designations such as "second difference not exceeding 1000," etc., is that it is customary to list the differences in the unit that comprises the right-hand digit of the function; thus,

$\begin{matrix} 3.604825 \ (f) \\ \\ 3.605136 \ (f) \end{matrix}$ $\quad 311 \ (d'_{\frac{1}{2}})$ means that $d'_{\frac{1}{2}}$ is .000311

An interpolation by the within processes consists of building up the function from a difference which is made up of increments caused by the calculated effects of second and higher differences on the first differences, these higher differences are sometimes given with a greater number of figures so as to produce greater accuracy of a total. Partial products of these differences and their corresponding coefficients are, likewise, often carried out to one more place than desired in the total. As the Marchant Method does this automatically, there is not much occasion to take advantage of this process in these methods. It does, however, explain why more digits are used in certain parts of the calculation than would appear to be required.

CHART AND FORMULAS

The Straight-line Interpolation of Example I is familiar to all computers.

In Example II, the Bessel Formula for Coefficients is used, as shown on the Chart. This disregards the effect of third and subsequent differences, consequently it should not be used when third differences exceed 60. If this amount is exceeded, the third difference affects the accuracy of the digit at right of the one being retained by more than 5, which is unsatisfactory.

In Examples III and IV the Everett Formula for Coefficients is used, as shown on the Chart. This takes into account third differences, but does not require that third differences be tabulated. Example IV differs from III only in that the effect of fourth differences is taken into account. Fourth differences need not be considered if they are less than 20, because such a difference affects the digit at right of the one being retained by less than 5.

The basic formula for Examples II and IIa is:

$$f_n = f_0 + nd'_{\frac{1}{2}} + B''_{0+1} \ (d''_0 + d''_1)$$

for Examples III and IIIa is:

$$f_n = f_0 + nd'_{\frac{1}{2}} + E''_0 d''_0 + E''_1 d''_1$$

and for Examples IV and IVa is:

$$f_n = f_0 + nd'_{\frac{1}{2}} + E''_0 M''_0 + E''_1 M''_1$$

in which

$$M''_0 = d''_0 - 0.184 d''''_0$$

$$M''_1 = d''_1 - 0184 d''''_1$$

EXACT VALUES OF COEFFICIENTS B''_{0+1} E''_0 AND E''_1

Dwight's Mathematical Tables (McGraw-Hill Publishing Company, 1941), pages 184 and 185, shows these by arguments of $n = .01$, from $n = 0$ to $n = 1.00$.

(a) In column headed pC_2 are values of $2\ B''_{0+1}$; that is to say,

actual values of B''_{0+1} for use in place of Chart readings are one-half of those shown in Dwight's Tables.

(b) In column headed pC_3 are values of E''_0 if signs are changed.

(c) The invert of column headed pC_3, shows values of E''_1 if signs are changed; that is to say, if there be added to Dwight's Tables a column of arguments which start $n = .99$ alongside of the tabular argument for $n = .01$, etc., the column pC_3 becomes a table of E''_1 with reference to the new argument column.

The following are calculated values of factors relating to the coefficients required for sub-tabulating to tenths:

n	B''_{0+1}	E''_0	E''_1
0.1	-.0225	-.0285	-.0165
0.2	-.04	-.048	-.032
0.3	-.0525	-.0595	-.0455
0.4	-.06	-.064	-.056
0.5	-.0625	-.0625	-.0625
0.6	-.06	-.056	-.064
0.7	-.0525	-.0455	-.0595
0.8	-.04	-.032	-.048
0.9	-.0225	-.0165	-.0285

A comprehensive set of tables of various types of coefficients, together with a practical discussion of all phases of interpolation by Central Difference methods, is found in:

"Interpolation and Allied Tables,"
reprinted from Nautical Almanac for
1937 with preface by L. J. Comrie

Published by: His Majesty's Stationery Office
Adastral House, Kingsway,
London W.C.2

This publication is recommended to all who have much of this work to do. It also discusses Inverse Curvilinear Interpolation using higher order differences.

DIRECT INTERPOLATION AND SUB-TABULATION (IF FOURTH DIFFERENCES DO NOT EXCEED 1000)

EXPLANATORY APPENDIX TO MARCHANT-METHOD MM-439E1

A NOTE ON OBTAINING 4TH DIFFERENCES FOR USE WITH "COMRIE THROW-BACK" IN EXAMPLE IV

Reference was made in the second paragraph of the "Remarks" section, Page 1, of Marchant Method MM-439E1, to the fact that in sub-tabulation it is not necessary to obtain third and fourth differences, except at infrequent intervals, and then only in order to obtain their general range as a guide to the selection of method or as a means of obtaining the 4th difference correction of Example IV.

Inasmuch as a 4th difference must be known before the "4th difference correction" can be determined, it might appear that the statement is inconsistent, because obviously 4th differences will normally vary somewhat from interval to interval.

Actually, however, in ordinary computing practice, it will be found that the 4th difference correction generally can be obtained without the necessity of completely tabulating the 3rd and 4th differences. This is because the large majority of functions which are tabulated to the number of places used in ordinary computing, - rarely more than 7 places - will have no great variation in 4th differences; that is to say, a small 5th difference.

By following the procedure below, the tabulation of 4th differences for every interval may usually be avoided.

A plan that does this is to obtain 4th differences at about every fifth or tenth interval and observe their trend, plotting them graphically and obtaining the 4th differences for the intermediate intervals from the curve so drawn.

This will ordinarily give quite as accurate a 4th difference as would actual differencing at each interval, because the graphical method eliminates the error forced into the 4th difference due to rounding up of the right-hand digit of the tabulated function. Such round-ups can affect the right-hand digit of the 4th difference by as much as 8.

In considering the precision of this approximation, it is to be noted that in the computation of the interpolates only "0.184 x 4th difference" is used. This is additional justification for the procedure of eliminating 4th differencing for every interval when the work falls within the class of Example IV.

CURVILINEAR INTERPOLATION BY LAGRANGEAN COEFFICIENTS

Example, with Table, Supplied by and Reproduced by courtesy of

George Rutledge, Ph.D., Professor of Mathematics
Massachusetts Institute of Technology

REMARKS *In the preparation of scientific and mathematical tables it is customary to make observations or calculations only sufficiently close together to clearly show the trend of the function. Interpolation for in-between points may then be done by curvilinear interpolation" so that the points found fall on a smooth curve connecting the known values. These interpolated values are then included in the table After issuance of the table, the ordinary user employs "straight line" interpolation for intermediate values (see Marchant Methods 439A1, 439B, 439C, 439A2, 439A3), though when great accuracy is desired the "curvilinear" method will be used.*

This method relates to "curvilinear interpolation" when given five equidistant points. Similar procedures enable interpolation to a closer degree when given 7 points, or up to 17, if desired. Further information is contained in the paper of Dr. Rutledge (with Prescott Crout), "Tables and Methods for Extending Tables for Interpolation Without Differences," Jour. of Math. and Phys., Vol. IX, No. 3, 1930.

Whereas the method herein refers to curvilinear interpolation without use of "differences" (see Marchant Method No. 419), the conclusion must not be drawn that interpolation by using differences is any less exact. Differencing has an advantage that it indicates errors in the tabulation from which the interpolation is made. Direct and Inverse Curvilinear Interpolation, using differences, is fully discussed in the works of Dr. L. H. Comrie, late Supt., H. M. Nautical Almanac Office and now Managing Director, Scientific Computing Service, Ltd., London. (List of references supplied upon application.)

EXAMPLE Let us assume that in preparation of a table of 7 place logarithms, calculations are made as follows:

For "n" equals	4.8	$\log_{10} n$ equals	0.68124 12
	4.9	" "	.69019 61
	5.0	" "	.69897 00
	5.1	" "	.70757 02
	5.2	" "	.71600 33
	5.3	" "	.72427 59

It is desired to interpolate for $\log_{10} n$ for "n" equals 5.04 (as one step in the preparation of a more detailed table).

NOTE As all values are known, the solution to a known result clearly exemplifies the method.

OUTLINE: By reference to the table of Rutledge Crout 5 point Lagrangean coefficients attached hereto, it will be observed that interpolation may be made forward from 5.0, with the interval (lambda) being 0.4, or it may be made backward from 5.1 with the interval being -0.6. The double computation is essential because it is only through agreement in the two results that any conclusion as to accuracy is justified.

The result desired is the algebraic sum of the products of the five logarithms and their respective "coefficients" as taken from the table for intervals 0.4 and -0.6 respectively. The point from which interpolation takes place is the median.

(over)

OPERATIONS: Decimals: Upper Dial 7; Middle Dial 14; Keyboard Dial 7.

INTERPOLATING FORWARD FROM 5.0

In this case the median is 5.0 and the 5 points are 4.8, 4.9, 5.0, 5.1 and 5.2.

(1) Set up in Keyboard Dial value of function for 4.8 (.68124 12) and multiply by coefficient K_{-2} for interval .4 (.0224000).

(2) Clear Upper and Keyboard Dials, set up in Keyboard Dial the value of function for 4.9 (.69019 61) and reverse multiply by coefficient K_{-1} (-.15360 00).

(3) Proceed as above for the following multiplications:

K_0 .69897 00 x .80640 00
K_1 .70757 02 x .35840 00
K_2 .71600 33 x -.03360 00 (multiply in reverse)

Middle Dial shows Log_{10} 5.04 equals 0.70243 05 plus.

(It may be noted that "straight-line" interpolation gives the erroneous value .70241 01)

INTERPOLATING BACKWARD FROM 5.1

In this case the median is 5.1, and the points are 4.9, 5.0, 5.1, 5.2 and 5.3.

As the interpolation is backward, the coefficients for lambda equal to -.6 as indicated by the bottom line of the table are used.

(1) Set up in Keyboard Dial value of function for 4.9 (.69019 61) for interval -.6 and reverse multiply by coefficient K_{-2} (-.04160 00).

(2) Clear Upper and Keyboard Dials, set up in Keyboard Dial the value of function for 5.0 (.69897 00) and multiply by coefficient K_{-1} (.58240 00).

(3) Proceed as above for the following multiplications:

K_0 .70757 02 x .58240 00
K_1 .71600 33 x -.14560 00 (multiply in reverse)
K_2 .72427 59 x .02240 00

Middle Dial shows Log_{10} 5.04 equals 0.70243 06 minus.

Agreement may be expected (on the basis of error formulas) in this range to 7 places with an uncertainty of 1 in the final place if 7 place data is used.* With 10 place data we may expect (on the basis of the same error formulas) agreement in the forward and backward interpolation to essentially 9 places. These error predictions vary according to the range or degree of curve between the five points.

NOTE: Using a similar 7 point interpolation method and data from 10 place British Association Tables of logarithms, the value for 5.04 is found by this method to be 0.70243 05364 plus, with agreement to 10 places in forward and backward interpolating.

(*) The Rutledge-Crout error formulas show that this error, when seven place data are used in this part of the table, cannot exceed .00000 007.

RUTLEDGE CROUT
FIVE POINT LAGRANGEAN COEFFICIENTS
(Exact Values)

Interval λ	K_{-2}	K_{-1}	K_0	K_1	K_2	Interval* λ
0.1	.00783 75	-.05985 00	.98752 50	.07315 00	-.00866 25	-0.1
.2	.01440 00	-.10560 00	.95040 00	.15840 00	-.01760 00	- .2
.3	.01933 75	-.13685 00	.88952 50	.25415 00	-.02616 25	- .3
.4	.02240 00	-.15360 00	.80640 00	.35840 00	-.03360 00	- .4
.5	.02343 75	-.15625 00	.70312 50	.46875 00	-.03906 25	- .5
.6	.02240 00	-.14560 00	.58240 00	.58240 00	-.04160 00	- .6
.7	.01933 75	-.12285 00	.44752 50	.69615 00	-.04016 25	- .7
.8	.01440 00	-.08960 00	.30240 00	.80640 00	-.03360 00	- .8
.9	.00783 75	-.04785 00	.15152 50	.90915 00	-.02066 25	- .9
λ	K_2	K_1	K_0	K_{-1}	K_{-2}	λ

*The values K_{-2}, K_{-1}, K_0, K_1, K_2 for negative values of λ are indicated by the bottom line of this table.

Thus for λ equals -.6, K_{-2} equals -.0416000, etc.

TABLE OF 5-POINT LAGRANGEAN INTERPOLATION COEFFICIENTS
(From 0 to 2, Argument 0.001, 7-Place)

An advance copy of a portion of a more extensive set of tables now in preparation

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PREFACE

The tables reproduced herein were computed at the request of the Chief of Ordnance, U. S. Army (O.O. 063.2/33, July 24, 1940, from the Chief of Ordnance to Dr. Arnold N. Lowan, Project Supervisor for Works Projects Administration for New York City) by the project for "Computation of Mathematical Tables," O.P. 65-2-97-33, conducted by the Works Projects Administration for New York City under the sponsorship of the National Bureau of Standards, Dr. Lyman J. Briggs, Director.

These 5-Point Values were originally issued in June, 1941 for private distribution in the War Department. They have just been released for public reference, but inasmuch as the volume of tables which is to include these 5-Point Values will not be ready for some time, permission has been obtained to reproduce them in this Advance Form.

From the Preface of the original privately circulated edition of these tables, signed by C. M. Wesson, Major General, Chief of Ordnance, we quote:

"Acknowledgment is made to Arnold N. Lowan, Ph. D., Murray Pfeferman, Milton Abramowitz, M.S.; Gertrude Blanch, Ph.D; Jack Laderman, M.A.; Jacob L. Miller, B.A.; Matilda M. Persily, B.A.; and Hyman Serbin, Ph.D., of the administrative and technical staff of the project, for their work in computing and editing this table. Further acknowledgment is made to Professor J.A. Shohat, Mathematics Department, University of Pennsylvania, and to Dr. L. W. Dederick of the Aberdeen Proving Ground, Maryland, for their many valuable suggestions."

Particular attention is directed to the "Introduction" written by Dr. Arnold N. Lowan, which appears on the following pages.

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BIBLIOGRAPHY OF TABLES

Truman L. Kelley. The Kelley Statistical Tables. New York, Macmillan Co. (1938). Lagrangean Interpolants. Cubic (four-point), [0(.001) 1;10D]; Quintic (six-point), [0(.01)1;10D] and Septic (eight-point) [0(.1)9;11D].

E. V. Huntington. Tables of Lagrangean Coefficients for Interpolating without Differences. Am. Acad. Arts & Sci., vol. 63 (1929) pp. 421-437. Cubic (four-point) interpolants. Tables 3 and 3a; [0(.01)1] for interpolating in main part of a table, and at the beginning or end, respectively. Quintic (six-point). Tables 5, 5a, 5b; [0(.01)1;8D] for interpolating in main part of a table in first interval, and in second interval of a table, respectively. Tables 30a, 30, 50a, 50b, 50 (cumulative): - short tables for continuous interpolation at intervals of 1/5, 2/5, 3/4, 4/5, 1.

G. Rutledge. Fundamental Table for Lagrangean Coefficients. Jour. Math. & Phys., vol. 8 (1929) pp. 1-12.

------ with P. Crout. Tables and Methods of Extending Tables for Interpolation without Differences, ibid, vol. 9 (1930) pp. 166-180. (Basic values for the computation of Lagrangean coefficients of various orders.)

Karl Pearson (editor). Tracts for Computers. No. II, Part I pp. 31-46. Cambridge Univ. Press, London (1920). Short tables of coefficients in the 4, 5, 6, 7, 8, 9, 10, 11-point Lagrangean formulae, with considerable expository material.

NOTE: The figures in the square brackets indicate the range, interval, and number of decimal places in the tabulated entries. For example: In Kelley's table, the cubic interpolants are given for n ranging from 0 to 1, at intervals of 0.001 to 10 decimal places.

INTRODUCTION

The general Lagrangean interpolation formula of degree (n-1) is expressible as follows:

$$(1)\quad f(x) = \frac{(x-a_2)(x-a_3)\ldots(x-a_n)}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)} f(a_1) + \frac{(x-a_1)(x-a_3)\ldots(x-a_n)}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)} f(a_2)$$

$$+ \ldots + \frac{(x-a_1)(x-a_2)\ldots(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1})} f(a_n) + R_n ,$$

$$(2)\quad R_n = \frac{(x-a_1)(x-a_2)\ldots(x-a_n)}{n!} f^{(n)}(y).$$

In the above expressions, a_1, a_2, ... a_n are n arguments or points for which the values of f(x) are known, and $f^{(n)}(y)$ is the nth derivative of f(x) at some point y in the interval enclosing the points a_1, a_2, ... a_n and x.

When f(x) is a polynomial of degree no higher than (n-1), R_n is identically zero. In the case where f(x) is a continuous function with n continuous derivatives, an approximate value of f(x) may be calculated from (1) by neglecting R_n. In that case, an upper bound of the error in the approximation is obtainable by replacing $f^{(n)}(y)$, in (2), by the upper bound of the numerical value of the nth derivative in the range under consideration. Formula (1) is sometimes called an "n-point" interpolation formula, because the value of f(x) is made to depend on n given values of the function.

The Lagrangean formula may also be used for interpolation in a table where the points a_1, a_2, ... a_n are uniformly spaced -- at intervals h, say. In that case formula (1) becomes considerably simplified. For the case when n = 5, let the five consecutive arguments be designated by a_{-2}, a_{-1}, a_0, a_1, a_2, and let

$$(3)\qquad x = a_0 + ph .$$

Replacing the above value of x in formulae (1) and (2), the following formulae are obtained when n = 5:

$$(4)\quad f(x) = \frac{p(p^2-1)(p-2)}{24} f(a_{-2}) - \frac{p(p-1)(p^2-4)}{6} f(a_{-1}) + \frac{(p^2-1)(p^2-4)}{4} f(a_0)$$

$$- \frac{p(p+1)(p^2-4)}{6} f(a_1) + \frac{p(p^2-1)(p+2)}{24} f(a_2) + R_5 ,$$

$$(4a)\qquad R_5 = \frac{p(p^2-1)(p^2-4)\, h^5 f^{(5)}(y)}{120}$$

In practice, the value of $f^{(5)}(y)$ is sometimes difficult to determine. However, when the interval h is relatively small, and the values of the function are known for several more adjacent (equally spaced) arguments, $h^5 f^{(5)}(y)$ may be estimated from the fifth differences of the entries in the region under consideration. Whenever R_5 is relatively small, these fifth differences will generally not vary appreciably among themselves. In that case, the fifth difference corresponding to the argument a_{-2}, say, may replace $h^5 f^5(y)$, if $f(a_3)$ is known. In other words

$$(4b)^*\quad R_5 \sim \frac{p(p^2-1)(p^2-4)}{120} [-f(a_{-2})+5f(a_{-1})-10f(a_0)+10f(a_1)-5f(a_2)+f(a_3)]$$

Formula (4), with R_5 omitted, is designated as the <u>five-point Lagrangean</u> interpolation formula, and the coefficients of $f(a_k)$, k = -2, -1, 0, 1, 2, are tabulated in this volume to seven decimals. The values of p range from 0 to 2 at intervals of 0.001. Specifically

$$A_{-2} = p(p^2-1)(p-2)/24 \; ; \; A_{-1} = -p(p-1)(p^2-4)/6 \; ;$$

$$A_0 = (p^2-1)(p^2-4)/4 \; ; \; A_1 = -p(p+1)(p^2-4)/6 \; ;$$

$$A_2 = p(p^2-1)(p+2)/24$$

It should be noted that for every value of p, $(A_{-2}+A_{-1}+A_0+A_1+A_2) = 1$ theoretically. Since the entries in this table are rounded numbers, the sum of the five coefficients may occasionally differ from unity by as much as two units in the last decimal place.

INTERPOLATION BY MEANS OF LAGRANGEAN COEFFICIENTS

The following examples will illustrate the application of formula (4). Let it be required to find x^8 corresponding to x = 4.3239 from a table listing x^8 at intervals of 0.1.

*The symbol $\sim$ will be used to denote approximate equality.

Solution. The work may be conveniently arranged as follows:

$p = .239$

a_k	$a_{-2} = 4.1$	$a_{-1} = 4.2$	$a_0 = 4.3$	$a_1 = 4.4$	$a_2 = 4.5$
$f(a_k)$	79849.3	96826.5	116882.0	140482.2	168151.3
$A_k(p)$	.0165349	-.1195211	.9294145	.1945949	-.0210231

$$(5)\quad f(x) = (79849.3)(.0165349) - (96826.5)(.1195211) + (116882.0)(.9294145) + (140482.2)(.1945949) - (168151.3)(.0210231) = 122181.4\ .$$

An upper bound of R_5 may be obtained either from (4a) or (4b). Using (4a), we have

$$h^5 f^{(5)}(y) < h^5 f^{(5)}(4.5) < 6.2;\ \frac{p(p^2-1)(p^2-4)}{120} < 0.0075$$

hence $\qquad 0 < R_5 < (6.2)(0.0075) < 0.05$

When R_5 is estimated from (4b), $h^5 f^{(5)}(y)$ may be replaced by the fifth difference of the entries in the region under consideration; the value of $f(x)$ for another point in the region - say $f(a_3)$ - is therefore needed. From a table of x^8, it is found that $f(4.6) = 200476.1$, and it may be verified that the fifth differences in this region are less than 6.4; hence the upper bound of R_5 previously obtained is verified from (4b). By direct computation, $(4.3239)^8 = 122181.41$. The computed value is therefore correct in this instance to seven significant figures. Rounding errors inherent in the terms of (5) may sometimes occasion an error of somewhat more than a unit in the seventh significant figure of the computed value of $f(x)$, even when R_5 is negligible; but this rounding error will generally be less than two units in the seventh significant figure of $f(x)$.

Sometimes it is required to interpolate between the last two entries of a table, or between the first two entries. In that case, interpolation must be made to depend on entries all of which are on one side of a_0 and a_1. The problem of interpolating near the end of a table may be easily solved. For example, in the previous problem, let it be assumed that the table ends with $x = 4.4$. The points on which interpolation is made to depend are now as follows: $a_{-2} = 4.0$; $a_{-1} = 4.1$; $a_0 = 4.2$; $a_1 = 4.3$; $a_2 = 4.4$. As before $x = a_0 + ph = 4.2 + 0.1239$ hence $ph = 0.1239$ and $p = 1.239$. Since the coefficients corresponding to the above value of p are tabulated, the problem may be solved by the same method as before.

The problem of interpolating near the beginning of a table remains to be considered. In the former example, let it be assumed that $f(x)$ is known for the arguments 4.3, 4.4, 4.5, 4.6 and 4.7. If the points are taken in the above order, $a_0 = 4.5$ and $x = a_0 - .1761$; hence $p = -1.761$. It may be readily verified that $A_k(-p) = A_{-k}(p)$; that is, the coefficients A_{-2} and A_2 as well as A_{-1} and A_1 must be interchanged. One way of accomplishing this is to reverse the order of the points as follows:

$a_{-2} = 4.7$, $a_{-1} = 4.6$, $a_0 = 4.5$, $a_1 = 4.4$, $a_2 = 4.3$, $x = a_0 - .1761$, $ph = -.1761$ and since h is negative, $p = 1.761$. Hence the coefficients A_k may be found from this table. The actual work follows:

$p = 1.761$

a_k	$a_{-2} = 4.7$	$a_{-1} = 4.6$	$a_0 = 4.5$	$a_1 = 4.4$	$a_2 = 4.3$
$f(a_k)$	238112.9	200476.1	168151.3	140482.2	116882.0
A_k	-.0368466	.2007678	-.4721634	.7284097	.5798324

$$f(x) = -(238112.9)(.0368466) + (200476.1)(.2007678) - (168151.3)(.4721634) + (140482.2)(.7284097) + (116882.0)(.5798324) = 122181.2\ .$$

The last place is incorrect by two units. This error is partly due to the higher value of R_5, arising from the larger value of p.

If p in formula (3) is not a tabulated argument of this table, interpolation for $f(x)$ may be performed by one of the following two methods:

a. Interpolation for coefficients. Let F_{-1}, F_0, F_1 be the values of any one of the five interpolants, corresponding to the arguments $p_0 - h$, p_0, and $p_0 + h$. Let, further, F_p designate the value of the interpolant for the argument $p = p_0 + th$. The value of F_p may be determined as follows:

$$(6)\qquad F_p = F_0 + t(F_1 - F_0) + \tfrac{1}{2}t(t-1)\,[F_{-1} - 2F_0 + F_1]\ .$$

If the term involving $\frac{1}{2}t(t-1)$ is neglected, the value of F_p will generally be correct to within two units in the sixth decimal place.

Example: Let it be required to find the coefficients A_k corresponding to $p = .2394762$.

Solution: $p_0 = .239$, $p_0 + h = .240$, $p_0 - h = .238$; $t = .4762$; $\frac{1}{2}t(t-1) =$

-.137. With the aid of the tables of A_k, the calculations may be conveniently arranged as follows:

$$p = .2394762$$

		A_{-2}	A_{-1}	A_0	A_1	A_2
$p = .238$	$F_{-1} =$	.0164834	-.1191919	.9299971	.1936477	-.0209364
$p = .239$	$F_0 =$	.0165349	-.1195211	.9294145	.1945949	-.0210231
$p = .240$	$F_1 =$	.0165862	-.1198490	.9288294	.1955430	-.0211098
$C = F_0 - F_{-1}$	$=$	.0000515	-.0003292	-.0005826	.0009472	-.0000867
$D = F_1 - F_0$	$=$	.0000513	-.0003279	-.0005851	.0009481	-.0000867
$D-C$	$=$		.0000013	-.0000025	.0000009	
$t\,D$	$=$	.00002442	-.00015614	-.00027862	.00045149	-.00004129
$\frac{1}{2}t(t-1)(D-C) = E$	$=$		-.00000018	+.00000034	-.00000012	
$F + tD + E = A_k(p)$	$=$	.0165593	-.1196774	.9291362	.1950463	-.0210644

As a check on the accuracy of the computations, it may be verified that the sum of the coefficients $A_k(p)$ is equal to unity. The value of the function may then be obtained by the method already illustrated.

b. Alternative method. In the given example, the values of $f(x)$ may be found for $p = 0.239$, 0.240, and 0.241 by the usual method. The results are as follows:

x	f(x)	diff.
4.3239	122181.4	
		22.5
4.3240	122203.9	
		22.7
4.3241	122226.6	

Since the first differences are almost constant, linear interpolation for $f(x)$ in the above values is adequate. The calculation follows:

$$f(4.32394762) = 122181.4 + .4762(22.5) = 122192.1\,,$$

If the first differences are appreciably different, a closer approximation to $f(x)$ may be obtained with the aid of formula (6), where now the F's refer to the function $f(x)$ under consideration.

ADVANTAGES OF THE LAGRANGEAN FORMULA

When adequate tables of Lagrangean interpolants and a calculating machine are at hand, interpolation by means of the Lagrangean formula is simple and expedient. The partial products $A_k f(a_k)$ can be accumulated in the machine and the final result obtained with a minimum of subsidiary hand computations. This method is particularly advantageous when interpolation is performed in a table which does not list differences. Several good tables of four-point interpolants are already available* and it is hoped the present tables will give the Lagrangean formulae the greater prominence they deserve.

ARNOLD N. LOWAN

*See Bibliography

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.000	+0.0000 000	-0.0000 000	+1.0000 000	+0.0000 000	-0.0000 000
01	.0000 833	.0006 660	+0.9999 988	.0006 673	.0000 834
02	.0001 665	.0013 307	.9999 950	.0013 360	.0001 668
03	.0002 496	.0019 940	.9999 888	.0020 060	.0002 504
04	.0003 327	.0026 560	.9999 800	.0026 773	.0003 340
0.005	+0.0004 156	-0.0033 166	+0.9999 688	+0.0033 500	-0.0004 177
06	.0004 985	.0039 760	.9999 550	.0040 240	.0005 015
07	.0005 813	.0046 339	.9999 388	.0046 993	.0005 853
08	.0006 640	.0052 906	.9999 200	.0053 759	.0006 693
09	.0007 466	.0059 459	.9998 988	.0060 539	.0007 533
0.010	+0.0008 291	-0.0065 998	+0.9998 750	+0.0067 332	-0.0008 374
11	.0009 115	.0072 524	.9998 488	.0074 138	.0009 216
12	.0009 939	.0079 037	.9998 200	.0080 957	.0010 059
13	.0010 761	.0085 536	.9997 888	.0087 790	.0010 902
14	.0011 583	.0092 022	.9997 550	.0094 635	.0011 746
0.015	+0.0012 403	-0.0098 494	+0.9997 188	+0.0101 494	-0.0012 591
16	.0013 223	.0104 953	.9996 800	.0108 366	.0013 437
17	.0014 042	.0111 399	.9996 388	.0115 252	.0014 283
18	.0014 860	.0117 830	.9995 950	.0122 150	.0015 130
19	.0015 677	.0124 249	.9995 488	.0129 062	.0015 978
0.020	+0.0016 493	-0.0130 654	+0.9995 000	+0.0135 986	-0.0016 827
21	.0017 309	.0137 045	.9994 488	.0142 924	.0017 676
22	.0018 123	.0143 423	.9993 951	.0149 875	.0018 526
23	.0018 936	.0149 787	.9993 388	.0156 839	.0019 377
24	.0019 749	.0156 138	.9992 801	.0163 816	.0020 228
0.025	+0.0020 560	-0.0162 475	+0.9992 188	+0.0170 807	-0.0021 081
26	.0021 371	.0168 798	.9991 551	.0177 810	.0021 933
27	.0022 180	.0175 108	.9990 889	.0184 826	.0022 787
28	.0022 989	.0181 404	.9990 202	.0191 856	.0023 641
29	.0023 796	.0187 687	.9989 489	.0198 898	.0024 496
0.030	+0.0024 603	-0.0193 956	+0.9988 752	+0.0205 954	-0.0025 352
31	.0025 408	.0200 212	.9987 990	.0213 022	.0026 209
32	.0026 213	.0206 454	.9987 203	.0220 104	.0027 066
33	.0027 017	.0212 682	.9986 390	.0227 198	.0027 923
34	.0027 819	.0218 897	.9985 553	.0234 306	.0028 782
0.035	+0.0028 621	-0.0225 098	+0.9984 691	+0.0241 426	-0.0029 641
36	.0029 422	.0231 285	.9983 804	.0248 559	.0030 500
37	.0030 221	.0237 459	.9982 892	.0255 706	.0031 361
38	.0031 020	.0243 619	.9981 955	.0262 865	.0032 222
39	.0031 818	.0249 765	.9980 993	.0270 037	.0033 083
0.040	+0.0032 614	-0.0255 898	+0.9980 006	+0.0277 222	-0.0033 946
41	.0033 410	.0262 017	.9978 995	.0284 420	.0034 808
42	.0034 205	.0268 122	.9977 958	.0291 631	.0035 672
43	.0034 998	.0274 213	.9976 896	.0298 855	.0036 536
44	.0035 791	.0280 291	.9975 809	.0306 092	.0037 401
0.045	+0.0036 582	-0.0286 355	+0.9974 698	+0.0313 341	-0.0038 266
46	.0037 372	.0292 405	.9973 561	.0320 604	.0039 132
47	.0038 162	.0298 442	.9972 400	.0327 879	.0039 999
48	.0038 950	.0304 465	.9971 213	.0335 167	.0040 866
49	.0039 737	.0310 474	.9970 002	.0342 468	.0041 733
0.050	+0.0040 523	-0.0316 469	+0.9968 766	+0.0349 781	-0.0042 602

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.050	+0.0040 523	-0.0316 469	+0.9968 766	+0.0349 781	-0.0042 602
51	.0041 309	.0322 450	.9967 504	.0357 108	.0043 470
52	.0042 093	.0328 418	.9966 218	.0364 447	.0044 340
53	.0042 875	.0334 372	.9964 907	.0371 799	.0045 210
54	.0043 657	.0340 312	.9963 571	.0379 163	.0046 080
0.055	+0.0044 438	-0.0346 238	+0.9962 210	+0.0386 541	-0.0046 951
56	.0045 218	.0352 150	.9960 825	.0393 931	.0047 823
57	.0045 996	.0358 049	.9959 414	.0401 334	.0048 695
58	.0046 774	.0363 934	.9957 978	.0408 749	.0049 568
59	.0047 550	.0369 805	.9956 518	.0416 178	.0050 441
0.060	+0.0048 325	-0.0375 662	+0.9955 032	+0.0423 618	-0.0051 315
61	.0049 100	.0381 505	.9953 522	.0431 072	.0052 189
62	.0049 873	.0387 334	.9951 987	.0438 538	.0053 064
63	.0050 644	.0393 150	.9950 427	.0446 017	.0053 939
64	.0051 415	.0398 951	.9948 842	.0453 508	.0054 815
0.065	+0.0052 185	-0.0404 739	+0.9947 232	+0.0461 013	-0.0055 691
66	.0052 953	.0410 512	.9945 597	.0468 529	.0056 568
67	.0053 721	.0416 272	.9943 938	.0476 058	.0057 445
68	.0054 487	.0422 018	.9942 253	.0483 600	.0058 322
69	.0055 252	.0427 750	.9940 544	.0491 155	.0059 201
0.070	+0.0056 016	-0.0433 468	+0.9938 810	+0.0498 722	-0.0060 079
71	.0056 779	.0439 173	.9937 051	.0506 301	.0060 958
72	.0057 540	.0444 863	.9935 267	.0513 893	.0061 838
73	.0058 301	.0450 539	.9933 458	.0521 498	.0062 718
74	.0059 060	.0456 201	.9931 625	.0529 115	.0063 598
0.075	+0.0059 818	-0.0461 850	+0.9929 767	+0.0536 744	-0.0064 479
76	.0060 575	.0467 484	.9927 883	.0544 386	.0065 360
77	.0061 330	.0473 104	.9925 975	.0552 041	.0066 242
78	.0062 085	.0478 711	.9924 043	.0559 707	.0067 124
79	.0062 838	.0484 303	.9922 085	.0567 387	.0068 007
0.080	+0.0063 590	-0.0489 882	+0.9920 102	+0.0575 078	-0.0068 890
81	.0064 341	.0495 446	.9918 095	.0582 783	.0069 773
82	.0065 091	.0500 996	.9916 063	.0590 499	.0070 657
83	.0065 840	.0506 533	.9914 006	.0598 228	.0071 541
84	.0066 587	.0512 055	.9911 924	.0605 969	.0072 425
0.085	+0.0067 333	-0.0517 563	+0.9909 818	+0.0613 723	-0.0073 310
86	.0068 078	.0523 058	.9907 687	.0621 489	.0074 195
87	.0068 821	.0528 538	.9905 531	.0629 267	.0075 081
88	.0069 564	.0534 004	.9903 350	.0637 058	.0075 967
89	.0070 305	.0539 456	.9901 144	.0644 860	.0076 853
0.090	+0.0071 045	-0.0544 894	+0.9898 914	+0.0652 676	-0.0077 740
91	.0071 784	.0550 318	.9896 659	.0660 503	.0078 627
92	.0072 521	.0555 728	.9894 379	.0668 343	.0079 515
93	.0073 257	.0561 124	.9892 075	.0676 195	.0080 402
94	.0073 992	.0566 506	.9889 745	.0684 059	.0081 290
0.095	+0.0074 726	-0.0571 873	+0.9887 391	+0.0691 935	-0.0082 179
96	.0075 458	.0577 227	.9885 012	.0699 824	.0083 067
97	.0076 189	.0582 566	.9882 609	.0707 725	.0083 956
98	.0076 919	.0587 892	.9880 181	.0715 638	.0084 846
99	.0077 648	.0593 203	.9877 728	.0723 563	.0085 735
0.100	+0.0078 375	-0.0598 500	+0.9875 250	+0.0731 500	-0.0086 625

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.100	+0.0078 375	-0.0598 500	+0.9875 250	+0.0731 500	-0.0086 625
01	.0079 101	.0603 783	.9872 748	.0739 449	.0087 515
02	.0079 826	.0609 052	.9870 221	.0747 411	.0088 406
03	.0080 549	.0614 306	.9867 669	.0755 385	.0089 296
04	.0081 271	.0619 547	.9865 092	.0763 370	.0090 187
0.105	+0.0081 992	-0.0624 773	+0.9862 491	+0.0771 368	-0.0091 078
06	.0082 712	.0629 985	.9859 866	.0779 378	.0091 970
07	.0083 430	.0635 183	.9857 215	.0787 400	.0092 862
08	.0084 147	.0640 367	.9854 540	.0795 434	.0093 754
09	.0084 863	.0645 537	.9851 840	.0803 480	.0094 646
0.110	+0.0085 577	-0.0650 692	+0.9849 116	+0.0811 538	-0.0095 538
11	.0086 290	.0655 834	.9846 367	.0819 608	.0096 431
12	.0087 001	.0660 961	.9843 593	.0827 690	.0097 324
13	.0087 712	.0666 074	.9840 795	.0835 783	.0098 217
14	.0088 421	.0671 172	.9837 972	.0843 889	.0099 110
0.115	+0.0089 128	-0.0676 257	+0.9835 125	+0.0852 007	-0.0100 003
16	.0089 835	.0681 327	.9832 253	.0860 137	.0100 897
17	.0090 540	.0686 383	.9829 356	.0868 278	.0101 791
18	.0091 243	.0691 425	.9826 435	.0876 432	.0102 685
19	.0091 946	.0696 452	.9823 489	.0884 597	.0103 579
0.120	+0.0092 646	-0.0701 466	+0.9820 518	+0.0892 774	-0.0104 474
21	.0093 346	.0706 465	.9817 523	.0900 963	.0105 368
22	.0094 044	.0711 449	.9814 504	.0909 164	.0106 263
23	.0094 741	.0716 420	.9811 460	.0917 377	.0107 158
24	.0095 436	.0721 376	.9808 391	.0925 602	.0108 053
0.125	+0.0096 130	-0.0726 318	+0.9805 298	+0.0933 838	-0.0108 948
26	.0096 823	.0731 246	.9802 180	.0942 086	.0109 843
27	.0097 514	.0736 160	.9799 038	.0950 346	.0110 738
28	.0098 204	.0741 059	.9795 871	.0958 617	.0111 634
29	.0098 893	.0745 944	.9792 680	.0966 901	.0112 529
0.130	+0.0099 580	-0.0750 814	+0.9789 464	+0.0975 196	-0.0113 425
31	.0100 266	.0755 671	.9786 224	+0983 502	.0114 321
32	.0100 950	.0760 513	.9782 959	.0991 821	.0115 217
33	.0101 633	.0765 340	.9779 670	.1000 151	.0116 113
34	.0102 314	.0770 154	.9776 356	.1008 492	.0117 009
0.135	+0.0102 994	-0.0774 953	+0.9773 018	+0.1016 846	-0.0117 905
36	.0103 673	.0779 738	.9769 655	.1025 211	.0118 801
37	.0104 350	.0784 508	.9766 268	.1033 587	.0119 698
38	.0105 026	.0789 264	.9762 857	.1041 975	.0120 594
39	.0105 700	.0794 006	.9759 421	.1050 375	.0121 490
0.140	+0.0106 373	-0.0798 734	+0.9755 960	+0.1058 786	-0.0122 387
41	.0107 045	.0803 447	.9752 476	.1067 209	.0123 283
42	.0107 715	.0808 145	.9748 966	.1075 644	.0124 180
43	.0108 384	.0812 830	.9745 433	.1084 089	.0125 076
44	.0109 051	.0817 500	.9741 875	.1092 547	.0125 973
0.145	+0.0109 717	-0.0822 156	+0.9738 293	+0.1101 016	-0.0126 869
46	.0110 381	.0826 797	.9734 686	.1109 496	.0127 766
47	.0111 044	.0831 424	.9731 055	.1117 988	.0128 662
48	.0111 705	.0836 037	.9727 399	.1126 491	.0129 559
49	.0112 365	.0840 635	.9723 720	.1135 005	.0130 455
0.150	+0.0113 023	-0.0845 219	+0.9720 016	+0.1143 531	-0.0131 352

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.150	+0.0113 023	-0.0845 219	+0.9720 016	+0.1143 531	-0.0131 352
51	.0113 680	.0849 788	.9716 287	.1152 069	.0132 248
52	.0114 336	.0854 343	.9712 534	.1160 617	.0133 144
53	.0114 990	.0858 884	.9708 757	.1169 177	.0134 041
54	.0115 642	.0863 410	.9704 956	.1177 749	.0134 937
0.155	+0.0116 294	-0.0867 922	+0.9701 131	+0.1186 332	-0.0135 833
56	.0116 943	.0872 420	.9697 281	.1194 926	.0136 730
57	.0117 591	.0876 903	.9693 406	.1203 531	.0137 626
58	.0118 238	.0881 371	.9689 508	.1212 147	.0138 522
59	.0118 883	.0885 826	.9685 585	.1220 775	.0139 418
0.160	+0.0119 526	-0.0890 266	+0.9681 638	+0.1229 414	-0.0140 314
61	.0120 168	.0894 691	.9677 667	.1238 065	.0141 209
62	.0120 809	.0899 102	.9673 672	.1246 726	.0142 105
63	.0121 448	.0903 499	.9669 652	.1255 399	.0143 001
64	.0122 086	.0907 881	.9665 608	.1264 083	.0143 896
0.165	+0.0122 722	-0.0912 248	+0.9661 541	+0.1272 778	-0.0144 791
66	.0123 356	.0916 602	.9657 448	.1281 484	.0145 687
67	.0123 989	.0920 941	.9653 332	.1290 201	.0146 582
68	.0124 621	.0925 265	.9649 191	.1298 930	.0147 477
69	.0125 250	.0929 575	.9645 027	.1307 669	.0148 372
0.170	+0.0125 879	-0.0933 870	+0.9640 838	+0.1316 420	-0.0149 266
71	.0126 506	.0938 151	.9636 625	.1325 181	.0150 161
72	.0127 131	.0942 418	.9632 388	.1333 954	.0151 055
73	.0127 755	.0946 670	.9628 127	.1342 738	.0151 949
74	.0128 377	.0950 908	.9623 842	.1351 532	.0152 843
0.175	+0.0128 998	-0.0955 131	+0.9619 532	+0.1360 338	-0.0153 737
76	.0129 617	.0959 340	.9615 199	.1369 155	.0154 630
77	.0130 234	.0963 534	.9610 841	.1377 982	.0155 524
78	.0130 850	.0967 714	.9606 460	.1386 821	.0156 417
79	.0131 465	.0971 879	.9602 054	.1395 670	.0157 310
0.180	+0.0132 077	-0.0976 030	+0.9597 624	+0.1404 530	-0.0158 203
81	.0132 689	.0980 166	.9593 171	.1413 402	.0159 095
82	.0133 298	.0984 288	.9588 693	.1422 284	.0159 987
83	.0133 906	.0988 395	.9584 191	.1431 177	.0160 879
84	.0134 513	.0992 488	.9579 666	.1440 080	.0161 771
0.185	+0.0135 118	-0.0996 566	+0.9575 116	+0.1448 995	-0.0162 663
86	.0135 721	.1000 630	.9570 542	.1457 920	.0163 554
87	.0136 323	.1004 679	.9565 945	.1466 857	.0164 445
88	.0136 923	.1008 714	.9561 323	.1475 804	.0165 336
89	.0137 522	.1012 735	.9556 677	.1484 761	.0166 226
0.190	+0.0138 119	-0.1016 740	+0.9552 008	+0.1493 730	-0.0167 116
91	.0138 714	.1020 732	.9547 315	.1502 709	.0168 006
92	.0139 308	.1024 708	.9542 597	.1511 699	.0168 896
93	.0139 900	.1028 671	.9537 856	.1520 699	.0169 785
94	.0140 491	.1032 618	.9533 091	.1529 710	.0170 674
0.195	+0.0141 080	-0.1036 552	+0.9528 302	+0.1538 732	-0.0171 562
96	.0141 667	.1040 470	.9523 489	.1547 764	.0172 450
97	.0142 253	.1044 375	.9518 653	.1556 807	.0173 338
98	.0142 837	.1048 264	.9513 792	.1565 861	.0174 226
99	.0143 419	.1052 139	.9508 908	.1574 925	.0175 113
0.200	+0.0144 000	-0.1056 000	+0.9504 000	+0.1584 000	-0.0176 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.200	+0.0144 000	-0.1056 000	+0.9504 000	+0.1584 000	-0.0176 000
01	.0144 579	.1059 846	.9499 068	.1593 085	.0176 886
02	.0145 157	.1063 678	.9494 112	.1602 181	.0177 773
03	.0145 733	.1067 495	.9489 133	.1611 287	.0178 658
04	.0146 307	.1071 297	.9484 130	.1620 404	.0179 544
0.205	+0.0146 880	-0.1075 085	+0.9479 103	+0.1629 531	-0.0180 429
06	.0147 450	.1078 858	.9474 052	.1638 669	.0181 313
07	.0148 020	.1082 617	.9468 978	.1647 817	.0182 197
08	.0148 587	.1086 361	.9463 879	.1656 976	.0183 081
09	.0149 153	.1090 091	.9458 758	.1666 144	.0183 964
0.210	+0.0149 718	-0.1093 806	+0.9453 612	+0.1675 324	-0.0184 847
11	.0150 281	.1097 507	.9448 443	.1684 513	.0185 730
12	.0150 842	.1101 193	.9443 250	.1693 713	.0186 612
13	.0151 401	.1104 865	.9438 033	.1702 923	.0187 493
14	.0151 959	.1108 522	.9432 793	.1712 144	.0188 374
0.215	+0.0152 515	-0.1112 164	+0.9427 529	+0.1721 375	-0.0189 255
16	.0153 069	.1115 792	.9422 242	.1730 616	.0190 135
17	.0153 622	.1119 405	.9416 931	.1739 867	.0191 015
18	.0154 173	.1123 004	.9411 596	.1749 129	.0191 894
19	.0154 722	.1126 588	.9406 238	.1758 400	.0192 772
0.220	+0.0155 269	-0.1130 158	+0.9400 856	+0.1767 682	-0.0193 651
21	.0155 815	.1133 713	.9395 451	.1776 974	.0194 528
22	.0156 360	.1137 253	.9390 022	.1786 277	.0195 405
23	.0156 902	.1140 779	.9384 570	.1795 589	.0196 282
24	.0157 443	.1144 290	.9379 094	.1804 912	.0197 158
0.225	+0.0157 982	-0.1147 787	+0.9373 595	+0.1814 244	-0.0198 034
26	.0158 519	.1151 269	.9368 072	.1823 587	.0198 909
27	.0159 055	.1154 737	.9362 526	.1832 939	.0199 783
28	.0159 589	.1158 190	.9356 956	.1842 302	.0200 657
29	.0160 121	.1161 628	.9351 363	.1851 675	.0201 530
0.230	+0.0160 652	-0.1165 052	+0.9345 746	+0.1861 058	-0.0202 403
31	.0161 181	.1168 462	.9340 106	.1870 450	.0203 275
32	.0161 708	.1171 856	.9334 443	.1879 853	.0204 147
33	.0162 233	.1175 237	.9328 756	.1889 266	.0205 018
34	.0162 757	.1178 602	.9323 046	.1898 688	.0205 888
0.235	+0.0163 279	-0.1181 953	+0.9317 312	+0.1908 121	-0.0206 758
36	.0163 799	.1185 290	.9311 555	.1917 563	.0207 627
37	.0164 317	.1188 612	.9305 775	.1927 015	.0208 496
38	.0164 834	.1191 919	.9299 971	.1936 477	.0209 364
39	.0165 349	.1195 211	.9294 145	.1945 949	.0210 231
0.240	+0.0165 862	-0.1198 490	+0.9288 294	+0.1955 430	-0.0211 098
41	.0166 374	.1201 753	.9282 421	.1964 922	.0211 964
42	.0166 884	.1205 002	.9276 524	.1974 423	.0212 829
43	.0167 392	.1208 236	.9270 604	.1983 934	.0213 694
44	.0167 898	.1211 456	.9264 661	.1993 454	.0214 557
0.245	+0.0168 402	-0.1214 661	+0.9258 695	+0.2002 985	-0.0215 421
46	.0168 905	.1217 852	.9252 705	.2012 525	.0216 283
47	.0169 406	.1221 028	.9246 693	.2022 074	.0217 145
48	.0169 905	.1224 190	.9240 657	.2031 634	.0218 006
49	.0170 403	.1227 336	.9234 598	.2041 203	.0218 867
0.250	+0.0170 898	-0.1230 469	+0.9228 516	+0.2050 781	-0.0219 727

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.250	+0.0170 898	-0.1230 469	+0.9228 516	+0.2050 781	-0.0219 727
51	.0171 392	.1233 586	.9222 410	.2060 369	.0220 586
52	.0171 884	.1236 690	.9216 282	.2069 967	.0221 444
53	.0172 375	.1239 778	.9210 130	.2079 574	.0222 301
54	.0172 863	.1242 852	.9203 956	.2089 191	.0223 158
0.255	+0.0173 350	-0.1245 911	+0.9197 758	+0.2098 817	-0.0224 014
56	.0173 835	.1248 956	.9191 537	.2108 453	.0224 869
57	.0174 318	.1251 986	.9185 294	.2118 098	.0225 724
58	.0174 800	.1255 002	.9179 027	.2127 753	.0226 578
59	.0175 280	.1258 003	.9172 737	.2137 417	.0227 430
0.260	+0.0175 757	-0.1260 990	+0.9166 424	+0.2147 090	-0.0228 283
61	.0176 233	.1263 961	.9160 089	.2156 773	.0229 134
62	.0176 708	.1266 919	.9153 730	.2166 465	.0229 984
63	.0177 180	.1269 862	.9147 348	.2176 167	.0230 834
64	.0177 651	.1272 790	.9140 944	.2185 878	.0231 683
0.265	+0.0178 120	-0.1275 703	+0.9134 516	+0.2195 598	-0.0232 531
66	.0178 587	.1278 602	.9128 066	.2205 327	.0233 378
67	.0179 052	.1281 487	.9121 593	.2215 066	.0234 224
68	.0179 515	.1284 356	.9115 097	.2224 814	.0235 070
69	.0179 977	.1287 212	.9108 578	.2234 571	.0235 914
0.270	+0.0180 437	-0.1290 052	+0.9102 036	+0.2244 338	-0.0236 758
71	.0180 895	.1292 878	.9095 471	.2254 113	.0237 601
72	.0181 351	.1295 690	.9088 884	.2263 898	.0238 443
73	.0181 805	.1298 487	.9082 274	.2273 692	.0239 284
74	.0182 258	.1301 269	.9075 641	.2283 495	.0240 124
0.275	+0.0182 708	-0.1304 037	+0.9068 985	+0.2293 307	-0.0240 963
76	.0183 157	.1306 790	.9062 307	.2303 128	.0241 802
77	.0183 604	.1309 529	.9055 606	.2312 958	.0242 639
78	.0184 050	.1312 253	.9048 882	.2322 797	.0243 476
79	.0184 493	.1314 963	.9042 136	.2332 645	.0244 311
0.280	+0.0184 934	-0.1317 658	+0.9035 366	+0.2342 502	-0.0245 146
81	.0185 374	.1320 338	.9028 575	.2352 369	.0245 979
82	.0185 812	.1323 004	.9021 760	.2362 244	.0246 812
83	.0186 248	.1325 655	.9014 923	.2372 128	.0247 643
84	0186 682	.1328 292	.9008 063	.2382 021	.0248 474
0.285	+0.0187 114	-0.1330 914	+0.9001 181	+0.2391 922	-0.0249 304
86	.0187 545	.1333 522	.8994 276	.2401 833	.0250 133
87	.0187 973	.1336 115	.8987 349	.2411 752	.0250 960
88	.0188 400	.1338 693	.8980 399	.2421 681	.0251 787
89	.0188 825	.1341 257	.8973 427	.2431 618	.0252 613
0.290	+0.0189 248	-0.1343 806	+0.8966 432	+0.2441 564	-0.0253 437
91	.0189 669	.1346 341	.8959 415	.2451 518	.0254 261
92	.0190 088	.1348 861	.8952 375	.2461 482	.0255 083
93	.0190 506	.1351 367	.8945 313	.2471 454	.0255 905
94	.0190 921	.1353 858	.8938 228	.2481 434	.0256 725
0.295	+0.0191 335	-0.1356 335	+0.8931 121	+0.2491 424	-0.0257 545
96	.0191 747	.1358 797	.8923 991	.2501 422	.0258 363
97	.0192 157	.1361 245	.8916 840	.2511 428	.0259 180
98	.0192 565	.1363 678	.8909 665	.2521 444	.0259 996
99	.0192 971	.1366 096	.8902 469	.2531 468	.0260 811
0.300	+0.0193 375	-0.1368 500	+0.8895 250	+0.2541 500	-0.0261 625

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.300	+0.0193 375	-0.1368 500	+0.8895 250	+0.2541 500	-0.0261 625
01	.0193 777	.1370 889	.8888 009	.2551 541	.0262 438
02	.0194 178	.1373 264	.8880 745	.2561 590	.0263 249
03	.0194 577	.1375 625	.8873 460	.2571 648	.0264 060
04	.0194 973	.1377 970	.8866 152	.2581 715	.0264 869
0.305	+0.0195 368	-0.1380 302	+0.8858 822	+0.2591 790	-0.0265 678
06	.0195 761	.1382 618	.8851 469	.2601 873	.0266 485
07	.0196 152	.1384 921	.8844 095	.2611 964	.0267 291
08	.0196 541	.1387 208	.8836 698	.2622 065	.0268 095
09	.0196 928	.1389 482	.8829 279	.2632 173	.0268 899
0.310	+0.0197 314	-0.1391 740	+0.8821 838	+0.2642 290	-0.0269 701
11	.0197 697	.1393 985	.8814 375	.2652 415	.0270 502
12	.0198 079	.1396 214	.8806 890	.2662 548	.0271 302
13	.0198 458	.1398 429	.8799 382	.2672 690	.0272 101
14	.0198 836	.1400 630	.8791 853	.2682 839	.0272 899
0.315	+0.0199 212	-0.1402 816	+0.8784 302	+0.2692 998	-0.0273 695
16	.0199 586	.1404 988	.8776 728	.2703 164	.0274 490
17	.0199 958	.1407 145	.8769 133	.2713 338	.0275 284
18	.0200 328	.1409 288	.8761 515	.2723 521	.0276 076
19	.0200 696	.1411 416	.8753 876	.2733 712	.0276 868
0.320	+0.0201 062	-0.1413 530	+0.8746 214	+0.2743 910	-0.0277 658
21	.0201 427	.1415 629	.8738 531	.2754 117	.0278 446
22	.0201 789	.1417 714	.8730 826	.2764 332	.0279 234
23	.0202 150	.1419 784	.8723 099	.2774 555	.0280 020
24	.0202 508	.1421 840	.8715 350	.2784 786	.0280 805
0.325	+0.0202 865	-0.1423 881	+0.8707 579	+0.2795 025	-0.0281 588
26	.0203 219	.1425 908	.8699 786	.2805 272	.0282 371
27	.0203 572	.1427 920	.8691 972	.2815 527	.0283 152
28	.0203 923	.1429 918	.8684 136	.2825 790	.0283 931
29	.0204 272	.1431 901	.8676 278	.2836 061	.0284 709
0.330	+0.0204 619	-0.1433 870	+0.8668 398	+0.2846 340	-0.0285 486
31	.0204 964	.1435 825	.8660 497	.2856 626	.0286 262
32	.0205 307	.1437 765	.8652 573	.2866 921	.0287 036
33	.0205 648	.1439 691	.8644 628	.2877 223	.0287 809
34	.0205 987	.1441 602	.8636 662	.2887 533	.0288 580
0.335	+0.0206 324	-0.1443 498	+0.8628 674	+0.2897 850	-0.0289 350
36	.0206 660	.1445 381	.8620 664	.2908 176	.0290 118
37	.0206 993	.1447 249	.8612 632	.2918 509	.0290 886
38	.0207 324	.1449 102	.8604 579	.2928 850	.0291 651
39	.0207 654	.1450 941	.8596 505	.2939 198	.0292 416
0.340	+0.0207 981	-0.1452 766	+0.8588 408	+0.2949 554	-0.0293 179
41	.0208 307	.1454 576	.8580 291	.2959 918	.0293 940
42	.0208 631	.1456 371	.8572 151	.2970 290	.0294 700
43	.0208 952	.1458 153	.8563 991	.2980 669	.0295 459
44	.0209 272	.1459 920	.8555 809	.2991 055	.0296 216
0.345	+0.0209 589	-0.1461 672	+0.8547 605	+0.3001 449	-0.0296 971
46	.0209 905	.1463 410	.8539 380	.3011 851	.0297 725
47	.0210 219	.1465 134	.8531 133	.3022 260	.0298 478
48	.0210 531	.1466 843	.8522 865	.3032 676	.0299 229
49	.0210 841	.1468 538	.8514 576	.3043 100	.0299 979
0.350	+0.0211 148	-0.1470 219	+0.8506 266	+0.3053 531	-0.0300 727

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.350	+0.0211 148	-0.1470 219	+0.8506 266	+0.3053 531	-0.0300 727
51	.0211 454	.1471 885	.8497 934	.3063 970	.0301 473
52	.0211 758	.1473 537	.8489 581	.3074 416	.0302 218
53	.0212 060	.1475 174	.8481 206	.3084 869	.0302 962
54	.0212 360	.1476 797	.8472 810	.3095 330	.0303 703
0.355	+0.0212 658	-0.1478 406	+0.8464 393	+0.3105 798	-0.0304 444
56	.0212 954	.1480 000	.8455 955	.3116 273	.0305 182
57	.0213 248	.1481 580	.8447 496	.3126 756	.0305 920
58	.0213 540	.1483 145	.8439 015	.3137 245	.0306 655
59	.0213 830	.1484 697	.8430 513	.3147 742	.0307 389
0.360	+0.0214 118	-0.1486 234	+0.8421 990	+0.3158 246	-0.0308 122
61	.0214 405	.1487 756	.8413 446	.3168 758	.0308 852
62	.0214 689	.1489 264	.8404 881	.3179 276	.0309 582
63	.0214 971	.1490 758	.8396 295	.3189 801	.0310 309
64	.0215 251	.1492 238	.8387 688	.3200 334	.0311 035
0.365	+0.0215 529	-0.1493 703	+0.8379 060	+0.3210 873	-0.0311 759
66	.0215 805	.1495 154	.8370 411	.3221 420	.0312 482
67	.0216 079	.1496 590	.8361 740	.3231 973	.0313 203
68	.0216 351	.1498 013	.8353 049	.3242 534	.0313 922
69	.0216 622	.1499 421	.8344 337	.3253 101	.0314 639
0.370	+0.0216 890	-0.1500 814	+0.8335 604	+0.3263 676	-0.0315 355
71	.0217 156	.1502 194	.8326 850	.3274 257	.0316 069
72	.0217 420	.1503 559	.8318 075	.3284 845	.0316 782
73	.0217 682	.1504 910	.8309 280	.3295 440	.0317 492
74	.0217 943	.1506 246	.8300 463	.3306 042	.0318 201
0.375	+0.0218 201	-0.1507 568	+0.8291 626	+0.3316 650	-0.0318 909
76	.0218 457	.1508 876	.8282 768	.3327 266	.0319 614
77	.0218 711	.1510 170	.8273 889	.3337 888	.0320 318
78	.0218 963	.1511 449	.8264 990	.3348 517	.0321 020
79	.0219 213	.1512 715	.8256 069	.3359 152	.0321 720
0.380	+0.0219 461	-0.1513 966	+0.8247 128	+0.3369 794	-0.0322 419
81	.0219 708	.1515 202	.8238 167	.3380 443	.0323 115
82	.0219 952	.1516 425	.8229 185	.3391 099	.0323 810
83	.0220 194	.1517 633	.8220 182	.3401 761	.0324 503
84	.0220 434	.1518 827	.8211 158	.3412 429	.0325 194
0.385	+0.0220 672	-0.1520 007	+0.8202 114	+0.3423 105	-0.0325 884
86	.0220 908	.1521 172	.8193 050	.3433 786	.0326 571
87	.0221 142	.1522 324	.8183 964	.3444 474	.0327 257
88	.0221 374	.1523 461	.8174 859	.3455 169	.0327 941
89	.0221 604	.1524 584	.8165 733	.3465 870	.0328 623
0.390	+0.0221 832	-0.1525 692	+0.8156 586	+0.3476 578	-0.0329 303
91	.0222 058	.1526 787	.8147 419	.3487 292	.0329 981
92	.0222 282	.1527 867	.8138 232	.3498 012	.0330 658
93	.0222 504	.1528 933	.8129 024	.3508 738	.0331 332
94	.0222 723	.1529 985	.8119 796	.3519 471	.0332 005
0.395	+0.0222 941	-0.1531 023	+0.8110 547	+0.3530 211	-0.0332 676
96	.0223 157	.1532 047	.8101 278	.3540 956	.0333 344
97	.0223 371	.1533 056	.8091 989	.3551 708	.0334 011
98	.0223 583	.1534 052	.8082 680	.3562 466	.0334 676
99	.0223 792	.1535 033	.8073 350	.3573 230	.0335 339
0.400	+0.0224 000	-0.1536 000	+0.8064 000	+0.3584 000	-0.0336 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.400	+0.0224 000	-0.1536 000	+0.8064 000	+0.3584 000	-0.0336 000
01	.0224 206	.1536 953	.8054 630	.3594 776	.0336 659
02	.0224 409	.1537 892	.8045 240	.3605 559	.0337 316
03	.0224 611	.1538 816	.8035 829	.3616 347	.0337 971
04	.0224 810	.1539 727	.8026 399	.3627 142	.0338 624
0.405	+0.0225 008	-0.1540 623	+0.8016 948	+0.3637 943	-0.0339 275
06	.0225 203	.1541 506	.8007 477	.3648 749	.0339 924
07	.0225 397	.1542 374	.7997 986	.3659 562	.0340 571
08	.0225 588	.1543 228	.7988 476	.3670 381	.0341 216
09	.0225 778	.1544 068	.7978 945	.3681 205	.0341 859
0.410	+0.0225 965	-0.1544 894	+0.7969 394	+0.3692 036	-0.0342 500
11	.0226 150	.1545 706	.7959 823	.3702 872	.0343 139
12	.0226 333	.1546 504	.7950 233	.3713 714	.0343 776
13	.0226 514	.1547 288	.7940 622	.3724 562	.0344 411
14	.0226 694	.1548 058	.7930 991	.3735 416	.0345 043
0.415	+0.0226 871	-0.1548 813	+0.7921 341	+0.3746 275	-0.0345 674
16	.0227 046	.1549 555	.7911 671	.3757 141	.0346 302
17	.0227 219	.1550 283	.7901 981	.3768 012	.0346 928
18	.0227 390	.1550 996	.7892 271	.3778 888	.0347 553
19	.0227 559	.1551 696	.7882 542	.3789 770	.0348 175
0.420	+0.0227 725	-0.1552 382	+0.7872 792	+0.3800 658	-0.0348 795
21	.0227 890	.1553 053	.7863 023	.3811 552	.0349 412
22	.0228 053	.1553 711	.7853 235	.3822 451	.0350 028
23	.0228 214	.1554 354	.7843 426	.3833 356	.0350 641
24	.0228 372	.1554 984	.7833 599	.3844 266	.0351 253
0.425	+0.0228 529	-0.1555 600	+0.7823 751	+0.3855 182	-0.0351 862
26	.0228 683	.1556 201	.7813 884	.3866 103	.0352 469
27	.0228 836	.1556 789	.7803 997	.3877 029	.0353 073
28	.0228 986	.1557 363	.7794 091	.3887 961	.0353 676
29	.0229 135	.1557 923	.7784 165	.3898 899	.0354 276
0.430	+0.0229 281	-0.1558 468	+0.7774 220	+0.3909 842	-0.0354 874
31	.0229 425	.1559 000	.7764 255	.3920 790	.0355 470
32	.0229 567	.1559 518	.7754 271	.3931 743	.0356 063
33	.0229 707	.1560 022	.7744 268	.3942 702	.0356 655
34	.0229 845	.1560 512	.7734 245	.3953 666	.0357 244
0.435	+0.0229 981	-0.1560 989	+0.7724 203	+0.3964 635	-0.0357 830
36	.0230 115	.1561 451	.7714 141	.3975 609	.0358 415
37	.0230 247	.1561 900	.7704 060	.3986 589	.0358 997
38	.0230 377	.1562 334	.7693 960	.3997 574	.0359 577
39	.0230 505	.1562 755	.7683 841	.4008 563	.0360 154
0.440	+0.0230 630	-0.1563 162	+0.7673 702	+0.4019 558	-0.0360 730
41	.0230 754	.1563 555	.7663 545	.4030 558	.0361 302
42	.0230 876	.1563 934	.7653 368	.4041 563	.0361 873
43	.0230 995	.1564 299	.7643 172	.4052 573	.0362 441
44	.0231 112	.1564 650	.7632 957	.4063 588	.0363 007
0.445	+0.0231 228	-0.1564 988	+0.7622 722	+0.4074 608	-0.0363 570
46	.0231 341	.1565 312	.7612 469	.4085 633	.0364 131
47	.0231 452	.1565 622	.7602 197	.4096 663	.0364 690
48	.0231 561	.1565 918	.7591 905	.4107 698	.0365 246
49	.0231 668	.1566 200	.7581 595	.4118 737	.0365 800
0.450	+0.0231 773	-0.1566 469	+0.7571 266	+0.4129 781	-0.0366 352

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.450	+0.0231 773	-0.1566 469	+0.7571 266	+0.4129 781	-0.0366 352
51	.0231 876	.1566 724	.7560 917	.4140 830	.0366 901
52	.0231 977	.1566 965	.7550 550	.4151 884	.0367 447
53	.0232 076	.1567 192	.7540 164	.4162 943	.0367 991
54	.0232 173	.1567 405	.7529 760	.4174 006	.0368 533
0.455	+0.0232 267	-0.1567 605	+0.7519 336	+0.4185 074	-0.0369 072
56	.0232 360	.1567 791	.7508 893	.4196 146	.0369 609
57	.0232 450	.1567 963	.7498 432	.4207 224	.0370 143
58	.0232 539	.1568 122	.7487 952	.4218 305	.0370 675
59	.0232 625	.1568 267	.7477 454	.4229 392	.0371 204
0.460	+0.0232 709	-0.1568 398	+0.7466 936	+0.4240 482	-0.0371 731
61	.0232 792	.1568 515	.7456 400	.4251 578	.0372 255
62	.0232 872	.1568 619	.7445 846	.4262 678	.0372 776
63	.0232 950	.1568 709	.7435 273	.4273 782	.0373 296
64	.0233 026	.1568 785	.7424 681	.4284 890	.0373 812
0.465	+0.0233 100	-0.1568 848	+0.7414 071	+0.4296 004	-0.0374 326
66	.0233 171	.1568 897	.7403 442	.4307 121	.0374 837
67	.0233 241	.1568 932	.7392 795	.4318 243	.0375 346
68	.0233 309	.1568 954	.7382 129	.4329 369	.0375 853
69	.0233 374	.1568 962	.7371 445	.4340 499	.0376 356
0.470	+0.0233 438	-0.1568 956	+0.7360 742	+0.4351 634	-0.0376 857
71	.0233 499	.1568 937	.7350 021	.4362 772	.0377 356
72	.0233 559	.1568 904	.7339 282	.4373 915	.0377 851
73	.0233 616	.1568 858	.7328 524	.4385 063	.0378 344
74	.0233 671	.1568 798	.7317 748	.4396 214	.0378 835
0.475	+0.0233 724	-0.1568 725	+0.7306 954	+0.4407 369	-0.0379 323
76	.0233 775	.1568 638	.7296 142	.4418 529	.0379 808
77	.0233 824	.1568 537	.7285 311	.4429 692	.0380 290
78	.0233 871	.1568 423	.7274 462	.4440 860	.0380 770
79	.0233 916	.1568 295	.7263 595	.4452 031	.0381 247
0.480	+0.0233 958	-0.1568 154	+0.7252 710	+0.4463 206	-0.0381 722
81	.0233 999	.1567 999	.7241 807	.4474 386	.0382 193
82	.0234 038	.1567 830	.7230 886	.4485 569	.0382 662
83	.0234 074	.1567 649	.7219 947	.4496 756	.0383 128
84	.0234 108	.1567 453	.7208 990	.4507 947	.0383 592
0.485	+0.0234 141	-0.1567 244	+0.7198 015	+0.4519 142	-0.0384 052
86	.0234 171	.1567 022	.7187 021	.4530 340	.0384 510
87	.0234 199	.1566 786	.7176 010	.4541 543	.0384 966
88	.0234 225	.1566 537	.7164 981	.4552 749	.0385 418
89	.0234 249	.1566 274	.7153 935	.4563 958	.0385 867
0.490	+0.0234 271	-0.1565 998	+0.7142 870	+0.4575 172	-0.0386 314
91	.0234 291	.1565 709	.7131 788	.4586 389	.0386 758
92	.0234 308	.1565 406	.7120 687	.4597 609	.0387 199
93	.0234 324	.1565 089	.7109 570	.4608 833	.0387 637
94	.0234 338	.1564 760	.7098 434	.4620 061	.0388 073
0.495	+0.0234 349	-0.1564 416	+0.7087 281	+0.4631 292	-0.0388 505
96	.0234 358	.1564 060	.7076 110	.4642 527	.0388 935
97	.0234 366	.1563 690	.7064 921	.4653 765	.0389 362
98	.0234 371	.1563 307	.7053 715	.4665 007	.0389 786
99	.0234 374	.1562 910	.7042 491	.4676 252	.0390 207
0.500	+0.0234 375	-0.1562 500	+0.7031 250	+0.4687 500	-0.0390 625

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.500	+0.0234 375	-0.1562 500	+0.7031 250	+0.4687 500	-0.0390 625
01	.0234 374	.1562 077	.7019 991	.4698 752	.0391 040
02	.0234 371	.1561 640	.7008 715	.4710 007	.0391 452
03	.0234 366	.1561 190	.6997 421	.4721 265	.0391 862
04	.0234 358	.1560 727	.6986 110	.4732 526	.0392 268
0.505	+0.0234 349	-0.1560 250	+0.6974 782	+0.4743 791	-0.0392 672
06	.0234 338	.1559 760	.6963 436	.4755 059	.0393 072
07	.0234 324	.1559 257	.6952 073	.4766 330	.0393 470
08	.0234 308	.1558 741	.6940 693	.4777 604	.0393 864
09	.0234 291	.1558 211	.6929 295	.4788 881	.0394 256
0.510	+0.0234 271	-0.1557 668	+0.6917 880	+0.4800 162	-0.0394 644
11	.0234 249	.1557 112	.6906 448	.4811 445	.0395 030
12	.0234 225	.1556 543	.6894 999	.4822 731	.0395 412
13	.0234 199	.1555 960	.6883 532	.4834 021	.0395 792
14	.0234 171	.1555 365	.6872 049	.4845 313	.0396 168
0.515	+0.0234 141	-0.1554 756	+0.6860 548	+0.4856 608	-0.0396 541
16	.0234 108	.1554 134	.6849 031	.4867 906	.0396 911
17	.0234 074	.1553 498	.6837 496	.4879 207	.0397 279
18	.0234 038	.1552 850	.6825 944	.4890 511	.0397 643
19	.0233 999	.1552 188	.6814 376	.4901 817	.0398 004
0.520	+0.0233 958	-0.1551 514	+0.6802 790	+0.4913 126	-0.0398 362
21	.0233 916	.1550 826	.6791 188	.4924 438	.0398 716
22	.0233 871	.1550 125	.6779 569	.4935 753	.0399 068
23	.0233 824	.1549 411	.6767 933	.4947 070	.0399 416
24	.0233 775	.1548 684	.6756 280	.4958 390	.0399 762
0.525	+0.0233 724	-0.1547 943	+0.6744 610	+0.4969 713	-0.0400 104
26	.0233 671	.1547 190	.6732 924	.4981 038	.0400 443
27	.0233 616	.1546 424	.6721 221	.4992 366	.0400 779
28	.0233 559	.1545 644	.6709 501	.5003 696	.0401 111
29	.0233 499	.1544 852	.6697 765	.5015 029	.0401 441
0.530	+0.0233 438	-0.1544 046	+0.6686 012	+0.5026 364	-0.0401 767
31	.0233 374	.1543 228	.6674 243	.5037 701	.0402 090
32	.0233 309	.1542 396	.6662 456	.5049 041	.0402 410
33	.0233 241	.1541 552	.6650 654	.5060 383	.0402 726
34	.0233 171	.1540 694	.6638 835	.5071 728	.0403 040
0.535	+0.0233 100	-0.1539 824	+0.6626 999	+0.5083 075	-0.0403 350
36	.0233 026	.1538 941	.6615 147	.5094 424	.0403 657
37	.0232 950	.1538 044	.6603 279	.5105 775	.0403 960
38	.0232 872	.1537 135	.6591 395	.5117 129	.0404 260
39	.0232 792	.1536 213	.6579 494	.5128 485	.0404 557
0.540	+0.0232 709	-0.1535 278	+0.6567 576	+0.5139 842	-0.0404 851
41	.0232 625	.1534 330	.6555 643	.5151 202	.0405 141
42	.0232 539	.1533 369	.6543 693	.5162 564	.0405 428
43	.0232 450	.1532 395	.6531 727	.5173 928	.0405 711
44	.0232 360	.1531 408	.6519 745	.5185 294	.0405 991
0.545	+0.0232 267	-0.1530 409	+0.6507 747	+0.5196 663	-0.0406 268
46	.0232 173	.1529 396	.6495 733	.5208 033	.0406 542
47	.0232 076	.1528 371	.6483 703	.5219 404	.0406 812
48	.0231 977	.1527 333	.6471 656	.5230 778	.0407 078
49	.0231 876	.1526 282	.6459 594	.5242 154	.0407 342
0.550	+0.0231 773	-0.1525 219	+0.6447 516	+0.5253 531	-0.0407 602

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.550	+0.0231 773	-0.1525 219	+0.6447 516	+0.5253 531	-0.0407 602
51	.0231 668	.1524 142	.6435 421	.5264 910	.0407 858
52	.0231 561	.1523 053	.6423 311	.5276 291	.0408 111
53	.0231 452	.1521 951	.6411 185	.5287 674	.0408 360
54	.0231 341	.1520 837	.6399 044	.5299 059	.0408 607
0.555	+0.0231 228	-0.1519 709	+0.6386 886	+0.5310 445	-0.0408 849
56	.0231 112	.1518 569	.6374 713	.5321 832	.0409 088
57	.0230 995	.1517 416	.6362 524	.5333 221	.0409 324
58	.0230 876	.1516 251	.6350 319	.5344 612	.0409 556
59	.0230 754	.1515 072	.6338 098	.5356 005	.0409 785
0.560	+0.0230 630	-0.1513 882	+0.6325 862	+0.5367 398	-0.0410 010
61	.0230 505	.1512 678	.6313 611	.5378 794	.0410 231
62	.0230 377	.1511 462	.6301 344	.5390 190	.0410 449
63	.0230 247	.1510 233	.6289 061	.5401 589	.0410 664
64	.0230 115	.1508 992	.6276 763	.5412 988	.0410 874
0.565	+0.0229 981	-0.1507 737	+0.6264 449	+0.5424 389	-0.0411 082
66	.0229 845	.1506 471	.6252 120	.5435 791	.0411 285
67	.0229 707	.1505 192	.6239 775	.5447 194	.0411 486
68	.0229 567	.1503 900	.6227 416	.5458 599	.0411 682
69	.0229 425	.1502 595	.6215 040	.5470 005	.0411 875
0.570	+0.0229 281	-0.1501 278	+0.6202 650	+0.5481 412	-0.0412 064
71	.0229 135	.1499 949	.6190 244	.5492 820	.0412 250
72	.0228 986	.1498 607	.6177 823	.5504 229	.0412 432
73	.0228 836	.1497 252	.6165 387	.5515 639	.0412 610
74	.0228 683	.1495 885	.6152 936	.5527 051	.0412 785
0.575	+0.0228 529	-0.1494 506	+0.6140 470	+0.5538 463	-0.0412 956
76	.0228 372	.1493 114	.6127 988	.5549 876	.0413 123
77	.0228 214	.1491 709	.6115 492	.5561 290	.0413 286
78	.0228 053	.1490 293	.6102 980	.5572 706	.0413 446
79	.0227 890	.1488 863	.6090 454	.5584 122	.0413 602
0.580	+0.0227 725	-0.1487 422	+0.6077 912	+0.5595 538	-0.0413 755
81	.0227 559	.1485 967	.6065 356	.5606 956	.0413 903
82	.0227 390	.1484 501	.6052 785	.5618 374	.0414 048
83	.0227 219	.1483 022	.6040 199	.5629 794	.0414 189
84	.0227 046	.1481 531	.6027 598	.5641 214	.0414 326
0.585	+0.0226 871	-0.1480 027	+0.6014 982	+0.5652 634	-0.0414 460
86	.0226 694	.1478 511	.6002 352	.5664 055	.0414 590
87	.0226 514	.1476 983	.5989 707	.5675 477	.0414 715
88	.0226 333	.1475 442	.5977 047	.5686 899	.0414 838
89	.0226 150	.1473 890	.5964 373	.5698 322	.0414 956
0.590	+0.0225 965	-0.1472 324	+0.5951 684	+0.5709 746	-0.0415 070
91	.0225 778	.1470 747	.5938 981	.5721 170	.0415 181
92	.0225 588	.1469 157	.5926 263	.5732 594	.0415 287
93	.0225 397	.1467 555	.5913 530	.5744 019	.0415 390
94	.0225 203	.1465 941	.5900 783	.5755 444	.0415 489
0.595	+0.0225 008	-0.1464 315	+0.5888 022	+0.5766 869	-0.0415 584
96	.0224 810	.1462 676	.5875 246	.5778 295	.0415 675
97	.0224 611	.1461 025	.5862 456	.5789 721	.0415 762
98	.0224 409	.1459 362	.5849 652	.5801 147	.0415 845
99	.0224 206	.1457 687	.5836 833	.5812 573	.0415 925
0.600	+0.0224 000	-0.1456 000	+0.5824 000	+0.5824 000	-0.0416 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.600	+0.0224 000	-0.1456 000	+0.5824 000	+0.5824 000	-0.0416 000
01	.0223 792	.1454 301	.5811 153	.5835 427	.0416 071
02	.0223 583	.1452 589	.5798 292	.5846 854	.0416 139
03	.0223 371	.1450 865	.5785 416	.5858 280	.0416 202
04	.0223 157	.1449 130	.5772 527	.5869 707	.0416 261
0.605	+0.0222 941	-0.1447 382	+0.5759 623	+0.5881 134	-0.0416 317
06	.0222 723	.1445 622	.5746 706	.5892 561	.0416 368
07	.0222 504	.1443 850	.5733 774	.5903 988	.0416 416
08	.0222 282	.1442 066	.5720 829	.5915 415	.0416 459
09	.0222 058	.1440 270	.5707 869	.5926 841	.0416 498
0.610	+0.0221 832	-0.1438 462	+0.5694 896	+0.5938 268	-0.0416 533
11	.0221 604	.1436 642	.5681 909	.5949 694	.0416 564
12	.0221 374	.1434 810	.5668 908	.5961 120	.0416 591
13	.0221 142	.1432 967	.5655 893	.5972 545	.0416 614
14	.0220 908	.1431 111	.5642 865	.5983 971	.0416 633
0.615	+0.0220 672	-0.1429 243	+0.5629 823	+0.5995 396	-0.0416 648
16	.0220 434	.1427 363	.5616 767	.6006 820	.0416 658
17	.0220 194	.1425 472	.5603 698	.6018 245	.0416 664
18	.0219 952	.1423 568	.5590 615	.6029 668	.0416 667
19	.0219 708	.1421 653	.5577 518	.6041 092	.0416 665
0.620	+0.0219 461	-0.1419 726	+0.5564 408	+0.6052 514	-0.0416 659
21	.0219 213	.1417 787	.5551 285	.6063 937	.0416 648
22	.0218 963	.1415 836	.5538 148	.6075 358	.0416 634
23	.0218 711	.1413 873	.5524 998	.6086 779	.0416 615
24	.0218 457	.1411 898	.5511 834	.6098 200	.0416 592
0.625	+0.0218 201	-0.1409 912	+0.5498 657	+0.6109 619	-0.0416 565
26	.0217 943	.1407 914	.5485 467	.6121 038	.0416 534
27	.0217 682	.1405 904	.5472 264	.6132 456	.0416 498
28	.0217 420	.1403 883	.5459 047	.6143 874	.0416 458
29	.0217 156	.1401 849	.5445 817	.6155 290	.0416 414
0.630	+0.0216 890	-0.1399 804	+0.5432 574	+0.6166 706	-0.0416 365
31	.0216 622	.1397 748	.5419 318	.6178 120	.0416 312
32	.0216 351	.1395 679	.5406 049	.6189 534	.0416 255
33	.0216 079	.1393 599	.5392 767	.6200 947	.0416 194
34	.0215 805	.1391 508	.5379 472	.6212 359	.0416 128
0.635	+0.0215 529	-0.1389 404	+0.5366 164	+0.6223 770	-0.0416 058
36	.0215 251	.1387 289	.5352 843	.6235 179	.0415 983
37	.0214 971	.1385 163	.5339 509	.6246 588	.0415 905
38	.0214 689	.1383 025	.5326 162	.6257 995	.0415 821
39	.0214 405	.1380 875	.5312 803	.6269 401	.0415 734
0.640	+0.0214 118	-0.1378 714	+0.5299 430	+0.6280 806	-0.0415 642
41	.0213 830	.1376 541	.5286 045	.6292 210	.0415 545
42	.0213 540	.1374 356	.5272 648	.6303 613	.0415 444
43	.0213 248	.1372 161	.5259 238	.6315 014	.0415 339
44	.0212 954	.1369 953	.5245 815	.6326 413	.0415 229
0.645	+0.0212 658	-0.1367 734	+0.5232 380	+0.6337 812	-0.0415 115
46	.0212 360	.1365 504	.5218 932	.6349 209	.0414 996
47	.0212 060	.1363 262	.5205 471	.6360 604	.0414 873
48	.0211 758	.1361 009	.5191 998	.6371 998	.0414 745
49	.0211 454	.1358 745	.5178 513	.6383 390	.0414 613
0.650	+0.0211 148	-0.1356 469	+0.5165 016	+0.6394 781	-0.0414 477

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.650	+0.0211 148	-0.1356 469	+0.5165 016	+0.6394 781	-0.0414 477
51	.0210 841	.1354 181	.5151 506	.6406 170	.0414 335
52	.0210 531	.1351 883	.5137 984	.6417 558	.0414 190
53	.0210 219	.1349 573	.5124 449	.6428 944	.0414 039
54	.0209 905	.1347 251	.5110 902	.6440 328	.0413 884
0.655	+0.0209 589	-0.1344 918	+0.5097 344	+0.6451 710	-0.0413 725
56	.0209 272	.1342 574	.5083 773	.6463 091	.0413 561
57	.0208 952	.1340 219	.5070 190	.6474 470	.0413 392
58	.0208 631	.1337 853	.5056 595	.6485 846	.0413 219
59	.0208 307	.1335 475	.5042 987	.6497 221	.0413 041
0.660	+0.0207 981	-0.1333 086	+0.5029 368	+0.6508 594	-0.0412 859
61	.0207 654	.1330 685	.5015 737	.6519 965	.0412 671
62	.0207 324	.1328 274	.5002 095	.6531 334	.0412 480
63	.0206 993	.1325 851	.4988 440	.6542 701	.0412 283
64	.0206 660	.1323 417	.4974 773	.6554 066	.0412 082
0.665	+0.0206 324	-0.1320 972	+0.4961 095	+0.6565 429	-0.0411 876
66	.0205 987	.1318 516	.4947 405	.6576 790	.0411 666
67	.0205 648	.1316 049	.4933 703	.6588 148	.0411 450
68	.0205 307	.1313 570	.4919 990	.6599 504	.0411 230
69	.0204 964	.1311 081	.4906 265	.6610 858	.0411 006
0.670	+0.0204 619	-0.1308 580	+0.4892 528	+0.6622 210	-0.0410 776
71	.0204 272	.1306 069	.4878 780	.6633 559	.0410 542
72	.0203 923	.1303 546	.4865 020	.6644 906	.0410 303
73	.0203 572	.1301 012	.4851 249	.6656 250	.0410 059
74	.0203 219	.1298 468	.4837 467	.6667 592	.0409 811
0.675	+0.0202 865	-0.1295 912	+0.4823 673	+0.6678 932	-0.0409 557
76	.0202 508	.1293 345	.4809 868	.6690 269	.0409 299
77	.0202 150	.1290 768	.4796 051	.6701 603	.0409 036
78	.0201 789	.1288 179	.4782 223	.6712 935	.0408 768
79	.0201 427	.1285 580	.4768 385	.6724 264	.0408 495
0.680	+0.0201 062	-0.1282 970	+0.4754 534	+0.6735 590	-0.0408 218
81	.0200 696	.1280 348	.4740 673	.6746 914	.0407 935
82	.0200 328	.1277 716	.4726 801	.6758 235	.0407 648
83	.0199 958	.1275 073	.4712 917	.6769 553	.0407 355
84	.0199 586	.1272 420	.4699 023	.6780 869	.0407 058
0.685	+0.0199 212	-0.1269 755	+0.4685 118	+0.6792 181	-0.0406 756
86	.0198 836	.1267 080	.4671 201	.6803 491	.0406 449
87	.0198 458	.1264 393	.4657 274	.6814 798	.0406 137
88	.0198 079	.1261 696	.4643 336	.6826 101	.0405 820
89	.0197 697	.1258 989	.4629 388	.6837 402	.0405 498
0.690	+0.0197 314	-0.1256 270	+0.4615 428	+0.6848 700	-0.0405 171
91	.0196 928	.1253 541	.4601 458	.6859 994	.0404 839
92	.0196 541	.1250 801	.4587 477	.6871 286	.0404 502
93	.0196 152	.1248 051	.4573 485	.6882 574	.0404 160
94	.0195 761	.1245 290	.4559 483	.6893 859	.0403 813
0.695	+0.0195 368	-0.1242 518	+0.4545 470	+0.6905 141	-0.0403 461
96	.0194 973	.1239 736	.4531 447	.6916 419	.0403 104
97	.0194 577	.1236 943	.4517 413	.6927 695	.0402 742
98	.0194 178	.1234 139	.4503 369	.6938 966	.0402 375
99	.0193 777	.1231 325	.4489 315	.6950 235	.0402 002
0.700	+0.0193 375	-0.1228 500	+0.4475 250	+0.6961 500	-0.0401 625

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.700	+0.0193 375	-0.1228 500	+0.4475 250	+0.6961 500	-0.0401 625
01	.0192 971	.1225 665	.4461 175	.6972 762	.0401 242
02	.0192 565	.1222 819	.4447 089	.6984 020	.0400 855
03	.0192 157	.1219 963	.4432 994	.6995 274	.0400 462
04	.0191 747	.1217 096	.4418 888	.7006 525	.0400 064
0.705	+0.0191 335	-0.1214 219	+0.4404 772	+0.7017 773	-0.0399 661
06	.0190 921	.1211 331	.4390 646	.7029 016	.0399 252
07	.0190 506	.1208 433	.4376 510	.7040 256	.0398 839
08	.0190 088	.1205 524	.4362 364	.7051 492	.0398 420
09	.0189 669	.1202 606	.4348 208	.7062 725	.0397 996
0.710	+0.0189 248	-0.1199 676	+0.4334 042	+0.7073 954	-0.0397 567
11	.0188 825	.1196 737	.4319 866	.7085 178	.0397 133
12	.0188 400	.1193 787	.4305 681	.7096 399	.0396 693
13	.0187 973	.1190 827	.4291 485	.7107 616	.0396 248
14	.0187 545	.1187 856	.4277 280	.7118 829	.0395 798
0.715	+0.0187 114	-0.1184 875	+0.4263 065	+0.7130 039	-0.0395 343
16	.0186 682	.1181 884	.4248 840	.7141 244	.0394 882
17	.0186 248	.1178 883	.4234 606	.7152 444	.0394 416
18	.0185 812	.1175 871	.4220 362	.7163 641	.0393 944
19	.0185 374	.1172 850	.4206 109	.7174 834	.0393 468
0.720	+0.0184 934	-0.1169 818	+0.4191 846	+0.7186 022	-0.0392 986
21	.0184 493	.1166 776	.4177 574	.7197 207	.0392 498
22	.0184 050	.1163 723	.4163 293	.7208 387	.0392 005
23	.0183 604	.1160 661	.4149 002	.7219 562	.0391 507
24	.0183 157	.1157 588	.4134 701	.7230 733	.0391 004
0.725	+0.0182 708	-0.1154 506	+0.4120 392	+0.7241 900	-0.0390 495
26	.0182 258	.1151 413	.4106 073	.7253 063	.0389 980
27	.0181 805	.1148 311	.4091 745	.7264 221	.0389 460
28	.0181 351	.1145 198	.4077 408	.7275 374	.0388 935
29	.0180 895	.1142 075	.4063 061	.7286 523	.0388 404
0.730	+0.0180 437	-0.1138 942	+0.4048 706	+0.7297 668	-0.0387 868
31	.0179 977	.1135 800	.4034 342	.7308 807	.0387 326
32	.0179 515	.1132 647	.4019 968	.7319 942	.0386 779
33	.0179 052	.1129 484	.4005 586	.7331 073	.0386 227
34	.0178 587	.1126 312	.3991 195	.7342 198	.0385 668
0.735	+0.0178 120	-0.1123 129	+0.3976 795	+0.7353 319	-0.0385 105
36	.0177 651	.1119 937	.3962 386	.7364 435	.0384 535
37	.0177 180	.1116 735	.3947 969	.7375 546	.0383 961
38	.0176 708	.1113 523	.3933 543	.7386 653	.0383 380
39	.0176 233	.1110 301	.3919 108	.7397 754	.0382 794
0.740	+0.0175 757	-0.1107 070	+0.3904 664	+0.7408 850	-0.0382 203
41	.0175 280	.1103 828	.3890 212	.7419 942	.0381 605
42	.0174 800	.1100 577	.3875 752	.7431 028	.0381 003
43	.0174 318	.1097 316	.3861 283	.7442 109	.0380 394
44	.0173 835	.1094 046	.3846 805	.7453 185	.0379 780
0.745	+0.0173 350	-0.1090 765	+0.3832 319	+0.7464 256	-0.0379 160
46	.0172 863	.1087 475	.3817 825	.7475 322	.0378 535
47	.0172 375	.1084 176	.3803 323	.7486 382	.0377 904
48	.0171 884	.1080 866	.3788 812	.7497 437	.0377 267
49	.0171 392	.1077 547	.3774 293	.7508 487	.0376 625
0.750	+0.0170 898	-0.1074 219	+0.3759 766	+0.7519 531	-0.0375 977

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.750	+0.0170 898	-0.1074 219	+0.3759 766	+0.7519 531	-0.0375 977
51	.0170 403	.1070 881	.3745 230	.7530 570	.0375 323
52	.0169 905	.1067 533	.3730 687	.7541 604	.0374 663
53	.0169 406	.1064 176	.3716 136	.7552 632	.0373 998
54	.0168 905	.1060 809	.3701 576	.7563 654	.0373 326
0.755	+0.0168 402	-0.1057 433	+0.3687 009	+0.7574 671	-0.0372 649
56	.0167 898	.1054 047	.3672 433	.7585 682	.0371 967
57	.0167 392	.1050 652	.3657 850	.7596 688	.0371 278
58	.0166 884	.1047 247	.3643 259	.7607 688	.0370 584
59	.0166 374	.1043 833	.3628 661	.7618 682	.0369 884
0.760	+0.0165 862	-0.1040 410	+0.3614 054	+0.7629 670	-0.0369 178
61	.0165 349	.1036 977	.3599 440	.7640 653	.0368 466
62	.0164 834	.1033 535	.3584 819	.7651 630	.0367 748
63	.0164 317	.1030 083	.3570 189	.7662 601	.0367 024
64	.0163 799	.1026 622	.3555 553	.7673 565	.0366 295
0.765	+0.0163 279	-0.1023 152	+0.3540 908	+0.7684 524	-0.0365 559
66	.0162 757	.1019 673	.3526 257	.7695 477	.0364 818
67	.0162 233	.1016 184	.3511 597	.7706 424	.0364 071
68	.0161 708	.1012 686	.3496 931	.7717 365	.0363 317
69	.0161 181	.1009 179	.3482 257	.7728 299	.0362 558
0.770	+0.0160 652	-0.1005 662	+0.3467 576	+0.7739 228	-0.0361 793
71	.0160 121	.1002 137	.3452 888	.7750 150	.0361 022
72	.0159 589	.0998 602	.3438 192	.7761 066	.0360 245
73	.0159 055	.0995 058	.3423 490	.7771 975	.0359 462
74	.0158 519	.0991 505	.3408 780	.7782 879	.0358 673
0.775	+0.0157 982	-0.0987 943	+0.3394 063	+0.7793 775	-0.0357 877
76	.0157 443	.0984 372	.3379 340	.7804 666	.0357 076
77	.0156 902	.0980 792	.3364 609	.7815 550	.0356 269
78	.0156 360	.0977 203	.3349 872	.7826 427	.0355 456
79	.0155 815	.0973 605	.3335 127	.7837 298	.0354 636
0.780	+0.0155 269	-0.0969 998	+0.3320 376	+0.7848 162	-0.0353 811
81	.0154 722	.0966 381	.3305 619	.7859 020	.0352 979
82	.0154 173	.0962 756	.3290 854	.7869 871	.0352 141
83	.0153 622	.0959 122	.3276 083	.7880 715	.0351 297
84	.0153 069	.0955 479	.3261 305	.7891 553	.0350 447
0.785	+0.0152 515	-0.0951 828	+0.3246 521	+0.7902 384	-0.0349 591
86	.0151 959	.0948 167	.3231 730	.7913 207	.0348 729
87	.0151 401	.0944 498	.3216 932	.7924 024	.0347 860
88	.0150 842	.0940 819	.3202 129	.7934 834	.0346 985
89	.0150 281	.0937 132	.3187 318	.7945 638	.0346 104
0.790	+0.0149 718	-0.0933 436	+0.3172 502	+0.7956 434	-0.0345 217
91	.0149 153	.0929 732	.3157 679	.7967 223	.0344 324
92	.0148 587	.0926 018	.3142 850	.7978 005	.0343 424
93	.0148 020	.0922 296	.3128 015	.7988 779	.0342 518
94	.0147 450	.0918 566	.3113 174	.7999 547	.0341 606
0.795	+0.0146 880	-0.0914 826	+0.3098 327	+0.8010 308	-0.0340 687
96	.0146 307	.0911 078	.3083 473	.8021 061	.0339 763
97	.0145 733	.0907 322	.3068 614	.8031 807	.0338 831
98	.0145 157	.0903 556	.3053 748	.8042 545	.0337 894
99	.0144 579	.0899 782	.3038 877	.8053 276	.0336 950
0.800	+0.0144 000	-0.0896 000	+0.3024 000	+0.8064 000	-0.0336 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.800	+0.0144 000	-0.0896 000	+0.3024 000	+0.8064 000	-0.0336 000
01	.0143 419	.0892 209	.3009 117	.8074 716	.0335 043
02	.0142 837	.0888 410	.2994 228	.8085 425	.0334 081
03	.0142 253	.0884 602	.2979 334	.8096 126	.0333 111
04	.0141 667	.0880 785	.2964 434	.8106 820	.0332 136
0.805	+0.0141 080	-0.0876 960	+0.2949 529	+0.8117 506	-0.0331 153
06	.0140 491	.0873 127	.2934 617	.8128 184	.0330 165
07	.0139 900	.0869 286	.2919 701	.8138 855	.0329 170
08	.0139 308	.0865 435	.2904 779	.8149 517	.0328 168
09	.0138 714	.0861 577	.2889 851	.8160 172	.0327 161
0.810	+0.0138 119	-0.0857 710	+0.2874 918	+0.8170 820	-0.0326 146
11	.0137 522	.0853 835	.2859 980	.8181 459	.0325 125
12	.0136 923	.0849 952	.2845 036	.8192 090	.0324 098
13	.0136 323	.0846 060	.2830 088	.8202 714	.0323 064
14	.0135 721	.0842 161	.2815 134	.8213 329	.0322 023
0.815	+0.0135 118	-0.0838 252	+0.2800 175	+0.8223 936	-0.0320 976
16	.0134 513	.0834 336	.2785 211	.8234 535	.0319 923
17	.0133 906	.0830 412	.2770 241	.8245 127	.0318 863
18	.0133 298	.0826 479	.2755 267	.8255 709	.0317 796
19	.0132 689	.0822 538	.2740 288	.8266 284	.0316 723
0.820	+0.0132 077	-0.0818 590	+0.2725 304	+0.8276 850	-0.0315 643
21	.0131 465	.0814 633	.2710 316	.8287 408	.0314 556
22	.0130 850	.0810 668	.2695 322	.8297 958	.0313 463
23	.0130 234	.0806 695	.2680 324	.8308 499	.0312 363
24	.0129 617	.0802 714	.2665 321	.8319 032	.0311 256
0.825	+0.0128 998	-0.0798 725	+0.2650 313	+0.8329 557	-0.0310 143
26	.0128 377	.0794 728	.2635 301	.8340 072	.0309 023
27	.0127 755	.0790 723	.2620 285	.8350 580	.0307 896
28	.0127 131	.0786 710	.2605 264	.8361 078	.0306 763
29	.0126 506	.0782 689	.2590 238	.8371 568	.0305 623
0.830	+0.0125 879	-0.0778 660	+0.2575 208	+0.8382 050	-0.0304 476
31	.0125 250	.0774 624	.2560 174	.8392 522	.0303 323
32	.0124 621	.0770 579	.2545 135	.8402 986	.0302 162
33	.0123 989	.0766 527	.2530 092	.8413 441	.0300 995
34	.0123 356	.0762 467	.2515 045	.8423 887	.0299 821
0.835	+0.0122 722	-0.0758 400	+0.2499 994	+0.8434 324	-0.0298 640
36	.0122 086	.0754 324	.2484 939	.8444 752	.0297 453
37	.0121 448	.0750 241	.2469 880	.8455 171	.0296 258
38	.0120 809	.0746 150	.2454 817	.8465 581	.0295 057
39	.0120 168	.0742 052	.2439 749	.8475 983	.0293 849
0.840	+0.0119 526	-0.0737 946	+0.2424 678	+0.8486 374	-0.0292 634
41	.0118 883	.0733 832	.2409 604	.8496 757	.0291 412
42	.0118 238	.0729 710	.2394 525	.8507 131	.0290 183
43	.0117 591	.0725 581	.2379 443	.8517 495	.0288 947
44	.0116 943	.0721 445	.2364 356	.8527 850	.0287 704
0.845	+0.0116 294	-0.0717 301	+0.2349 267	+0.8538 195	-0.0286 455
46	.0115 642	.0713 149	.2334 173	.8548 531	.0285 198
47	.0114 990	.0708 990	.2319 077	.8558 858	.0283 934
48	.0114 336	.0704 824	.2303 976	.8569 175	.0282 664
49	.0113 680	.0700 650	.2288 873	.8579 483	.0281 386
0.850	+0.0113 023	-0.0696 469	+0.2273 766	+0.8589 781	-0.0280 102

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.850	+0.0113 023	-0.0696 469	+0.2273 766	+0.8589 781	-0.0280 102
51	.0112 365	.0692 280	.2258 655	.8600 070	.0278 810
52	.0111 705	.0688 084	.2243 542	.8610 349	.0277 511
53	.0111 044	.0683 881	.2228 425	.8620 618	.0276 205
54	.0110 381	.0679 670	.2213 305	.8630 877	.0274 893
0.855	+0.0109 717	-0.0675 452	+0.2198 181	+0.8641 127	-0.0273 573
56	.0109 051	.0671 227	.2183 055	.8651 367	.0272 246
57	.0108 384	.0666 994	.2167 926	.8661 596	.0270 912
58	.0107 715	.0662 755	.2152 794	.8671 816	.0269 570
59	.0107 045	.0658 508	.2137 658	.8682 026	.0268 222
0.860	+0.0106 373	-0.0654 254	+0.2122 520	+0.8692 226	-0.0266 867
61	.0105 700	.0649 992	.2107 380	.8702 416	.0265 504
62	.0105 026	.0645 724	.2092 236	.8712 596	.0264 134
63	.0104 350	.0641 449	.2077 090	.8722 766	.0262 757
64	.0103 673	.0637 166	.2061 941	.8732 925	.0261 373
0.865	+0.0102 994	-0.0632 877	+0.2046 789	+0.8743 075	-0.0259 981
66	.0102 314	.0628 580	.2031 635	.8753 214	.0258 583
67	.0101 633	.0624 277	.2016 478	.8763 342	.0257 177
68	.0100 950	.0619 966	.2001 319	.8773 460	.0255 763
69	.0100 266	.0615 649	.1986 158	.8783 568	.0254 343
0.870	+0.0099 580	-0.0611 324	+0.1970 994	+0.8793 666	-0.0252 915
71	.0098 893	.0606 993	.1955 828	.8803 753	.0251 480
72	.0098 204	.0602 655	.1940 660	.8813 829	.0250 038
73	.0097 514	.0598 310	.1925 489	.8823 895	.0248 588
74	.0096 823	.0593 958	.1910 316	.8833 950	.0247 131
0.875	+0.0096 130	-0.0589 600	+0.1895 142	+0.8843 994	-0.0245 667
76	.0095 436	.0585 234	.1879 965	.8854 028	.0244 195
77	.0094 741	.0580 862	.1864 786	.8864 051	.0242 716
78	.0094 044	.0576 483	.1849 605	.8874 063	.0241 229
79	.0093 346	.0572 098	.1834 423	.8884 064	.0239 735
0.880	+0.0092 646	-0.0567 706	+0.1819 238	+0.8894 054	-0.0238 234
81	.0091 946	.0563 307	.1804 052	.8904 034	.0236 725
82	.0091 243	.0558 901	.1788 864	.8914 002	.0235 208
83	.0090 540	.0554 489	.1773 675	.8923 959	.0233 685
84	.0089 835	.0550 071	.1758 484	.8933 906	.0232 153
0.885	+0.0089 128	-0.0545 645	+0.1743 291	+0.8943 841	-0.0230 615
86	.0088 421	.0541 214	.1728 097	.8953 765	.0229 068
87	.0087 712	.0536 776	.1712 901	.8963 677	.0227 515
88	.0087 001	.0532 331	.1697 704	.8973 579	.0225 953
89	.0086 290	.0527 880	.1682 506	.8983 469	.0224 385
0.890	+0.0085 577	-0.0523 422	+0.1667 306	+0.8993 348	-0.0222 808
91	.0084 863	.0518 958	.1652 105	.9003 215	.0221 224
92	.0084 147	.0514 488	.1636 903	.9013 071	.0219 633
93	.0083 430	.0510 012	.1621 700	.9022 915	.0218 033
94	.0082 712	.0505 529	.1606 495	.9032 748	.0216 427
0.895	+0.0081 992	-0.0501 039	+0.1591 290	+0.9042 569	-0.0214 812
96	.0081 271	.0496 544	.1576 084	.9052 379	.0213 190
97	.0080 549	.0492 042	.1560 877	.9062 177	.0211 560
98	.0079 826	.0487 534	.1545 669	.9071 963	.0209 923
99	.0079 101	.0483 020	.1530 460	.9081 737	.0208 278
0.900	+0.0078 375	-0.0478 500	+0.1515 250	+0.9091 500	-0.0206 625

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.900	+0.0078 375	-0.0478 500	+0.1515 250	+0.9091 500	-0.0206 625
01	.0077 648	.0473 974	.1500 040	.9101 251	.0204 964
02	.0076 919	.0469 441	.1484 829	.9110 990	.0203 296
03	.0076 189	.0464 903	.1469 617	.9120 716	.0201 620
04	.0075 458	.0460 358	.1454 405	.9130 431	.0199 936
0.905	+0.0074 726	-0.0455 807	+0.1439 192	+0.9140 134	-0.0198 245
06	.0073 992	.0451 251	.1423 979	.9149 825	.0196 546
07	.0073 257	.0446 688	.1408 766	.9159 503	.0194 838
08	.0072 521	.0442 119	.1393 552	.9169 170	.0193 124
09	.0071 784	.0437 545	.1378 338	.9178 824	.0191 401
0.910	+0.0071 045	-0.0432 964	+0.1363 124	+0.9188 466	-0.0189 670
11	.0070 305	.0428 378	.1347 910	.9198 095	.0187 932
12	.0069 564	.0423 786	.1332 695	.9207 712	.0186 185
13	.0068 821	.0419 188	.1317 481	.9217 317	.0184 431
14	.0068 078	.0414 584	.1302 266	.9226 909	.0182 669
0.915	+0.0067 333	-0.0409 975	+0.1287 052	+0.9236 489	-0.0180 899
16	.0066 587	.0405 359	.1271 837	.9246 056	.0179 121
17	.0065 840	.0400 738	.1256 623	.9255 611	.0177 335
18	.0065 091	.0396 112	.1241 409	.9265 153	.0175 541
19	.0064 341	.0391 480	.1226 196	.9274 682	.0173 739
0.920	+0.0063 590	-0.0386 842	+0.1210 982	+0.9284 198	-0.0171 930
21	.0062 838	.0382 198	.1195 769	.9293 702	.0170 112
22	.0062 085	.0377 549	.1180 557	.9303 193	.0168 286
23	.0061 330	.0372 894	.1165 345	.9312 671	.0166 452
24	.0060 575	.0368 234	.1150 134	.9322 136	.0164 610
0.925	+0.0059 818	-0.0363 568	+0.1134 923	+0.9331 588	-0.0162 760
26	.0059 060	.0358 897	.1119 713	.9341 027	.0160 902
27	.0058 301	.0354 221	.1104 503	.9350 453	.0159 036
28	.0057 540	.0349 539	.1089 295	.9359 866	.0157 162
29	.0056 779	.0344 851	.1074 087	.9369 265	.0155 280
0.930	+0.0056 016	-0.0340 158	+0.1058 880	+0.9378 652	-0.0153 389
31	.0055 252	.0335 460	.1043 674	.9388 025	.0151 491
32	.0054 487	.0330 757	.1028 469	.9397 385	.0149 584
33	.0053 721	.0326 048	.1013 265	.9406 731	.0147 669
34	.0052 953	.0321 334	.0998 062	.9416 064	.0145 746
0.935	+0.0052 185	-0.0316 615	+0.0982 861	+0.9425 384	-0.0143 815
36	.0051 415	.0311 891	.0967 661	.9434 690	.0141 875
37	.0050 644	.0307 161	.0952 461	.9443 982	.0139 927
38	.0049 873	.0302 426	.0937 264	.9453 261	.0137 971
39	.0049 100	.0297 687	.0922 067	.9462 527	.0136 007
0.940	+0.0048 325	-0.0292 942	+0.0906 872	+0.9471 778	-0.0134 035
41	.0047 550	.0288 192	.0891 679	.9481 016	.0132 054
42	.0046 774	.0283 437	.0876 487	.9490 240	.0130 065
43	.0045 996	.0278 677	.0861 297	.9499 451	.0128 067
44	.0045 218	.0273 912	.0846 108	.9508 647	.0126 062
0.945	+0.0044 438	-0.0269 142	+0.0830 922	+0.9517 830	-0.0124 048
46	.0043 657	.0264 367	.0815 737	.9526 998	.0122 025
47	.0042 875	.0259 587	.0800 553	.9536 152	.0119 994
48	.0042 093	.0254 802	.0785 372	.9545 293	.0117 955
49	.0041 309	.0250 013	.0770 193	.9554 419	.0115 908
0.950	+0.0040 523	-0.0245 219	+0.0755 016	+0.9563 531	-0.0113 852

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.950	+0.0040 523	-0.0245 219	+0.0755 016	+0.9563 531	-0.0113 852
51	.0039 737	.0240 420	.0739 840	.9572 629	.0111 787
52	.0038 950	.0235 616	.0724 667	.9581 713	.0109 714
53	.0038 162	.0230 807	.0709 496	.9590 782	.0107 633
54	.0037 372	.0225 994	.0694 328	.9599 837	.0105 543
0.955	+0.0036 582	-0.0221 176	+0.0679 162	+0.9608 878	-0.0103 445
56	.0035 791	.0216 354	.0663 998	.9617 904	.0101 338
57	.0034 998	.0211 526	.0648 836	.9626 915	.0099 223
58	.0034 205	.0206 695	.0633 677	.9635 912	.0097 099
59	.0033 410	.0201 858	.0618 520	.9644 895	.0094 967
0.960	+0.0032 614	-0.0197 018	+0.0603 366	+0.9653 862	-0.0092 826
61	.0031 818	.0192 172	.0588 215	.9662 815	.0090 676
62	.0031 020	.0187 322	.0573 066	.9671 754	.0088 518
63	.0030 221	.0182 468	.0557 921	.9680 677	.0086 351
64	.0029 422	.0177 610	.0542 778	.9689 586	.0084 176
0.965	+0.0028 621	-0.0172 746	+0.0527 638	+0.9698 480	-0.0081 992
66	.0027 819	.0167 879	.0512 500	.9707 359	.0079 799
67	.0027 017	.0163 007	.0497 366	.9716 222	.0077 598
68	.0026 213	.0158 131	.0482 235	.9725 071	.0075 388
69	.0025 408	.0153 251	.0467 107	.9733 905	.0073 170
0.970	+0.0024 603	-0.0148 366	+0.0451 982	+0.9742 724	-0.0070 942
71	.0023 796	.0143 478	.0436 860	.9751 527	.0068 706
72	.0022 989	.0138 585	.0421 742	.9760 315	.0066 461
73	.0022 180	.0133 687	.0406 627	.9769 088	.0064 208
74	.0021 371	.0128 786	.0391 515	.9777 846	.0061 945
0.975	+0.0020 560	-0.0123 881	+0.0376 407	+0.9786 588	-0.0059 674
76	.0019 749	.0118 971	.0361 303	.9795 315	.0057 394
77	.0018 936	.0114 058	.0346 202	.9804 026	.0055 106
78	.0018 123	.0109 140	.0331 104	.9812 722	.0052 808
79	.0017 309	.0104 219	.0316 010	.9821 402	.0050 502
0.980	+0.0016 493	-0.0099 294	+0.0300 920	+0.9830 066	-0.0048 187
81	.0015 677	.0094 364	.0285 834	.9838 715	.0045 863
82	.0014 860	.0089 431	.0270 752	.9847 348	.0043 530
83	.0014 042	.0084 494	.0255 674	.9855 966	.0041 188
84	.0013 223	.0079 553	.0240 599	.9864 567	.0038 837
0.985	+0.0012 403	-0.0074 608	+0.0225 529	+0.9873 153	-0.0036 477
86	.0011 583	.0069 660	.0210 463	.9881 723	.0034 109
87	.0010 761	.0064 707	.0195 401	.9890 277	.0031 731
88	.0009 939	.0059 751	.0180 343	.9898 814	.0029 344
89	.0009 115	.0054 792	.0165 289	.9907 336	.0026 949
0.990	+0.0008 291	-0.0049 828	+0.0150 240	+0.9915 842	-0.0024 544
91	.0007 466	.0044 861	.0135 195	.9924 331	.0022 131
92	.0006 640	.0039 891	.0120 155	.9932 804	.0019 708
93	.0005 813	.0034 917	.0105 119	.9941 261	.0017 276
94	.0004 985	.0029 939	.0090 088	.9949 702	.0014 836
0.995	+0.0004 156	-0.0024 958	+0.0075 061	+0.9958 126	-0.0012 386
96	.0003 327	.0019 973	.0060 039	.9966 534	.0009 927
97	.0002 496	.0014 985	.0045 022	.9974 925	.0007 459
98	.0001 665	.0009 993	.0030 010	.9983 300	.0004 982
99	.0000 833	.0004 998	.0015 002	+0.9991 658	.0002 495
1.000	+0.0000 000	-0.0000 000	+0.0000 000	+1.0000 000	-0.0000 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.000	-0.0000 000	+0.0000 000	-0.0000 000	+1.0000 000	+0.0000 000
01	.0000 834	.0005 002	.0014 997	.0008 325	.0002 505
02	.0001 668	.0010 007	.0029 990	.0016 633	.0005 018
03	.0002 504	.0015 015	.0044 977	.0024 925	.0007 541
04	.0003 340	.0020 026	.0059 959	.0033 199	.0010 073
1.005	-0.0004 177	+0.0025 041	-0.0074 936	+1.0041 457	+0.0012 615
06	.0005 015	.0030 059	.0089 908	.0049 698	.0015 166
07	.0005 853	.0035 080	.0104 874	.0057 922	.0017 725
08	.0006 693	.0040 104	.0119 835	.0066 129	.0020 295
09	.0007 533	.0045 131	.0134 790	.0074 319	.0022 873
1.010	-0.0008 374	+0.0050 162	-0.0149 740	+1.0082 492	+0.0025 461
11	.0009 216	.0055 195	.0164 684	.0090 647	.0028 058
12	.0010 059	.0060 231	.0179 623	.0098 786	.0030 664
13	.0010 902	.0065 271	.0194 555	.0106 907	.0033 280
14	.0011 746	.0070 313	.0209 482	.0115 010	.0035 905
1.015	-0.0012 591	+0.0075 358	-0.0224 404	+1.0123 097	+0.0038 540
16	.0013 437	.0080 406	.0239 319	.0131 166	.0041 184
17	.0014 283	.0085 457	.0254 228	.0139 217	.0043 837
18	.0015 130	.0090 511	.0269 131	.0147 251	.0046 500
19	.0015 978	.0095 567	.0284 029	.0155 268	.0049 172
1.020	-0.0016 827	+0.0100 626	-0.0298 920	+1.0163 266	+0.0051 853
21	.0017 676	.0105 688	.0313 804	.0171 248	.0054 544
22	.0018 526	.0110 753	.0328 683	.0179 211	.0057 245
23	.0019 377	.0115 820	.0343 555	.0187 156	.0059 955
24	.0020 228	.0120 890	.0358 421	.0195 084	.0062 675
1.025	-0.0021 081	+0.0125 963	-0.0373 280	+1.0202 994	+0.0065 404
26	.0021 933	.0131 038	.0388 133	.0210 886	.0068 142
27	.0022 787	.0136 116	.0402 979	.0218 760	.0070 891
28	.0023 641	.0141 196	.0417 819	.0226 616	.0073 648
29	.0024 496	.0146 279	.0432 652	.0234 454	.0076 416
1.030	-0.0025 352	+0.0151 364	-0.0447 478	+1.0242 274	+0.0079 193
31	.0026 209	.0156 451	.0462 297	.0250 075	.0081 979
32	.0027 066	.0161 541	.0477 110	.0257 859	.0084 776
33	.0027 923	.0166 633	.0491 915	.0265 624	.0087 582
34	.0028 782	.0171 728	.0506 714	.0273 370	.0090 397
1.035	-0.0029 641	+0.0176 825	-0.0521 505	+1.0281 099	+0.0093 222
36	.0030 500	.0181 924	.0536 289	.0288 808	.0096 057
37	.0031 361	.0187 025	.0551 066	.0296 500	.0098 902
38	.0032 222	.0192 129	.0565 836	.0304 173	.0101 756
39	.0033 083	.0197 235	.0580 599	.0311 827	.0104 621
1.040	-0.0033 946	+0.0202 342	-0.0595 354	+1.0319 462	+0.0107 494
41	.0034 808	.0207 452	.0610 101	.0327 079	.0110 378
42	.0035 672	.0212 564	.0624 841	.0334 677	.0113 272
43	.0036 536	.0217 678	.0639 574	.0342 257	.0116 175
44	.0037 401	.0222 794	.0654 299	.0349 817	.0119 088
1.045	-0.0038 266	+0.0227 913	-0.0669 016	+1.0357 359	+0.0122 011
46	.0039 132	.0233 033	.0683 725	.0364 881	.0124 944
47	.0039 999	.0238 154	.0698 427	.0372 385	.0127 886
48	.0040 866	.0243 278	.0713 121	.0379 870	.0130 839
49	.0041 733	.0248 404	.0727 807	.0387 335	.0133 801
1.050	-0.0042 602	+0.0253 531	-0.0742 484	+1.0394 781	+0.0136 773

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.050	-0.0042 602	+0.0253 531	-0.0742 484	+1.0394 781	+0.0136 773
51	.0043 470	.0258 660	.0757 154	.0402 208	.0139 756
52	.0044 340	.0263 791	.0771 816	.0409 616	.0142 748
53	.0045 210	.0268 924	.0786 469	.0417 005	.0145 750
54	.0046 080	.0274 059	.0801 114	.0424 374	.0148 762
1.055	-0.0046 951	+0.0279 195	-0.0815 751	+1.0431 723	+0.0151 784
56	.0047 823	.0284 332	.0830 379	.0439 053	.0154 816
57	.0048 695	.0289 471	.0844 999	.0446 364	.0157 859
58	.0049 568	.0294 612	.0859 611	.0453 655	.0160 911
59	.0050 441	.0299 755	.0874 213	.0460 927	.0163 973
1.060	-0.0051 315	+0.0304 898	-0.0888 808	+1.0468 178	+0.0167 045
61	.0052 189	.0310 044	.0903 393	.0475 410	.0170 128
62	.0053 064	.0315 190	.0917 970	.0482 623	.0173 220
63	.0053 939	.0320 339	.0932 538	.0489 815	.0176 323
64	.0054 815	.0325 488	.0947 097	.0496 988	.0179 436
1.065	-0.0055 691	+0.0330 639	-0.0961 647	+1.0504 140	+0.0182 559
66	.0056 568	.0335 791	.0976 188	.0511 273	.0185 692
67	.0057 445	.0340 944	.0990 719	.0518 385	.0188 835
68	.0058 322	.0346 099	.1005 242	.0525 477	.0191 988
69	.0059 201	.0351 255	.1019 756	.0532 550	.0195 152
1.070	-0.0060 079	+0.0356 412	-0.1034 260	+1.0539 602	+0.0198 326
71	.0060 958	.0361 570	.1048 755	.0546 633	.0201 510
72	.0061 838	.0366 729	.1063 240	.0553 645	.0204 704
73	.0062 718	.0371 889	.1077 716	.0560 636	.0207 909
74	.0063 598	.0377 051	.1092 183	.0567 606	.0211 124
1.075	-0.0064 479	+0.0382 213	-0.1106 640	+1.0574 557	+0.0214 349
76	.0065 360	.0387 376	.1121 087	.0581 486	.0217 585
77	.0066 242	.0392 540	.1135 524	.0588 395	.0220 831
78	.0067 124	.0397 706	.1149 952	.0595 284	.0224 087
79	.0068 007	.0402 872	.1164 370	.0602 151	.0227 353
1.080	-0.0068 890	+0.0408 038	-0.1178 778	+1.0608 998	+0.0230 630
81	.0069 773	.0413 206	.1193 175	.0615 825	.0233 918
82	.0070 657	.0418 374	.1207 563	.0622 630	.0237 216
83	.0071 541	.0423 544	.1221 941	.0629 414	.0240 524
84	.0072 425	.0428 714	.1236 308	.0636 178	.0243 843
1.085	-0.0073 310	+0.0433 884	-0.1250 666	+1.0642 920	+0.0247 172
86	.0074 195	.0439 055	.1265 013	.0649 642	.0250 511
87	.0075 081	.0444 227	.1279 349	.0656 342	.0253 861
88	.0075 967	.0449 399	.1293 675	.0663 021	.0257 222
89	.0076 853	.0454 572	.1307 991	.0669 679	.0260 593
1.090	-0.0077 740	+0.0459 746	-0.1322 296	+1.0676 316	+0.0263 975
91	.0078 627	.0464 920	.1336 590	.0682 931	.0267 367
92	.0079 515	.0470 094	.1350 874	.0689 525	.0270 770
93	.0080 402	.0475 269	.1365 147	.0696 097	.0274 183
94	.0081 290	.0480 444	.1379 409	.0702 648	.0277 607
1.095	-0.0082 179	+0.0485 619	-0.1393 660	+1.0709 178	+0.0281 042
96	.0083 067	.0490 795	.1407 900	.0715 686	.0284 487
97	.0083 956	.0495 971	.1422 129	.0722 172	.0287 943
98	.0084 846	.0501 147	.1436 347	.0728 636	.0291 410
99	.0085 735	.0506 323	.1450 554	.0735 079	.0294 887
1.100	-0.0086 625	+0.0511 500	-0.1464 750	+1.0741 500	+0.0298 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.100	-0.0086 625	+0.0511 500	-0.1464 750	+1.0741 500	+0.0298 375
01	.0087 515	.0516 677	.1478 934	.0747 899	.0301 874
02	.0088 406	.0521 854	.1493 107	.0754 276	.0305 383
03	.0089 296	.0527 030	.1507 269	.0760 631	.0308 903
04	.0090 187	.0532 207	.1521 419	.0766 964	.0312 434
1.105	-0.0091 078	+0.0537 384	-0.1535 557	+1.0773 276	+0.0315 976
06	.0091 970	.0542 561	.1549 684	.0779 564	.0319 528
07	.0092 862	.0547 738	.1563 799	.0785 831	.0323 092
08	.0093 754	.0552 915	.1577 903	.0792 076	.0326 666
09	.0094 646	.0558 091	.1591 994	.0798 298	.0330 251
1.110	-0.0095 538	+0.0563 268	-0.1606 074	+1.0804 498	+0.0333 847
11	.0096 431	.0568 444	.1620 142	.0810 675	.0337 454
12	.0097 324	.0573 620	.1634 197	.0816 830	.0341 071
13	.0098 217	.0578 795	.1648 241	.0822 962	.0344 700
14	.0099 110	.0583 971	.1662 272	.0829 072	.0348 339
1.115	-0.0100 003	+0.0589 146	-0.1676 291	+1.0835 160	+0.0351 990
16	.0100 897	.0594 320	.1690 298	.0841 224	.0355 651
17	.0101 791	.0599 495	.1704 293	.0847 266	.0359 323
18	.0102 685	.0604 668	.1718 275	.0853 285	.0363 007
19	.0103 579	.0609 842	.1732 245	.0859 281	.0366 701
1.120	-0.0104 474	+0.0615 014	-0.1746 202	+1.0865 254	+0.0370 406
21	.0105 368	.0620 187	.1760 146	.0871 205	.0374 123
22	.0106 263	.0625 358	.1774 078	.0877 132	.0377 850
23	.0107 158	.0630 529	.1787 997	.0883 036	.0381 589
24	.0108 053	.0635 700	.1801 903	.0888 917	.0385 338
1.125	-0.0108 948	+0.0640 869	-0.1815 796	+1.0894 775	+0.0389 099
26	.0109 843	.0646 038	.1829 676	.0900 610	.0392 871
27	.0110 738	.0651 206	.1843 543	.0906 422	.0396 654
28	.0111 634	.0656 374	.1857 397	.0912 210	.0400 448
29	.0112 529	.0661 540	.1871 238	.0917 974	.0404 253
1.130	-0.0113 425	+0.0666 706	-0.1885 066	+1.0923 716	+0.0408 070
31	.0114 321	.0671 870	.1898 880	.0929 433	.0411 898
32	.0115 217	.0677 034	.1912 681	.0935 128	.0415 736
33	.0116 113	.0682 197	.1926 469	.0940 798	.0419 587
34	.0117 009	.0687 359	.1940 243	.0946 445	.0423 448
1.135	-0.0117 905	+0.0692 520	-0.1954 003	+1.0952 068	+0.0427 321
36	.0118 801	.0697 679	.1967 750	.0957 668	.0431 205
37	.0119 698	.0702 838	.1981 483	.0963 243	.0435 100
38	.0120 594	.0707 995	.1995 203	.0968 795	.0439 006
39	.0121 490	.0713 151	.2008 908	.0974 323	.0442 924
1.140	-0.0122 387	+0.0718 306	-0.2022 600	+1.0979 826	+0.0446 853
41	.0123 283	.0723 460	.2036 277	.0985 306	.0450 794
42	.0124 180	.0728 613	.2049 941	.0990 762	.0454 746
43	.0125 076	.0733 764	.2063 590	.0996 193	.0458 709
44	.0125 973	.0738 913	.2077 225	.1001 600	.0462 684
1.145	-0.0126 869	+0.0744 062	-0.2090 846	+1.1006 983	+0.0466 670
46	.0127 766	.0749 209	.2104 453	.1012 342	.0470 668
47	.0128 662	.0754 354	.2118 045	.1017 676	.0474 677
48	.0129 559	.0759 498	.2131 623	.1022 985	.0478 698
49	.0130 455	.0764 640	.2145 186	.1028 271	.0482 730
1.150	-0.0131 352	+0.0769 781	-0.2158 734	+1.1033 531	+0.0486 773

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.150	-0.0131 352	+0.0769 781	-0.2158 734	+1.1033 531	+0.0486 773
51	.0132 248	.0774 920	.2172 268	.1038 767	.0490 829
52	.0133 144	.0780 058	.2185 787	.1043 979	.0494 895
53	.0134 041	.0785 194	.2199 292	.1049 165	.0498 974
54	.0134 937	.0790 328	.2212 781	.1054 327	.0503 063
1.155	-0.0135 833	+0.0795 460	-0.2226 256	+1.1059 464	+0.0507 165
56	.0136 730	.0800 591	.2239 715	.1064 576	.0511 278
57	.0137 626	.0805 720	.2253 160	.1069 663	.0515 402
58	.0138 522	.0810 846	.2266 589	.1074 725	.0519 539
59	.0139 418	.0815 971	.2280 003	.1079 762	.0523 687
1.160	-0.0140 314	+0.0821 094	-0.2293 402	+1.1084 774	+0.0527 846
61	.0141 209	.0826 215	.2306 785	.1089 761	.0532 018
62	.0142 105	.0831 334	.2320 153	.1094 723	.0536 201
63	.0143 001	.0836 451	.2333 505	.1099 659	.0540 396
64	.0143 896	.0841 566	.2346 842	.1104 570	.0544 602
1.165	-0.0144 791	+0.0846 679	-0.2360 163	+1.1109 455	+0.0548 820
66	.0145 687	.0851 790	.2373 469	.1114 315	.0553 050
67	.0146 582	.0856 898	.2386 758	.1119 150	.0557 292
68	.0147 477	.0862 004	.2400 032	.1123 959	.0561 546
69	.0148 372	.0867 108	.2413 290	.1128 742	.0565 811
1.170	-0.0149 266	+0.0872 210	-0.2426 532	+1.1133 500	+0.0570 089
71	.0150 161	.0877 309	.2439 758	.1138 232	.0574 378
72	.0151 055	.0882 406	.2452 967	.1142 938	.0578 679
73	.0151 949	.0887 500	.2466 161	.1147 618	.0582 992
74	.0152 843	.0892 592	.2479 338	.1152 272	.0587 317
1.175	-0.0153 737	+0.0897 682	-0.2492 499	+1.1156 900	+0.0591 654
76	.0154 630	.0902 769	.2505 643	.1161 503	.0596 003
77	.0155 524	.0907 853	.2518 771	.1166 079	.0600 363
78	.0156 417	.0912 935	.2531 883	.1170 629	.0604 736
79	.0157 310	.0918 014	.2544 978	.1175 153	.0609 121
1.180	-0.0158 203	+0.0923 090	-0.2558 056	+1.1179 650	+0.0613 517
81	.0159 095	.0928 164	.2571 117	.1184 122	.0617 926
82	.0159 987	.0933 235	.2584 161	.1188 567	.0622 347
83	.0160 879	.0938 303	.2597 189	.1192 985	.0626 780
84	.0161 771	.0943 369	.2610 199	.1197 377	.0631 225
1.185	-0.0162 663	+0.0948 431	-0.2623 193	+1.1201 743	+0.0635 682
86	.0163 554	.0953 491	.2636 169	.1206 081	.0640 151
87	.0164 445	.0958 548	.2649 128	.1210 394	.0644 632
88	.0165 336	.0963 601	.2662 070	.1214 679	.0649 126
89	.0166 226	.0968 652	.2674 995	.1218 938	.0653 631
1.190	-0.0167 116	+0.0973 700	-0.2687 902	+1.1223 170	+0.0658 149
91	.0168 006	.0978 744	.2700 792	.1227 375	.0662 679
92	.0168 896	.0983 786	.2713 664	.1231 553	.0667 221
93	.0169 785	.0988 824	.2726 518	.1235 704	.0671 775
94	.0170 674	.0993 859	.2739 355	.1239 828	.0676 342
1.195	-0.0171 562	+0.0998 891	-0.2752 174	+1.1243 925	+0.0680 921
96	.0172 450	.1003 919	.2764 975	.1247 994	.0685 512
97	.0173 338	.1008 945	.2777 758	.1252 037	.0690 116
98	.0174 226	.1013 966	.2790 524	.1256 052	.0694 731
99	.0175 113	.1018 985	.2803 271	.1260 040	.0699 360
1.200	-0.0176 000	+0.1024 000	-0.2816 000	+1.1264 000	+0.0704 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.200	-0.0176 000	+0.1024 000	-0.2816 000	+1.1264 000	+0.0704 000
01	.0176 886	.1029 012	.2828 711	.1267 933	.0708 653
02	.0177 773	.1034 020	.2841 404	.1271 838	.0713 318
03	.0178 658	.1039 024	.2854 078	.1275 716	.0717 996
04	.0179 544	.1044 025	.2866 734	.1279 566	.0722 686
1.205	-0.0180 429	+0.1049 023	-0.2879 371	+1.1283 389	+0.0727 388
06	.0181 313	.1054 016	.2891 990	.1287 184	.0732 103
07	.0182 197	.1059 006	.2904 590	.1290 950	.0736 831
08	.0183 081	.1063 992	.2917 171	.1294 689	.0741 571
09	.0183 964	.1068 975	.2929 734	.1298 401	.0746 323
1.210	-0.0184 847	+0.1073 954	-0.2942 278	+1.1302 084	+0.0751 088
11	.0185 730	.1078 928	.2954 803	.1305 739	.0755 865
12	.0186 612	.1083 899	.2967 309	.1309 366	.0760 655
13	.0187 493	.1088 866	.2979 796	.1312 964	.0765 458
14	.0188 374	.1093 829	.2992 263	.1316 535	.0770 273
1.215	-0.0189 255	+0.1098 789	-0.3004 712	+1.1320 077	+0.0775 101
16	.0190 135	.1103 744	.3017 141	.1323 591	.0779 941
17	.0191 015	.1108 694	.3029 551	.1327 077	.0784 794
18	.0191 894	.1113 641	.3041 941	.1330 534	.0789 660
19	.0192 772	.1118 584	.3054 312	.1333 962	.0794 538
1.220	-0.0193 651	+0.1123 522	-0.3066 664	+1.1337 362	+0.0799 429
21	.0194 528	.1128 457	.3078 995	.1340 734	.0804 333
22	.0195 405	.1133 387	.3091 307	.1344 076	.0809 250
23	.0196 282	.1138 312	.3103 599	.1347 390	.0814 179
24	.0197 158	.1143 233	.3115 872	.1350 675	.0819 121
1.225	-0.0198 034	+0.1148 150	-0.3128 124	+1.1353 932	+0.0824 076
26	.0198 909	.1153 063	.3140 356	.1357 159	.0829 043
27	.0199 783	.1157 971	.3152 569	.1360 357	.0834 024
28	.0200 657	.1162 874	.3164 761	.1363 527	.0839 017
29	.0201 530	.1167 773	.3176 932	.1366 667	.0844 023
1.230	-0.0202 403	+0.1172 668	-0.3189 084	+1.1369 778	+0.0849 042
31	.0203 275	.1177 557	.3201 215	.1372 859	.0854 074
32	.0204 147	.1182 442	.3213 326	.1375 912	.0859 118
33	.0205 018	.1187 323	.3225 416	.1378 935	.0864 176
34	.0205 888	.1192 198	.3237 485	.1381 929	.0869 247
1.235	-0.0206 758	+0.1197 069	-0.3249 534	+1.1384 893	+0.0874 330
36	.0207 627	.1201 935	.3261 562	.1387 828	.0879 426
37	.0208 496	.1206 796	.3273 570	.1390 733	.0884 536
38	.0209 364	.1211 653	.3285 556	.1393 608	.0889 658
39	.0210 231	.1216 504	.3297 521	.1396 454	.0894 794
1.240	-0.0211 098	+0.1221 350	-0.3309 466	+1.1399 270	+0.0899 942
41	.0211 964	.1226 192	.3321 389	.1402 057	.0905 104
42	.0212 829	.1231 028	.3333 291	.1404 813	.0910 279
43	.0213 694	.1235 859	.3345 171	.1407 539	.0915 466
44	.0214 557	.1240 685	.3357 031	.1410 236	.0920 667
1.245	-0.0215 421	+0.1245 506	-0.3368 869	+1.1412 902	+0.0925 881
46	.0216 283	.1250 322	.3380 685	.1415 539	.0931 108
47	.0217 145	.1255 132	.3392 480	.1418 145	.0936 349
48	.0218 006	.1259 937	.3404 253	.1420 720	.0941 602
49	.0218 867	.1264 737	.3416 005	.1423 266	.0946 869
1.250	-0.0219 727	+0.1269 531	-0.3427 734	+1.1425 781	+0.0952 148

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.250	-0.0219 727	+0.1269 531	-0.3427 734	+1.1425 781	+0.0952 148
51	.0220 586	.1274 320	.3439 442	.1428 266	.0957 442
52	.0221 444	.1279 104	.3451 128	.1430 720	.0962 748
53	.0222 301	.1283 882	.3462 792	.1433 144	.0968 067
54	.0223 158	.1288 654	.3474 434	.1435 537	.0973 400
1.255	-0.0224 014	+0.1293 421	-0.3486 053	+1.1437 900	+0.0978 746
56	.0224 869	.1298 182	.3497 650	.1440 232	.0984 106
57	.0225 724	.1302 938	.3509 225	.1442 533	.0989 479
58	.0226 578	.1307 688	.3520 778	.1444 803	.0994 865
59	.0227 430	.1312 432	.3532 308	.1447 042	.1000 264
1.260	-0.0228 283	+0.1317 170	-0.3543 816	+1.1449 250	+0.1005 677
61	.0229 134	.1321 903	.3555 301	.1451 428	.1011 104
62	.0229 984	.1326 630	.3566 763	.1453 574	.1016 543
63	.0230 834	.1331 351	.3578 202	.1455 689	.1021 997
64	.0231 683	.1336 065	.3589 619	.1457 773	.1027 463
1.265	-0.0232 531	+0.1340 774	-0.3601 012	+1.1459 826	+0.1032 943
66	.0233 378	.1345 477	.3612 383	.1461 847	.1038 437
67	.0234 224	.1350 174	.3623 731	.1463 837	.1043 944
68	.0235 070	.1354 865	.3635 055	.1465 795	.1049 465
69	.0235 914	.1359 549	.3646 356	.1467 722	.1054 999
1.270	-0.0236 758	+0.1364 228	-0.3657 634	+1.1469 618	+0.1060 547
71	.0237 601	.1368 900	.3668 888	.1471 481	.1066 108
72	.0238 443	.1373 566	.3680 119	.1473 314	.1071 683
73	.0239 284	.1378 225	.3691 327	.1475 114	.1077 272
74	.0240 124	.1382 879	.3702 511	.1476 882	.1082 874
1.275	-0.0240 963	+0.1387 525	-0.3713 671	+1.1478 619	+0.1088 490
76	.0241 802	.1392 166	.3724 807	.1480 324	.1094 119
77	.0242 639	.1396 800	.3735 920	.1481 997	.1099 762
78	.0243 476	.1401 427	.3747 008	.1483 637	.1105 419
79	.0244 311	.1406 048	.3758 073	.1485 246	.1111 090
1.280	-0.0245 146	+0.1410 662	-0.3769 114	+1.1486 822	+0.1116 774
81	.0245 979	.1415 270	.3780 130	.1488 367	.1122 473
82	.0246 812	.1419 871	.3791 122	.1489 878	.1128 184
83	.0247 643	.1424 465	.3802 090	.1491 358	.1133 910
84	.0248 474	.1429 053	.3813 033	.1492 805	.1139 650
1.285	-0.0249 304	+0.1433 634	-0.3823 952	+1.1494 220	+0.1145 403
86	.0250 133	.1438 207	.3834 847	.1495 602	.1151 170
87	.0250 960	.1442 774	.3845 717	.1496 951	.1156 951
88	.0251 787	.1447 334	.3856 562	.1498 268	.1162 746
89	.0252 613	.1451 888	.3867 382	.1499 552	.1168 555
1.290	-0.0253 437	+0.1456 434	-0.3878 178	+1.1500 804	+0.1174 378
91	.0254 261	.1460 973	.3888 949	.1502 022	.1180 215
92	.0255 083	.1465 505	.3899 694	.1503 208	.1186 065
93	.0255 905	.1470 029	.3910 415	.1504 360	.1191 930
94	.0256 725	.1474 547	.3921 110	.1505 480	.1197 808
1.295	-0.0257 545	+0.1479 058	-0.3931 780	+1.1506 566	+0.1203 701
96	.0258 363	.1483 561	.3942 425	.1507 620	.1209 608
97	.0259 180	.1488 057	.3953 045	.1508 640	.1215 528
98	.0259 996	.1492 545	.3963 639	.1509 626	.1221 463
99	.0260 811	.1497 026	.3974 207	.1510 580	.1227 412
1.300	-0.0261 625	+0.1501 500	-0.3984 750	+1.1511 500	+0.1233 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.300	-0.0261 625	+0.1501 500	-0.3984 750	+1.1511 500	+0.1233 375
01	.0262 438	.1505 966	.3995 267	.1512 387	.1239 352
02	.0263 249	.1510 425	.4005 758	.1513 240	.1245 343
03	.0264 060	.1514 876	.4016 224	.1514 059	.1251 349
04	.0264 869	.1519 320	.4026 664	.1514 845	.1257 368
1.305	-0.0265 678	+0.1523 756	-0.4037 077	+1.1515 597	+0.1263 402
06	.0266 485	.1528 184	.4047 465	.1516 315	.1269 450
07	.0267 291	.1532 605	.4057 826	.1517 000	.1275 512
08	.0268 095	.1537 017	.4068 161	.1517 650	.1281 588
09	.0268 899	.1541 422	.4078 470	.1518 267	.1287 679
1.310	-0.0269 701	+0.1545 820	-0.4088 752	+1.1518 850	+0.1293 784
11	.0270 502	.1550 209	.4099 008	.1519 398	.1299 903
12	.0271 302	.1554 590	.4109 237	.1519 913	.1306 037
13	.0272 101	.1558 964	.4119 440	.1520 393	.1312 184
14	.0272 899	.1563 329	.4129 616	.1520 839	.1318 347
1.315	-0.0273 695	+0.1567 686	-0.4139 765	+1.1521 250	+0.1324 523
16	.0274 490	.1572 035	.4149 887	.1521 627	.1330 714
17	.0275 284	.1576 377	.4159 982	.1521 970	.1336 920
18	.0276 076	.1580 709	.4170 051	.1522 278	.1343 139
19	.0276 868	.1585 034	.4180 092	.1522 552	.1349 374
1.320	-0.0277 658	+0.1589 350	-0.4190 106	+1.1522 790	+0.1355 622
21	.0278 446	.1593 658	.4200 092	.1522 995	.1361 886
22	.0279 234	.1597 958	.4210 052	.1523 164	.1368 163
23	.0280 020	.1602 249	.4219 983	.1523 299	.1374 455
24	.0280 805	.1606 532	.4229 888	.1523 398	.1380 762
1.325	-0.0281 588	+0.1610 807	-0.4239 765	+1.1523 463	+0.1387 083
26	.0282 371	.1615 072	.4249 614	.1523 493	.1393 419
27	.0283 152	.1619 330	.4259 435	.1523 487	.1399 770
28	.0283 931	.1623 578	.4269 229	.1523 447	.1406 135
29	.0284 709	.1627 818	.4278 994	.1523 371	.1412 515
1.330	-0.0285 486	+0.1632 050	-0.4288 732	+1.1523 260	+0.1418 909
31	.0286 262	.1636 272	.4298 442	.1523 113	.1425 318
32	.0287 036	.1640 486	.4308 123	.1522 931	.1431 741
33	.0287 809	.1644 691	.4317 776	.1522 714	.1438 180
34	.0288 580	.1648 887	.4327 401	.1522 461	.1444 633
1.335	-0.0289 350	+0.1653 074	-0.4336 998	+1.1522 173	+0.1451 101
36	.0290 118	.1657 252	.4346 566	.1521 849	.1457 583
37	.0290 886	.1661 421	.4356 105	.1521 489	.1464 081
38	.0291 651	.1665 581	.4365 616	.1521 093	.1470 593
39	.0292 416	.1669 733	.4375 098	.1520 662	.1477 120
1.340	-0.0293 179	+0.1673 874	-0.4384 552	+1.1520 194	+0.1483 661
41	.0293 940	.1678 007	.4393 976	.1519 691	.1490 218
42	.0294 700	.1682 131	.4403 372	.1519 152	.1496 789
43	.0295 459	.1686 245	.4412 738	.1518 576	.1503 376
44	.0296 216	.1690 350	.4422 076	.1517 964	.1509 977
1.345	-0.0296 971	+0.1694 445	-0.4431 384	+1.1517 317	+0.1516 593
46	.0297 725	.1698 531	.4440 663	.1516 632	.1523 224
47	.0298 478	.1702 608	.4449 912	.1515 912	.1529 870
48	.0299 229	.1706 675	.4459 133	.1515 155	.1536 531
49	.0299 979	.1710 733	.4468 323	.1514 361	.1543 207
1.350	-0.0300 727	+0.1714 781	-0.4477 484	+1.1513 531	+0.1549 898

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.350	-0.0300 727	+0.1714 781	-0.4477 484	+1.1513 531	+0.1549 898
51	.0301 473	.1718 820	.4486 616	.1512 665	.1556 604
52	.0302 218	.1722 849	.4495 717	.1511 761	.1563 326
53	.0302 962	.1726 868	.4504 789	.1510 821	.1570 062
54	.0303 703	.1730 877	.4513 831	.1509 844	.1576 813
1.355	-0.0304 444	+0.1734 877	-0.4522 843	+1.1508 831	+0.1583 579
56	.0305 182	.1738 867	.4531 825	.1507 780	.1590 361
57	.0305 920	.1742 846	.4540 776	.1506 692	.1597 158
58	.0306 655	.1746 816	.4549 698	.1505 567	.1603 969
59	.0307 389	.1750 776	.4558 589	.1504 405	.1610 796
1.360	-0.0308 122	+0.1754 726	-0.4567 450	+1.1503 206	+0.1617 638
61	.0308 852	.1758 666	.4576 280	.1501 970	.1624 496
62	.0309 582	.1762 596	.4585 079	.1500 696	.1631 368
63	.0310 309	.1766 516	.4593 848	.1499 385	.1638 256
64	.0311 035	.1770 425	.4602 587	.1498 037	.1645 159
1.365	-0.0311 759	+0.1774 325	-0.4611 294	+1.1496 651	+0.1652 078
66	.0312 482	.1778 214	.4619 971	.1495 227	.1659 011
67	.0313 203	.1782 092	.4628 616	.1493 766	.1665 961
68	.0313 922	.1785 960	.4637 231	.1492 267	.1672 925
69	.0314 639	.1789 818	.4645 814	.1490 730	.1679 905
1.370	-0.0315 355	+0.1793 666	-0.4654 366	+1.1489 156	+0.1686 900
71	.0316 069	.1797 503	.4662 887	.1487 543	.1693 910
72	.0316 782	.1801 329	.4671 376	.1485 893	.1700 936
73	.0317 492	.1805 145	.4679 834	.1484 204	.1707 978
74	.0318 201	.1808 950	.4688 261	.1482 478	.1715 035
1.375	-0.0318 909	+0.1812 744	-0.4696 655	+1.1480 713	+0.1722 107
76	.0319 614	.1816 528	.4705 018	.1478 910	.1729 195
77	.0320 318	.1820 301	.4713 350	.1477 069	.1736 298
78	.0321 020	.1824 063	.4721 649	.1475 189	.1743 417
79	.0321 720	.1827 814	.4729 916	.1473 271	.1750 551
1.380	-0.0322 419	+0.1831 554	-0.4738 152	+1.1471 314	+0.1757 701
81	.0323 115	.1835 284	.4746 355	.1469 319	.1764 867
82	.0323 810	.1839 002	.4754 526	.1467 286	.1772 048
83	.0324 503	.1842 709	.4762 664	.1465 213	.1779 245
84	.0325 194	.1846 406	.4770 771	.1463 102	.1786 457
1.385	-0.0325 884	+0.1850 091	-0.4778 845	+1.1460 952	+0.1793 686
86	.0326 571	.1853 765	.4786 886	.1458 763	.1800 929
87	.0327 257	.1857 427	.4794 895	.1456 535	.1808 189
88	.0327 941	.1861 079	.4802 871	.1454 269	.1815 464
89	.0328 623	.1864 719	.4810 814	.1451 963	.1822 755
1.390	-0.0329 303	+0.1868 348	-0.4818 724	+1.1449 618	+0.1830 062
91	.0329 981	.1871 965	.4826 601	.1447 233	.1837 384
92	.0330 658	.1875 571	.4834 446	.1444 810	.1844 723
93	.0331 332	.1879 165	.4842 257	.1442 347	.1852 077
94	.0332 005	.1882 748	.4850 035	.1439 845	.1859 447
1.395	-0.0332 676	+0.1886 319	-0.4857 779	+1.1437 303	+0.1866 833
96	.0333 344	.1889 879	.4865 490	.1434 722	.1874 234
97	.0334 011	.1893 427	.4873 168	.1432 101	.1881 652
98	.0334 676	.1896 963	.4880 813	.1429 440	.1889 085
99	.0335 339	.1900 487	.4888 423	.1426 740	.1896 535
1.400	-0.0336 000	+0.1904 000	-0.4896 000	+1.1424 000	+0.1904 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.400	-0.0336 000	+0.1904 000	-0.4896 000	+1.1424 000	+0.1904 000
01	.0336 659	.1907 501	.4903 543	.1421 220	.1911 481
02	.0337 316	.1910 990	.4911 052	.1418 400	.1918 979
03	.0337 971	.1914 466	.4918 528	.1415 540	.1926 492
04	.0338 624	.1917 931	.4925 969	.1412 640	.1934 021
1.405	-0.0339 275	+0.1921 384	-0.4933 376	+1.1409 700	+0.1941 567
06	.0339 924	.1924 825	.4940 749	.1406 720	.1949 128
07	.0340 571	.1928 253	.4948 087	.1403 699	.1956 706
08	.0341 216	.1931 670	.4955 391	.1400 639	.1964 299
09	.0341 859	.1935 074	.4962 661	.1397 537	.1971 909
1.410	-0.0342 500	+0.1938 466	-0.4969 896	+1.1394 396	+0.1979 535
11	.0343 139	.1941 845	.4977 096	.1391 213	.1987 177
12	.0343 776	.1945 212	.4984 262	.1387 991	.1994 835
13	.0344 411	.1948 567	.4991 393	.1384 727	.2002 509
14	.0345 043	.1951 909	.4998 489	.1381 423	.2010 200
1.415	-0.0345 674	+0.1955 239	-0.5005 550	+1.1378 078	+0.2017 907
16	.0346 302	.1958 556	.5012 576	.1374 692	.2025 630
17	.0346 928	.1961 861	.5019 567	.1371 265	.2033 369
18	.0347 553	.1965 153	.5026 522	.1367 797	.2041 125
19	.0348 175	.1968 432	.5033 443	.1364 288	.2048 897
1.420	-0.0348 795	+0.1971 698	-0.5040 328	+1.1360 738	+0.2056 685
21	.0349 412	.1974 952	.5047 177	.1357 147	.2064 490
22	.0350 028	.1978 193	.5053 991	.1353 515	.2072 311
23	.0350 641	.1981 421	.5060 769	.1349 841	.2080 148
24	.0351 253	.1984 636	.5067 511	.1346 126	.2088 002
1.425	-0.0351 862	+0.1987 838	-0.5074 218	+1.1342 369	+0.2095 873
26	.0352 469	.1991 027	.5080 888	.1338 571	.2103 759
27	.0353 073	.1994 203	.5087 523	.1334 731	.2111 662
28	.0353 676	.1997 366	.5094 122	.1330 850	.2119 582
29	.0354 276	.2000 515	.5100 684	.1326 927	.2127 518
1.430	-0.0354 874	+0.2003 652	-0.5107 210	+1.1322 962	+0.2135 471
31	.0355 470	.2006 775	.5113 700	.1318 955	.2143 440
32	.0356 063	.2009 885	.5120 153	.1314 906	.2151 426
33	.0356 655	.2012 981	.5126 570	.1310 815	.2159 428
34	.0357 244	.2016 064	.5132 950	.1306 682	.2167 447
1.435	-0.0357 830	+0.2019 134	-0.5139 293	+1.1302 508	+0.2175 483
36	.0358 415	.2022 190	.5145 600	.1298 290	.2183 535
37	.0358 997	.2025 232	.5151 870	.1294 031	.2191 604
38	.0359 577	.2028 261	.5158 103	.1289 729	.2199 689
39	.0360 154	.2031 277	.5164 299	.1285 385	.2207 791
1.440	-0.0360 730	+0.2034 278	-0.5170 458	+1.1280 998	+0.2215 910
41	.0361 302	.2037 266	.5176 579	.1276 569	.2224 046
42	.0361 873	.2040 240	.5182 663	.1272 097	.2232 199
43	.0362 441	.2043 201	.5188 710	.1267 583	.2240 368
44	.0363 007	.2046 147	.5194 720	.1263 026	.2248 554
1.445	-0.0363 570	+0.2049 080	-0.5200 691	+1.1258 426	+0.2256 757
46	.0364 131	.2051 998	.5206 626	.1253 783	.2264 976
47	.0364 690	.2054 902	.5212 522	.1249 097	.2273 213
48	.0365 246	.2057 793	.5218 381	.1244 368	.2281 466
49	.0365 800	.2060 669	.5224 202	.1239 596	.2289 736
1.450	-0.0366 352	+0.2063 531	-0.5229 984	+1.1234 781	+0.2298 023

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.450	-0.0366 352	+0.2063 531	-0.5229 984	+1.1234 781	+0.2298 023
51	.0366 901	.2066 379	.5235 729	.1229 923	.2306 328
52	.0367 447	.2069 213	.5241 436	.1225 021	.2314 649
53	.0367 991	.2072 032	.5247 104	.1220 076	.2322 987
54	.0368 533	.2074 837	.5252 734	.1215 088	.2331 342
1.455	-0.0369 072	+0.2077 628	-0.5258 325	+1.1210 056	+0.2339 714
56	.0369 609	.2080 404	.5263 878	.1204 981	.2348 103
57	.0370 143	.2083 165	.5269 393	.1199 862	.2356 509
58	.0370 675	.2085 912	.5274 869	.1194 699	.2364 932
59	.0371 204	.2088 645	.5280 306	.1189 493	.2373 372
1.460	-0.0371 731	+0.2091 362	-0.5285 704	+1.1184 242	+0.2381 829
61	.0372 255	.2094 065	.5291 063	.1178 948	.2390 304
62	.0372 776	.2096 754	.5296 383	.1173 610	.2398 795
63	.0373 296	.2099 427	.5301 664	.1168 228	.2407 304
64	.0373 812	.2102 086	.5306 906	.1162 802	.2415 830
1.465	-0.0374 326	+0.2104 730	-0.5312 108	+1.1157 331	+0.2424 373
66	.0374 837	.2107 359	.5317 271	.1151 816	.2432 934
67	.0375 346	.2109 972	.5322 395	.1146 257	.2441 511
68	.0375 853	.2112 571	.5327 479	.1140 654	.2450 106
69	.0376 356	.2115 155	.5332 523	.1135 006	.2458 718
1.470	-0.0376 857	+0.2117 724	-0.5337 528	+1.1129 314	+0.2467 348
71	.0377 356	.2120 277	.5342 493	.1123 577	.2475 995
72	.0377 851	.2122 815	.5347 418	.1117 795	.2484 659
73	.0378 344	.2125 338	.5352 303	.1111 969	.2493 340
74	.0378 835	.2127 846	.5357 147	.1106 098	.2502 039
1.475	-0.0379 323	+0.2130 338	-0.5361 952	+1.1100 182	+0.2510 755
76	.0379 808	.2132 815	.5366 717	.1094 221	.2519 489
77	.0380 290	.2135 276	.5371 441	.1088 215	.2528 240
78	.0380 770	.2137 722	.5376 124	.1082 164	.2537 009
79	.0381 247	.2140 152	.5380 767	.1076 068	.2545 795
1.480	-0.0381 722	+0.2142 566	-0.5385 370	+1.1069 926	+0.2554 598
81	.0382 193	.2144 965	.5389 931	.1063 740	.2563 419
82	.0382 662	.2147 348	.5394 452	.1057 508	.2572 258
83	.0383 128	.2149 716	.5398 932	.1051 231	.2581 114
84	.0383 592	.2152 067	.5403 371	.1044 908	.2589 988
1.485	-0.0384 052	+0.2154 403	-0.5407 769	+1.1038 539	+0.2598 879
86	.0384 510	.2156 723	.5412 126	.1032 125	.2607 788
87	.0384 966	.2159 027	.5416 442	.1025 666	.2616 715
88	.0385 418	.2161 314	.5420 716	.1019 160	.2625 659
89	.0385 867	.2163 586	.5424 949	.1012 609	.2634 621
1.490	-0.0386 314	+0.2165 842	-0.5429 140	+1.1006 012	+0.2643 601
91	.0386 758	.2168 081	.5433 290	.0999 368	.2652 598
92	.0387 199	.2170 304	.5437 398	.0992 679	.2661 613
93	.0387 637	.2172 511	.5441 464	.0985 944	.2670 646
94	.0388 073	.2174 702	.5445 488	.0979 163	.2679 697
1.495	-0.0388 505	+0.2176 876	-0.5449 471	+1.0972 335	+0.2688 765
96	.0388 935	.2179 034	.5453 411	.0965 461	.2697 851
97	.0389 362	.2181 175	.5457 309	.0958 540	.2706 956
98	.0389 786	.2183 300	.5461 165	.0951 573	.2716 077
99	.0390 207	.2185 408	.5464 979	.0944 560	.2725 217
1.500	-0.0390 625	+0.2187 500	-0.5468 750	+1.0937 500	+0.2734 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.500	-0.0390 625	+0.2187 500	-0.5468 750	+1.0937 500	+0.2734 375
01	.0391 040	.2189 575	.5472 479	.0930 393	.2743 551
02	.0391 452	.2191 633	.5476 165	.0923 240	.2752 744
03	.0391 862	.2193 675	.5479 808	.0916 040	.2761 956
04	.0392 268	.2195 699	.5483 409	.0908 793	.2771 185
1.505	-0.0392 672	+0.2197 707	-0.5486 967	+1.0901 499	+0.2780 433
06	.0393 072	.2199 698	.5490 482	.0894 157	.2789 698
07	.0393 470	.2201 672	.5493 954	.0886 769	.2798 982
08	.0393 864	.2203 629	.5497 382	.0879 334	.2808 283
09	.0394 256	.2205 569	.5500 768	.0871 851	.2817 603
1.510	-0.0394 644	+0.2207 492	-0.5504 110	+1.0864 322	+0.2826 941
11	.0395 030	.2209 397	.5507 409	.0856 744	.2836 297
12	.0395 412	.2211 286	.5510 664	.0849 120	.2845 671
13	.0395 792	.2213 157	.5513 876	.0841 448	.2855 063
14	.0396 168	.2215 010	.5517 044	.0833 728	.2864 473
1.515	-0.0396 541	+0.2216 847	-0.5520 168	+1.0825 961	+0.2873 902
16	.0396 911	.2218 666	.5523 248	.0818 145	.2883 349
17	.0397 279	.2220 467	.5526 285	.0810 283	.2892 814
18	.0397 643	.2222 251	.5529 277	.0802 372	.2902 297
19	.0398 004	.2224 018	.5532 226	.0794 413	.2911 799
1.520	-0.0398 362	+0.2225 766	-0.5535 130	+1.0786 406	+0.2921 318
21	.0398 716	.2227 498	.5537 989	.0778 352	.2930 857
22	.0399 068	.2229 211	.5540 805	.0770 249	.2940 413
23	.0399 416	.2230 906	.5543 576	.0762 098	.2949 988
24	.0399 762	.2232 584	.5546 302	.0753 898	.2959 581
1.525	-0.0400 104	+0.2234 244	-0.5548 983	+1.0745 650	+0.2969 193
26	.0400 443	.2235 886	.5551 620	.0737 354	.2978 823
27	.0400 779	.2237 510	.5554 212	.0729 009	.2988 471
28	.0401 111	.2239 116	.5556 759	.0720 616	.2998 138
29	.0401 441	.2240 704	.5559 261	.0712 174	.3007 824
1.530	-0.0401 767	+0.2242 274	-0.5561 718	+1.0703 684	+0.3017 528
31	.0402 090	.2243 825	.5564 130	.0695 144	.3027 250
32	.0402 410	.2245 359	.5566 496	.0686 556	.3036 991
33	.0402 726	.2246 874	.5568 817	.0677 919	.3046 751
34	.0403 040	.2248 370	.5571 092	.0669 233	.3056 529
1.535	-0.0403 350	+0.2249 849	-0.5573 322	+1.0660 497	+0.3066 326
36	.0403 657	.2251 308	.5575 506	.0651 713	.3076 141
37	.0403 960	.2252 750	.5577 644	.0642 879	.3085 975
38	.0404 260	.2254 173	.5579 737	.0633 996	.3095 828
39	.0404 557	.2255 577	.5581 783	.0625 064	.3105 699
1.540	-0.0404 851	+0.2256 962	-0.5583 784	+1.0616 082	+0.3115 589
41	.0405 141	.2258 329	.5585 738	.0607 051	.3125 498
42	.0405 428	.2259 677	.5587 646	.0597 970	.3135 426
43	.0405 711	.2261 007	.5589 508	.0588 840	.3145 372
44	.0405 991	.2262 317	.5591 323	.0579 660	.3155 337
1.545	-0.0406 268	+0.2263 609	-0.5593 092	+1.0570 430	+0.3165 321
46	.0406 542	.2264 881	.5594 814	.0561 150	.3175 324
47	.0406 812	.2266 135	.5596 489	.0551 821	.3185 345
48	.0407 078	.2267 370	.5598 118	.0542 441	.3195 386
49	.0407 342	.2268 585	.5599 700	.0533 011	.3205 445
1.550	-0.0407 602	+0.2269 781	-0.5601 234	+1.0523 531	+0.3215 523

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.550	-0.0407 602	+0.2269 781	-0.5601 234	+1.0523 531	+0.3215 523
51	.0407 858	.2270 958	.5602 722	.0514 001	.3225 621
52	.0408 111	.2272 116	.5604 163	.0504 421	.3235 737
53	.0408 360	.2273 255	.5605 556	.0494 790	.3245 872
54	.0408 607	.2274 374	.5606 902	.0485 109	.3256 026
1.555	-0.0408 849	+0.2275 473	-0.5608 200	+1.0475 377	+0.3266 199
56	.0409 088	.2276 553	.5609 451	.0465 595	.3276 391
57	.0409 324	.2277 614	.5610 654	.0455 762	.3286 602
58	.0409 556	.2278 655	.5611 810	.0445 878	.3296 833
59	.0409 785	.2279 677	.5612 918	.0435 944	.3307 082
1.560	-0.0410 010	+0.2280 678	-0.5613 978	+1.0425 958	+0.3317 350
61	.0410 231	.2281 660	.5614 989	.0415 922	.3327 638
62	.0410 449	.2282 623	.5615 953	.0405 835	.3337 945
63	.0410 664	.2283 565	.5616 869	.0395 697	.3348 271
64	.0410 874	.2284 488	.5617 736	.0385 507	.3358 616
1.565	-0.0411 082	+0.2285 390	-0.5618 555	+1.0375 266	+0.3368 980
66	.0411 285	.2286 273	.5619 325	.0364 974	.3379 364
67	.0411 486	.2287 135	.5620 047	.0354 631	.3389 767
68	.0411 682	.2287 977	.5620 720	.0344 236	.3400 189
69	.0411 875	.2288 800	.5621 344	.0333 790	.3410 630
1.570	-0.0412 064	+0.2289 602	-0.5621 920	+1.0323 292	+0.3421 091
71	.0412 250	.2290 383	.5622 447	.0312 742	.3431 571
72	.0412 432	.2291 145	.5622 924	.0302 141	.3442 070
73	.0412 610	.2291 886	.5623 352	.0291 487	.3452 589
74	.0412 785	.2292 606	.5623 732	.0280 782	.3463 127
1.575	-0.0412 956	+0.2293 307	-0.5624 062	+1.0270 025	+0.3473 685
76	.0413 123	.2293 986	.5624 342	.0259 216	.3484 262
77	.0413 286	.2294 645	.5624 573	.0248 355	.3494 859
78	.0413 446	.2295 284	.5624 754	.0237 442	.3505 475
79	.0413 602	.2295 901	.5624 886	.0226 476	.3516 110
1.580	-0.0413 755	+0.2296 498	-0.5624 968	+1.0215 458	+0.3526 765
81	.0413 903	.2297 075	.5625 000	.0204 388	.3537 440
82	.0414 048	.2297 630	.5624 981	.0193 265	.3548 134
83	.0414 189	.2298 164	.5624 913	.0182 090	.3558 848
84	.0414 326	.2298 678	.5624 795	.0170 862	.3569 581
1.585	-0.0414 460	+0.2299 170	-0.5624 626	+1.0159 582	+0.3580 334
86	.0414 590	.2299 642	.5624 407	.0148 248	.3591 107
87	.0414 715	.2300 092	.5624 138	.0136 862	.3601 900
88	.0414 838	.2300 521	.5623 818	.0125 423	.3612 712
89	.0414 956	.2300 929	.5623 447	.0113 931	.3623 543
1.590	-0.0415 070	+0.2301 316	-0.5623 026	+1.0102 386	+0.3634 395
91	.0415 181	.2301 681	.5622 554	.0090 787	.3645 266
92	.0415 287	.2302 025	.5622 031	.0079 136	.3656 157
93	.0415 390	.2302 347	.5621 456	.0067 431	.3667 068
94	.0415 489	.2302 648	.5620 831	.0055 673	.3677 999
1.595	-0.0415 584	+0.2302 928	-0.5620 154	+1.0043 862	+0.3688 949
96	.0415 675	.2303 186	.5619 427	.0031 997	.3699 920
97	.0415 762	.2303 422	.5618 647	.0020 078	.3710 910
98	.0415 845	.2303 636	.5617 817	+1.0008 106	.3721 920
99	.0415 925	.2303 829	.5616 934	+0.9996 080	.3732 950
1.600	-0.0416 000	+0.2304 000	-0.5616 000	+0.9984 000	+0.3744 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.600	-0.0416 000	+0.2304 000	-0.5616 000	+0.9984 000	+0.3744 000
01	.0416 071	.2304 149	.5615 014	.9971 866	.3755 070
02	.0416 139	.2304 276	.5613 976	.9959 679	.3766 160
03	.0416 202	.2304 381	.5612 886	.9947 437	.3777 270
04	.0416 261	.2304 464	.5611 745	.9935 142	.3788 400
1.605	-0.0416 317	+0.2304 526	-0.5610 550	+0.9922 792	+0.3799 550
06	.0416 368	.2304 564	.5609 304	.9910 388	.3810 720
07	.0416 416	.2304 581	.5608 005	.9897 929	.3821 910
08	.0416 459	.2304 576	.5606 654	.9885 417	.3833 121
09	.0416 498	.2304 548	.5605 250	.9872 849	.3844 351
1.610	-0.0416 533	+0.2304 498	-0.5603 794	+0.9860 228	+0.3855 602
11	.0416 564	.2304 425	.5602 285	.9847 551	.3866 873
12	.0416 591	.2304 330	.5600 723	.9834 820	.3878 164
13	.0416 614	.2304 212	.5599 108	.9822 034	.3889 475
14	.0416 633	.2304 072	.5597 440	.9809 194	.3900 806
1.615	-0.0416 648	+0.2303 910	-0.5595 718	+0.9796 298	+0.3912 158
16	.0416 658	.2303 724	.5593 944	.9783 348	.3923 530
17	.0416 664	.2303 516	.5592 116	.9770 342	.3934 922
18	.0416 667	.2303 285	.5590 235	.9757 281	.3946 335
19	.0416 665	.2303 031	.5588 300	.9744 166	.3957 768
1.620	-0.0416 659	+0.2302 754	-0.5586 312	+0.9730 994	+0.3969 221
21	.0416 648	.2302 455	.5584 269	.9717 768	.3980 695
22	.0416 634	.2302 132	.5582 173	.9704 486	.3992 189
23	.0416 615	.2301 786	.5580 024	.9691 148	.4003 704
24	.0416 592	.2301 417	.5577 820	.9677 755	.4015 239
1.625	-0.0416 565	+0.2301 025	-0.5575 562	+0.9664 307	+0.4026 794
26	.0416 534	.2300 610	.5573 249	.9650 802	.4038 370
27	.0416 498	.2300 172	.5570 883	.9637 242	.4049 967
28	.0416 458	.2299 710	.5568 462	.9623 626	.4061 584
29	.0416 414	.2299 224	.5565 986	.9609 954	.4073 222
1.630	-0.0416 365	+0.2298 716	-0.5563 456	+0.9596 226	+0.4084 880
31	.0416 312	.2298 183	.5560 871	.9582 441	.4096 559
32	.0416 255	.2297 628	.5558 231	.9568 601	.4108 258
33	.0416 194	.2297 048	.5555 537	.9554 704	.4119 978
34	.0416 128	.2296 445	.5552 787	.9540 751	.4131 719
1.635	-0.0416 058	+0.2295 818	-0.5549 983	+0.9526 742	+0.4143 480
36	.0415 983	.2295 168	.5547 123	.9512 676	.4155 262
37	.0415 905	.2294 493	.5544 208	.9498 554	.4167 065
38	.0415 821	.2293 795	.5541 237	.9484 375	.4178 889
39	.0415 734	.2293 073	.5538 211	.9470 139	.4190 733
1.640	-0.0415 642	+0.2292 326	-0.5535 130	+0.9455 846	+0.4202 598
41	.0415 545	.2291 556	.5531 992	.9441 497	.4214 484
42	.0415 444	.2290 762	.5528 799	.9427 091	.4226 391
43	.0415 339	.2289 943	.5525 550	.9412 627	.4238 319
44	.0415 229	.2289 100	.5522 245	.9398 107	.4250 267
1.645	-0.0415 115	+0.2288 233	-0.5518 884	+0.9383 529	+0.4262 237
46	.0414 996	.2287 342	.5515 467	.9368 894	.4274 227
47	.0414 873	.2286 426	.5511 994	.9354 202	.4286 239
48	.0414 745	.2285 485	.5508 464	.9339 453	.4298 271
49	.0414 613	.2284 521	.5504 877	.9324 646	.4310 324
1.650	-0.0414 477	+0.2283 531	-0.5501 234	+0.9309 781	+0.4322 398

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.650	-0.0414 477	+0.2283 531	-0.5501 234	+0.9309 781	+0.4322 398
51	.0414 335	.2282 517	.5497 535	.9294 859	.4334 494
52	.0414 190	.2281 479	.5493 778	.9279 879	.4346 610
53	.0414 039	.2280 415	.5489 965	.9264 842	.4358 748
54	.0413 884	.2279 327	.5486 095	.9249 746	.4370 906
1.655	-0.0413 725	+0.2278 214	-0.5482 168	+0.9234 593	+0.4383 086
56	.0413 561	.2277 076	.5478 183	.9219 381	.4395 286
57	.0413 392	.2275 913	.5474 141	.9204 112	.4407 508
58	.0413 219	.2274 725	.5470 042	.9188 784	.4419 752
59	.0413 041	.2273 512	.5465 886	.9173 399	.4432 016
1.660	-0.0412 859	+0.2272 274	-0.5461 672	+0.9157 954	+0.4444 301
61	.0412 671	.2271 011	.5457 400	.9142 452	.4456 608
62	.0412 480	.2269 723	.5453 070	.9126 891	.4468 936
63	.0412 283	.2268 409	.5448 683	.9111 271	.4481 286
64	.0412 082	.2267 070	.5444 237	.9095 593	.4493 656
1.665	-0.0411 876	+0.2265 705	-0.5439 734	+0.9079 857	+0.4506 048
66	.0411 666	.2264 315	.5435 172	.9064 061	.4518 462
67	.0411 450	.2262 900	.5430 552	.9048 207	.4530 896
68	.0411 230	.2261 459	.5425 874	.9032 293	.4543 352
69	.0411 006	.2259 992	.5421 137	.9016 321	.4555 830
1.670	-0.0410 776	+0.2258 500	-0.5416 342	+0.9000 290	+0.4568 329
71	.0410 542	.2256 982	.5411 488	.8984 199	.4580 849
72	.0410 303	.2255 438	.5406 575	.8968 049	.4593 391
73	.0410 059	.2253 868	.5401 604	.8951 840	.4605 955
74	.0409 811	.2252 272	.5396 573	.8935 572	.4618 540
1.675	-0.0409 557	+0.2250 650	-0.5391 483	+0.8919 244	+0.4631 146
76	.0409 299	.2249 003	.5386 335	.8902 857	.4643 774
77	.0409 036	.2247 329	.5381 126	.8886 410	.4656 424
78	.0408 768	.2245 629	.5375 859	.8869 903	.4669 095
79	.0408 495	.2243 903	.5370 532	.8853 337	.4681 788
1.680	-0.0408 218	+0.2242 150	-0.5365 146	+0.8836 710	+0.4694 502
81	.0407 935	.2240 372	.5359 699	.8820 024	.4707 239
82	.0407 648	.2238 567	.5354 193	.8803 278	.4719 997
83	.0407 355	.2236 735	.5348 628	.8786 472	.4732 776
84	.0407 058	.2234 877	.5343 002	.8769 605	.4745 578
1.685	-0.0406 756	+0.2232 993	-0.5337 316	+0.8752 679	+0.4758 401
86	.0406 449	.2231 081	.5331 570	.8735 692	.4771 246
87	.0406 137	.2229 144	.5325 764	.8718 645	.4784 112
88	.0405 820	.2227 179	.5319 897	.8701 537	.4797 001
89	.0405 498	.2225 188	.5313 970	.8684 369	.4809 912
1.690	-0.0405 171	+0.2223 170	-0.5307 982	+0.8667 140	+0.4822 844
91	.0404 839	.2221 125	.5301 934	.8649 850	.4835 798
92	.0404 502	.2219 053	.5295 824	.8632 500	.4848 774
93	.0404 160	.2216 954	.5289 654	.8615 089	.4861 772
94	.0403 813	.2214 828	.5283 423	.8597 616	.4874 792
1.695	-0.0403 461	+0.2212 675	-0.5277 131	+0.8580 083	+0.4887 834
96	.0403 104	.2210 494	.5270 777	.8562 489	.4900 898
97	.0402 742	.2208 287	.5264 363	.8544 834	.4913 984
98	.0402 375	.2206 052	.5257 887	.8527 117	.4927 093
99	.0402 002	.2203 790	.5251 349	.8509 339	.4940 223
1.700	-0.0401 625	+0.2201 500	-0.5244 750	+0.8491 500	+0.4953 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.700	-0.0401 625	+0.2201 500	-0.5244 750	+0.8491 500	+0.4953 375
01	.0401 242	.2199 183	.5238 089	.8473 599	.4966 549
02	.0400 855	.2196 838	.5231 366	.8455 637	.4979 746
03	.0400 462	.2194 466	.5224 582	.8437 613	.4992 965
04	.0400 064	.2192 066	.5217 735	.8419 527	.5006 206
1.705	-0.0399 661	+0.2189 639	-0.5210 827	+0.8401 380	+0.5019 469
06	.0399 252	.2187 184	.5203 856	.8383 171	.5032 754
07	.0398 839	.2184 700	.5196 823	.8364 900	.5046 061
08	.0398 420	.2182 189	.5189 727	.8346 566	.5059 391
09	.0397 996	.2179 651	.5182 569	.8328 171	.5072 743
1.710	-0.0397 567	+0.2177 084	-0.5175 348	+0.8309 714	+0.5086 118
11	.0397 133	.2174 489	.5168 064	.8291 194	.5099 515
12	.0396 693	.2171 866	.5160 718	.8272 612	.5112 934
13	.0396 248	.2169 214	.5153 309	.8253 967	.5126 375
14	.0395 798	.2166 535	.5145 837	.8235 261	.5139 839
1.715	-0.0395 343	+0.2163 827	-0.5138 301	+0.8216 491	+0.5153 326
16	.0394 882	.2161 091	.5130 703	.8197 659	.5166 834
17	.0394 416	.2158 327	.5123 041	.8178 764	.5180 366
18	.0393 944	.2155 534	.5115 315	.8159 806	.5193 919
19	.0393 468	.2152 712	.5107 526	.8140 786	.5207 496
1.720	-0.0392 986	+0.2149 862	-0.5099 674	+0.8121 702	+0.5221 094
21	.0392 498	.2146 984	.5091 757	.8102 556	.5234 716
22	.0392 005	.2144 076	.5083 777	.8083 346	.5248 360
23	.0391 507	.2141 140	.5075 733	.8064 073	.5262 026
24	.0391 004	.2138 175	.5067 625	.8044 737	.5275 715
1.725	-0.0390 495	+0.2135 182	-0.5059 452	+0.8025 338	+0.5289 427
26	.0389 980	.2132 159	.5051 215	.8005 875	.5303 162
27	.0389 460	.2129 107	.5042 914	.7986 349	.5316 919
28	.0388 935	.2126 027	.5034 549	.7966 759	.5330 699
29	.0388 404	.2122 917	.5026 119	.7947 105	.5344 501
1.730	-0.0387 868	+0.2119 778	-0.5017 624	+0.7927 388	+0.5358 327
31	.0387 326	.2116 609	.5009 064	.7907 606	.5372 175
32	.0386 779	.2113 412	.5000 440	.7887 761	.5386 046
33	.0386 227	.2110 185	.4991 750	.7867 852	.5399 940
34	.0385 668	.2106 929	.4982 996	.7847 879	.5413 856
1.735	-0.0385 105	+0.2103 643	-0.4974 176	+0.7827 842	+0.5427 796
36	.0384 535	.2100 328	.4965 291	.7807 740	.5441 758
37	.0383 961	.2096 983	.4956 341	.7787 574	.5455 744
38	.0383 380	.2093 608	.4947 325	.7767 344	.5469 752
39	.0382 794	.2090 204	.4938 243	.7747 050	.5483 783
1.740	-0.0382 203	+0.2086 770	-0.4929 096	+0.7726 690	+0.5497 837
41	.0381 605	.2083 307	.4919 882	.7706 267	.5511 915
42	.0381 003	.2079 813	.4910 603	.7685 778	.5526 015
43	.0380 394	.2076 289	.4901 258	.7665 225	.5540 138
44	.0379 780	.2072 736	.4891 847	.7644 607	.5554 285
1.745	-0.0379 160	+0.2069 152	-0.4882 369	+0.7623 924	+0.5568 454
46	.0378 535	.2065 539	.4872 825	.7603 175	.5582 647
47	.0377 904	.2061 895	.4863 215	.7582 362	.5596 862
48	.0377 267	.2058 220	.4853 538	.7561 484	.5611 101
49	.0376 625	.2054 516	.4843 795	.7540 540	.5625 363
1.750	-0.0375 977	+0.2050 781	-0.4833 984	+0.7519 531	+0.5639 648

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.750	-0.0375 977	+0.2050 781	-0.4833 984	+0.7519 531	+0.5639 648
51	.0375 323	.2047 016	.4824 107	.7498 457	.5653 957
52	.0374 663	.2043 220	.4814 163	.7477 317	.5668 289
53	.0373 998	.2039 394	.4804 152	.7456 112	.5682 644
54	.0373 326	.2035 537	.4794 073	.7434 840	.5697 022
1.755	-0.0372 649	+0.2031 650	-0.4783 927	+0.7413 504	+0.5711 424
56	.0371 967	.2027 732	.4773 714	.7392 101	.5725 849
57	.0371 278	.2023 783	.4763 434	.7370 632	.5740 297
58	.0370 584	.2019 803	.4753 085	.7349 098	.5754 769
59	.0369 884	.2015 792	.4742 669	.7327 497	.5769 264
1.760	-0.0369 178	+0.2011 750	-0.4732 186	+0.7305 830	+0.5783 782
61	.0368 466	.2007 678	.4721 634	.7284 097	.5798 324
62	.0367 748	.2003 574	.4711 014	.7262 298	.5812 890
63	.0367 024	.1999 439	.4700 326	.7240 433	.5827 479
64	.0366 295	.1995 273	.4689 570	.7218 500	.5842 091
1.765	-0.0365 559	+0.1991 076	-0.4678 745	+0.7196 502	+0.5856 728
66	.0364 818	.1986 847	.4667 853	.7174 437	.5871 387
67	.0364 071	.1982 587	.4656 891	.7152 305	.5886 070
68	.0363 317	.1978 295	.4645 861	.7130 106	.5900 777
69	.0362 558	.1973 972	.4634 762	.7107 840	.5915 508
1.770	-0.0361 793	+0.1969 618	-0.4623 594	+0.7085 508	+0.5930 262
71	.0361 022	.1965 231	.4612 357	.7063 108	.5945 040
72	.0360 245	.1960 814	.4601 051	.7040 641	.5959 841
73	.0359 462	.1956 364	.4589 676	.7018 108	.5974 666
74	.0358 673	.1951 882	.4578 232	.6995 506	.5989 515
1.775	-0.0357 877	+0.1947 369	-0.4566 718	+0.6972 838	+0.6004 388
76	.0357 076	.1942 824	.4555 134	.6950 102	.6019 285
77	.0356 269	.1938 247	.4543 481	.6927 299	.6034 205
78	.0355 456	.1933 637	.4531 759	.6904 428	.6049 149
79	.0354 636	.1928 996	.4519 966	.6881 489	.6064 117
1.780	-0.0353 811	+0.1924 322	-0.4508 104	+0.6858 482	+0.6079 109
81	.0352 979	.1919 617	.4496 171	.6835 408	.6094 125
82	.0352 141	.1914 878	.4484 168	.6812 266	.6109 165
83	.0351 297	.1910 108	.4472 095	.6789 056	.6124 229
84	.0350 447	.1905 305	.4459 952	.6765 777	.6139 317
1.785	-0.0349 591	+0.1900 470	-0.4447 738	+0.6742 431	+0.6154 428
86	.0348 729	.1895 602	.4435 454	.6719 016	.6169 564
87	.0347 860	.1890 701	.4423 099	.6695 533	.6184 724
88	.0346 985	.1885 768	.4410 673	.6671 982	.6199 908
89	.0346 104	.1880 802	.4398 176	.6648 362	.6215 116
1.790	-0.0345 217	+0.1875 804	-0.4385 608	+0.6624 674	+0.6230 348
91	.0344 324	.1870 772	.4372 969	.6600 917	.6245 604
92	.0343 424	.1865 708	.4360 259	.6577 091	.6260 884
93	.0342 518	.1860 610	.4347 477	.6553 196	.6276 189
94	.0341 606	.1855 480	.4334 624	.6529 233	.6291 518
1.795	-0.0340 687	+0.1850 316	-0.4321 700	+0.6505 200	+0.6306 871
96	.0339 763	.1845 120	.4308 704	.6481 098	.6322 248
97	.0338 831	.1839 890	.4295 636	.6456 928	.6337 650
98	.0337 894	.1834 626	.4282 496	.6432 688	.6353 075
99	.0336 950	.1829 330	.4269 284	.6408 379	.6368 526
1.800	-0.0336 000	+0.1824 000	-0.4256 000	+0.6384 000	+0.6384 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.800	-0.0336 000	+0.1824 000	-0.4256 000	+0.6384 000	+0.6384 000
01	.0335 043	.1818 637	.4242 644	.6359 552	.6399 499
02	.0334 081	.1813 240	.4229 215	.6335 034	.6415 022
03	.0333 111	.1807 809	.4215 715	.6310 447	.6430 570
04	.0332 136	.1802 345	.4202 141	.6285 790	.6446 142
1.805	-0.0331 153	+0.1796 847	-0.4188 495	+0.6261 063	+0.6461 738
06	.0330 165	.1791 315	.4174 777	.6236 267	.6477 359
07	.0329 170	.1785 750	.4160 985	.6211 400	.6493 005
08	.0328 168	.1780 150	.4147 120	.6186 463	.6508 675
09	.0327 161	.1774 517	.4133 183	.6161 457	.6524 370
1.810	-0.0326 146	+0.1768 850	-0.4119 172	+0.6136 380	+0.6540 089
11	.0325 125	.1763 148	.4105 088	.6111 232	.6555 833
12	.0324 098	.1757 413	.4090 930	.6086 015	.6571 601
13	.0323 064	.1751 643	.4076 699	.6060 727	.6587 394
14	.0322 023	.1745 839	.4062 395	.6035 368	.6603 212
1.815	-0.0320 976	+0.1740 000	-0.4048 017	+0.6009 939	+0.6619 054
16	.0319 923	.1734 127	.4033 565	.5984 439	.6634 921
17	.0318 863	.1728 220	.4019 038	.5958 868	.6650 813
18	.0317 796	.1722 278	.4004 438	.5933 227	.6666 730
19	.0316 723	.1716 302	.3989 764	.5907 514	.6682 671
1.820	-0.0315 643	+0.1710 290	-0.3975 016	+0.5881 730	+0.6698 637
21	.0314 556	.1704 245	.3960 193	.5855 876	.6714 628
22	.0313 463	.1698 164	.3945 295	.5829 950	.6730 644
23	.0312 363	.1692 049	.3930 323	.5803 953	.6746 685
24	.0311 256	.1685 898	.3915 277	.5777 884	.6762 751
1.825	-0.0310 143	+0.1679 713	-0.3900 155	+0.5751 744	+0.6778 841
26	.0309 023	.1673 493	.3884 959	.5725 533	.6794 957
27	.0307 896	.1667 237	.3869 687	.5699 250	.6811 097
28	.0306 763	.1660 947	.3854 341	.5672 895	.6827 263
29	.0305 623	.1654 621	.3838 919	.5646 468	.6843 453
1.830	-0.0304 476	+0.1648 260	-0.3823 422	+0.5619 970	+0.6859 669
31	.0303 323	.1641 863	.3807 849	.5593 399	.6875 909
32	.0302 162	.1635 431	.3792 201	.5566 757	.6892 175
33	.0300 995	.1628 964	.3776 477	.5540 042	.6908 466
34	.0299 821	.1622 461	.3760 678	.5513 256	.6924 782
1.835	-0.0298 640	+0.1615 923	-0.3744 802	+0.5486 397	+0.6941 123
36	.0297 453	.1609 349	.3728 850	.5459 465	.6957 489
37	.0296 258	.1602 739	.3712 823	.5432 461	.6973 881
38	.0295 057	.1596 093	.3696 719	.5405 385	.6990 297
39	.0293 849	.1589 412	.3680 538	.5378 236	.7006 739
1.840	-0.0292 634	+0.1582 694	-0.3664 282	+0.5351 014	+0.7023 206
41	.0291 412	.1575 941	.3647 948	.5323 720	.7039 699
42	.0290 183	.1569 152	.3631 538	.5296 353	.7056 217
43	.0288 947	.1562 326	.3615 051	.5268 912	.7072 760
44	.0287 704	.1555 464	.3598 488	.5241 399	.7089 328
1.845	-0.0286 455	+0.1548 567	-0.3581 847	+0.5213 813	+0.7105 922
46	.0285 198	.1541 632	.3565 129	.5186 153	.7122 542
47	.0283 934	.1534 662	.3548 334	.5158 420	.7139 186
48	.0282 664	.1527 655	.3531 462	.5130 614	.7155 857
49	.0281 386	.1520 611	.3514 512	.5102 734	.7172 552
1.850	-0.0280 102	+0.1513 531	-0.3497 484	+0.5074 781	+0.7189 273

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.850	-0.0280 102	+0.1513 531	-0.3497 484	+0.5074 781	+0.7189 273
51	.0278 810	.1506 415	.3480 379	.5046 754	.7206 020
52	.0277 511	.1499 261	.3463 196	.5018 654	.7222 792
53	.0276 205	.1492 071	.3445 936	.4990 480	.7239 590
54	.0274 893	.1484 844	.3428 597	.4962 231	.7256 414
1.855	-0.0273 573	+0.1477 581	-0.3411 180	+0.4933 909	+0.7273 263
56	.0272 246	.1470 280	.3393 685	.4905 513	.7290 138
57	.0270 912	.1462 942	.3376 111	.4877 043	.7307 038
58	.0269 570	.1455 567	.3358 459	.4848 498	.7323 964
59	.0268 222	.1448 155	.3340 729	.4819 880	.7340 916
1.860	-0.0266 867	+0.1440 706	-0.3322 920	+0.4791 186	+0.7357 893
61	.0265 504	.1433 220	.3305 032	.4762 419	.7374 897
62	.0264 134	.1425 696	.3287 065	.4733 577	.7391 926
63	.0262 757	.1418 135	.3269 019	.4704 660	.7408 981
64	.0261 373	.1410 537	.3250 894	.4675 668	.7426 061
1.865	-0.0259 981	+0.1402 901	-0.3232 690	+0.4646 602	+0.7443 168
66	.0258 583	.1395 227	.3214 406	.4617 461	.7460 301
67	.0257 177	.1387 516	.3196 043	.4588 245	.7477 459
68	.0255 763	.1379 767	.3177 600	.4558 954	.7494 643
69	.0254 343	.1371 980	.3159 078	.4529 587	.7511 854
1.870	-0.0252 915	+0.1364 156	-0.3140 476	+0.4500 146	+0.7529 090
71	.0251 480	.1356 293	.3121 794	.4470 629	.7546 352
72	.0250 038	.1348 393	.3103 032	.4441 037	.7563 640
73	.0248 588	.1340 454	.3084 190	.4411 369	.7580 955
74	.0247 131	.1332 478	.3065 267	.4381 626	.7598 295
1.875	-0.0245 667	+0.1324 463	-0.3046 265	+0.4351 807	+0.7615 662
76	.0244 195	.1316 410	.3027 181	.4321 912	.7633 054
77	.0242 716	.1308 319	.3008 018	.4291 942	.7650 473
78	.0241 229	.1300 189	.2988 773	.4261 895	.7667 918
79	.0239 735	.1292 021	.2969 448	.4231 773	.7685 389
1.880	-0.0238 234	+0.1283 814	-0.2950 042	+0.4201 574	+0.7702 886
81	.0236 725	.1275 569	.2930 554	.4171 300	.7720 410
82	.0235 208	.1267 286	.2910 986	.4140 949	.7737 960
83	.0233 685	.1258 963	.2891 336	.4110 522	.7755 536
84	.0232 153	.1250 602	.2871 605	.4080 018	.7773 138
1.885	-0.0230 615	+0.1242 202	-0.2851 793	+0.4049 438	+0.7790 767
86	.0229 068	.1233 763	.2831 899	.4018 782	.7808 422
87	.0227 515	.1225 285	.2811 923	.3988 048	.7826 104
88	.0225 953	.1216 769	.2791 865	.3957 238	.7843 812
89	.0224 385	.1208 213	.2771 726	.3926 351	.7861 546
1.890	-0.0222 808	+0.1199 618	-0.2751 504	+0.3895 388	+0.7879 307
91	.0221 224	.1190 983	.2731 200	.3864 347	.7897 094
92	.0219 633	.1182 310	.2710 814	.3833 229	.7914 908
93	.0218 033	.1173 597	.2690 346	.3802 034	.7932 748
94	.0216 427	.1164 845	.2669 795	.3770 761	.7950 615
1.895	-0.0214 812	+0.1156 053	-0.2649 161	+0.3739 412	+0.7968 508
96	.0213 190	.1147 222	.2628 445	.3707 985	.7986 428
97	.0211 560	.1138 351	.2607 646	.3676 480	.8004 375
98	.0209 923	.1129 440	.2586 764	.3644 898	.8022 348
99	.0208 278	.1120 490	.2565 798	.3613 238	.8040 348
1.900	-0.0206 625	+0.1111 500	-0.2544 750	+0.3581 500	+0.8058 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.900	-0.0206 625	+0.1111 500	-0.2544 750	+0.3581 500	+0.8058 375
01	.0204 964	.1102 470	.2523 618	.3549 684	.8076 428
02	.0203 296	.1093 400	.2502 403	.3517 791	.8094 508
03	.0201 620	.1084 290	.2481 105	.3485 819	.8112 615
04	.0199 936	.1075 140	.2459 722	.3453 769	.8130 749
1.905	-0.0198 245	+0.1065 950	-0.2438 256	+0.3421 642	+0.8148 909
06	.0196 546	.1056 720	.2416 706	.3389 435	.8167 097
07	.0194 838	.1047 449	.2395 073	.3357 151	.8185 311
08	.0193 124	.1038 139	.2373 355	.3324 788	.8203 552
09	.0191 401	.1028 787	.2351 552	.3292 346	.8221 820
1.910	-0.0189 670	+0.1019 396	-0.2329 666	+0.3259 826	+0.8240 115
11	.0187 932	.1009 963	.2307 695	.3227 227	.8258 437
12	.0186 185	.1000 491	.2285 640	.3194 549	.8276 786
13	.0184 431	.0990 977	.2263 499	.3161 792	.8295 161
14	.0182 669	.0981 423	.2241 274	.3128 956	.8313 564
1.915	-0.0180 899	+0.0971 828	-0.2218 964	+0.3096 042	+0.8331 994
16	.0179 121	.0962 192	.2196 570	.3063 048	.8350 451
17	.0177 335	.0952 515	.2174 090	.3029 974	.8368 935
18	.0175 541	.0942 797	.2151 524	.2996 822	.8387 446
19	.0173 739	.0933 038	.2128 874	.2963 590	.8405 985
1.920	-0.0171 930	+0.0923 238	-0.2106 138	+0.2930 278	+0.8424 550
21	.0170 112	.0913 397	.2083 316	.2896 887	.8443 143
22	.0168 286	.0903 515	.2060 409	.2863 417	.8461 763
23	.0166 452	.0893 591	.2037 415	.2829 866	.8480 410
24	.0164 610	.0883 626	.2014 336	.2796 236	.8499 085
1.925	-0.0162 760	+0.0873 619	-0.1991 171	+0.2762 525	+0.8517 787
26	.0160 902	.0863 571	.1967 920	.2728 735	.8536 516
27	.0159 036	.0853 481	.1944 582	.2694 865	.8555 272
28	.0157 162	.0843 350	.1921 158	.2660 914	.8574 056
29	.0155 280	.0833 177	.1897 647	.2626 883	.8592 867
1.930	-0.0153 389	+0.0822 962	-0.1874 050	+0.2592 772	+0.8611 706
31	.0151 491	.0812 705	.1850 366	.2558 580	.8630 572
32	.0149 584	.0802 406	.1826 595	.2524 308	.8649 465
33	.0147 669	.0792 065	.1802 737	.2489 955	.8668 386
34	.0145 746	.0781 682	.1778 792	.2455 521	.8687 335
1.935	-0.0143 815	+0.0771 258	-0.1754 760	+0.2421 006	+0.8706 311
36	.0141 875	.0760 790	.1730 641	.2386 411	.8725 315
37	.0139 927	.0750 281	.1706 434	.2351 734	.8744 346
38	.0137 971	.0739 729	.1682 140	.2316 977	.8763 405
39	.0136 007	.0729 135	.1657 758	.2282 138	.8782 491
1.940	-0.0134 035	+0.0718 498	-0.1633 288	+0.2247 218	+0.8801 605
41	.0132 054	.0707 819	.1608 730	.2212 217	.8820 747
42	.0130 065	.0697 097	.1584 084	.2177 134	.8839 917
43	.0128 067	.0686 333	.1559 350	.2141 970	.8859 114
44	.0126 062	.0675 526	.1534 528	.2106 725	.8878 339
1.945	-0.0124 048	+0.0664 676	-0.1509 617	+0.2071 397	+0.8897 592
46	.0122 025	.0653 783	.1484 618	.2035 988	.8916 872
47	.0119 994	.0642 847	.1459 530	.2000 497	.8936 181
48	.0117 955	.0631 868	.1434 354	.1964 924	.8955 517
49	.0115 908	.0620 846	.1409 089	.1929 268	.8974 881
1.950	-0.0113 852	+0.0609 781	-0.1383 734	+0.1893 531	+0.8994 273

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.950	-0.0113 852	+0.0609 781	-0.1383 734	+0.1893 531	+0.8994 273
51	.0111 787	.0598 673	.1358 291	.1857 712	.9013 693
52	.0109 714	.0587 521	.1332 759	.1821 810	.9033 141
53	.0107 633	.0576 326	.1307 137	.1785 826	.9052 617
54	.0105 543	.0565 088	.1281 426	.1749 759	.9072 121
1.955	-0.0103 445	+0.0553 806	-0.1255 625	+0.1713 610	+0.9091 653
56	.0101 338	.0542 481	.1229 734	.1677 378	.9111 213
57	.0099 223	.0531 112	.1203 754	.1641 064	.9130 801
58	.0097 099	.0519 699	.1177 684	.1604 666	.9150 418
59	.0094 967	.0508 243	.1151 524	.1568 186	.9170 062
1.960	-0.0092 826	+0.0496 742	-0.1125 274	+0.1531 622	+0.9189 734
61	.0090 676	.0485 198	.1098 933	.1494 976	.9209 435
62	.0088 518	.0473 610	.1072 502	.1458 246	.9229 164
63	.0086 351	.0461 978	.1045 981	.1421 433	.9248 921
64	.0084 176	.0450 302	.1019 369	.1384 537	.9268 706
1.965	-0.0081 992	+0.0438 581	-0.0992 666	+0.1347 557	+0.9288 520
66	.0079 799	.0426 816	.0965 873	.1310 494	.9308 362
67	.0077 598	.0415 007	.0938 988	.1273 347	.9328 232
68	.0075 388	.0403 154	.0912 013	.1236 116	.9348 131
69	.0073 170	.0391 256	.0884 946	.1198 802	.9368 058
1.970	-0.0070 942	+0.0379 314	-0.0857 788	+0.1161 404	+0.9388 013
71	.0068 706	.0367 327	.0830 539	.1123 921	.9407 997
72	.0066 461	.0355 295	.0803 198	.1086 355	.9428 009
73	.0064 208	.0343 219	.0775 765	.1048 704	.9448 049
74	.0061 945	.0331 098	.0748 240	.1010 970	.9468 119
1.975	-0.0059 674	+0.0318 932	-0.0720 624	+0.0973 150	+0.9488 216
76	.0057 394	.0306 721	.0692 916	.0935 247	.9508 343
77	.0055 106	.0294 465	.0665 115	.0897 259	.9528 497
78	.0052 808	.0282 164	.0637 222	.0859 186	.9548 681
79	.0050 502	.0269 818	.0609 237	.0821 029	.9568 893
1.980	-0.0048 187	+0.0257 426	-0.0581 160	+0.0782 786	+0.9589 133
81	.0045 863	.0244 990	.0552 989	.0744 459	.9609 403
82	.0043 530	.0232 508	.0524 726	.0706 047	.9629 701
83	.0041 188	.0219 981	.0496 371	.0667 550	.9650 027
84	.0038 837	.0207 408	.0467 922	.0628 968	.9670 383
1.985	-0.0036 477	+0.0194 789	-0.0439 380	+0.0590 301	+0.9690 767
86	.0034 109	.0182 125	.0410 745	.0551 548	.9711 180
87	.0031 731	.0169 416	.0382 016	.0512 710	.9731 622
88	.0029 344	.0156 660	.0353 195	.0473 786	.9752 093
89	.0026 949	.0143 859	.0324 279	.0434 777	.9772 592
1.990	-0.0024 544	+0.0131 012	-0.0295 270	+0.0395 682	+0.9793 121
91	.0022 131	.0118 118	.0266 167	.0356 501	.9813 678
92	.0019 708	.0105 179	.0236 970	.0317 234	.9834 265
93	.0017 276	.0092 194	.0207 679	.0277 882	.9854 880
94	.0014 836	.0079 163	.0178 294	.0238 443	.9875 524
1.995	-0.0012 386	+0.0066 085	-0.0148 815	+0.0198 919	+0.9896 197
96	.0009 927	.0052 961	.0119 241	.0159 308	.9916 900
97	.0007 459	.0039 790	.0089 573	.0119 610	.9937 631
98	.0004 982	.0026 573	.0059 810	.0079 827	.9958 392
99	.0002 495	.0013 310	.0029 953	.0039 957	+0.9979 181
2.000	-0.0000 000	+0.0000 000	-0.0000 000	+0.0000 000	+1.0000 000

CURVILINEAR INTERPOLATION BY THE METHOD OF CONSTANT SECOND DIFFERENCES

EXEMPLIFIED BY COMPUTING TABLE OF CAPACITY OF CURVED BOTTOM TANK AT INCH-OF-DEPTH SOUNDINGS

REMARKS: *In much engineering and scientific work, where straight-line interpolation is not regarded as sufficiently accurate, a satisfactory trend of tabular values may be obtained by considering Second Differences as constant for the sub-divisions of the intervals between calculated values of the table.*

EXAMPLE: The calculated volume of a tank with outwardly flaring curved sides, at various depths in even feet, is as below:

		First Diff (d')	Second Diff (d")	Avg. d"
7.0 ft.	197.87 bbls.		*3.23*	
		33.92		2.93
6.0	163.95		*2.62*	
		31.30		2.36
5.0	132.65		*2.09*	
		29.21		1.87
4.0	103.44		*1.64*	
		27.57		1.47
3.0	75.87		*1.29*	
		26.28		1.17
2.0	49.59		*1.04*	
		25.24		.97
1.0	24.35		*.89*	
		24.35		.87
0	0		*.84*	

It is desired to know the volume at intermediate depths at one-inch intervals.

OPERATIONS: Decimals: Upper Dial 5 and 0; Middle Dial 14, 9, and 4; Keyboard Dial 9 and 4. Use 10-column Marchant, with Upper Green Shift Key down on M-series models.

(1) Compute First and Second Differences (d' and d") by Marchant Method MM-419 and tabulate them as shown in italics above.

(2) Extrapolate d" for 0 ft. and 7.0 ft. as shown in script numerals. This may be done by calculating from 3rd differences or by plotting a few values of d" toward each end and extending their curve to the end points (in this case, 0.0 and 7.0 respectively).

(3) Compute the Average d" in each Interval; e.g., the Avg. d" for 6.0 to 7.0 ft. is $\frac{1}{2}(3.23 + 2.62)$, or 2.93, etc., and copy these at right of table.

(over)

CALCULATION OF FIRST INCREMENTS AND CONSTANT SECOND DIFFERENCES IN EACH TABLE INTERVAL

OUTLINE: First Increment in each Interval between computed values is:

(a) $$\frac{d'}{h} - \frac{\text{Avg. } d''}{h^2} \times \frac{h-1}{2}$$

in which d' = First Difference of Interval (See preceding page)
Avg. d" = Avg. Second Difference during Interval
h = Number of Increments in Interval

For machine calculation, the above formula is transposed to:

(b) $$\frac{1}{h^2}\left(d'h - \text{Avg. } d''\left(\frac{h-1}{2}\right)\right)$$

Constant Second Difference for h increments per Interval is:

(c) $$\frac{\text{Avg. } d''}{h^2}$$

For interpolation of volume between 0.0 and 1.0 ft., d' = 24.35, Avg. d" = .87, h = 12 (12 in. = 1 ft.), and similarly for other Intervals (See preceding page).

For calculation of right-hand factor of Formula (b) above, proceed as follows for all intervals of table:

(1) Set up d' (24.35) at 4th Keyboard-Dial decimal and multiply by h(12) at 5th Upper-Dial decimal.

(2) Clear Upper and Keyboard Dials, set up Avg. d" (.87) at 4th Keyboard-Dial decimal, and reverse-multiply at 5th Upper-Dial decimal by $\frac{1}{2}(h-1)$, or 5.5.

Upper Dial shows....994.50000 (complement of 5.5)

$d'h - \text{Avg. } d''\left(\frac{h-1}{2}\right)$, or 287.415 appears at 9th Middle-Dial decimal

(3) Similarly obtain this amount for all other Intervals of the table, recording them in Col. A on Work Sheet at top of Page 3.

(4) Set up in Keyboard Dial at 9th decimal $1/h^2$ (reciprocal of 144 = .006944444) as constant, and multiply by amounts in Col. A, producing First Increments, as in Col. B of Work Sheet.

(continued on next page)

WORK SHEET

Argument	A	B	C
7.0 ft		First Increments	Constant 2nd Diff.
	390.925	2.7148	.0203
6.			
	362.620	2.5182	.0164
5.			
	340.235	2.3627	.0130
4.			
	322.755	2.2414	.0102
3.			
	308.925	2.1453	.0081
2.			
	297.545	2.0663	.0067
1.			
	287.415	1.9959	.0060
0			

(5) Without clearing Keyboard Dial, continue to multiply by the respective Avg. d" values (.87, .97, 1.17, etc.), thus producing the values in Col. C of table of Work Sheet.

INTEGRATION FOR TABULAR VALUES

(6) Clear dials and, with carriage in 1st position, set up in Keyboard Dial at 4th decimal the First Increment of Interval (0 to 1.0 ft.) (1.9959) and add. Clear Upper Dial.

(7) Likewise, set up First Increment (1.9959) at 9th and corresponding Constant Second Difference (.0060) at 4th Keyboard-Dial decimal.
Keyboard Dial reads 1.99590.0060
Multiply by "1" at "0" Upper-Dial decimal.
Record 1.9959 as Table Value for 0 ft 1 in.

(8) Change Keyboard Dial at left so it reads same as Middle Dial at right (2.0019) and multiply by "1".
Record 3.9978 as Table Value for 0 ft. 2 in.

(9) Repeat Step 8 for all values up to 1 ft. 0 in., developing the table below:

1 ft.	0 in.	24.3468	rounded off as	24.35
	11 "	22.2849	" " "	22.28
	10 "	20.2229	" " "	20.22
	9 "	18.1791	" " "	18.18
	8 "	16.1352	" " "	16.14
	7 "	14.0973	" " "	14.10
	6 "	12.0654	" " "	12.07
	5 "	10.0395	" " "	10.04
	4 "	8.0196	" " "	8.02
	3 "	6.0057	" " "	6.01
	2 "	3.9978	" " "	4.00
	1 "	1.9959	" " "	2.00
	0			

Upper Dial will show 12, equaling h, as control, though the approach to 24.35 shows completion of steps in First Interval.

(over)

(10) Clear all dials, and repeat Steps 6 to 9 for the sub-divisions of the next interval. In this case, the start of the integration is made with 24.35 in Middle Dial at 9th decimal, and 2.0663 at 4th decimal. The Keyboard Dial reads 2.06630.0067 and the Upper Dial is clear.

(11) Proceed similarly for the remaining intervals.

NOTE A: The example shows d' and d" as positive and table is ascending. The method is just as suitable for negative differences, or with d' and d" of opposite sign. If Constant Second Difference is opposite in sign from First Increment, set up complement of Constant Second Difference, preceding by 9's and reducing right-hand digit of First Increment by "1". If tabular values are descending (i.e., d' is minus), reverse-multiply in the integration steps.

NOTE B: A quick check of correctness of First Increment and Constant Second Difference is that they satisfy the following:

$$\text{Interval Step} = h \times d' + \tfrac{1}{2}h(h-1)d''$$

or, in this case

$$24.35 = 12 \times 1.9959 + 66 \times .0060 = 24.3468$$

which is as satisfactory as can be obtained with places available.

NOTE C: Rounding-off of right-hand digit of Constant Second Difference may cause a maximum error of its value of, in this instance, .0005. By integrations, this is magnified to $\frac{1}{2}(h-1)(h-2)$ times .0005 at table value h-1; or, in this case, 55 x .0005 = .0275, as a maximum. This explains why values obtained by integration to 4 places at right of decimal are rounded-off to 2 places.

NOTE D: The use of the Upper Dial as a control of count in the integration is most helpful if it is desired to go immediately to any specific intermediate value. The Upper Dial always records the particular tabular value as numbered from point of origin of the Interval.

1026 6436 8

INVERSE CURVILINEAR INTERPOLATION

SHORT METHOD IN WHICH CONSTANT 2ND DIFFERENCES ARE ASSUMED

REMARKS: *This method is an adaptation to the Marchant of the Comrie Process which is described in the British Nautical Almanac for 1937. It is very satisfactory for inverse interpolation in the ordinary table which usually has tabulated values so close together that differences greater than the second can be ignored. The method requires the use of a pre-calculated table of B" (d''_0 + d''_1), or that slide-rule be used for this auxiliary calculation.*

A supplemental Note provides information as to the size of 3rd difference that may be ignored without its affecting the last place of the interpolate by more than ½.

EXAMPLE: In the following table find "x" which corresponds to y = 0.984 637 2. Differences are also shown which were computed as per Marchant Method MM-419.

x	y	d'	d"	d'''
1.0	0.993 377 5 (f_{-1})			
		-65 787 ($d'_{-½}$)		
2.0	0.986 798 8 (f_0)		+ 436 (d''_0)	
		-65 351 ($d'_{½}$)		-3 ($d'''_{½}$)
3.0	0.980 263 7 (f_1)		+ 433 (d''_1)	
		-64 918 ($d'_{1½}$)		
4.0	0.973 771 9 (f_2)			

OPERATIONS: Decimals; Upper Dial 7, Middle Dial 14, Keyboard Dial 7. Use any Marchant model (This problem is exemplified on a semi-automatic model.) As this is a decreasing function, set Manual Counter Control toward the operator. If it were increasing, it would be away from operator.

(1) Set up (f_0) (0.986 798 8) in Keyboard Dial and, with carriage in 8th position, depress Add Bar.

(2) Set up ($d'_{½}$) (0.006 535 1) and touch Short-Cut Bar once.

(f_1) (0.980 263 7) appears in Middle Dial as proof.

(3) Depress X Bar once and clear Upper Dial.

(f_0) (0.986 798 8) is restored in Middle Dial.

(4) Shift to 7th position and hold down Short-Cut Bar until Middle Dial amount falls just below (f_x) (0.984 637 2), and then tap X Bar once.

Left figure of approximate interval ratio "n" (0.3) appears in Upper Dial.

(5) Shift to 6th position and hold down Short-Cut Bar until Middle Dial amount falls just below (f_x) (0.984 637 2), and then tap X Bar once.

Left figures of approximate interval ratio "n" (0.33) appear in Upper Dial. Middle Dial shows "0.984 642 217" but the amount need not be separately noted.

(6) Consult Chart of Bessellian Coefficients (see Marchant Method MM439E1) and note that for n = 0.33 the coefficient "B" is -0.055 for Direct Interpolation. However, for inverse interpolation the sign is +. The approximate correction to be applied is + 0.055 X (436 + 433) = 47.8, which value is obtained from pre-calculated table of B" ($d''_0 + d''_1$) or by slide rule. This amount (47.8), being positive, is added to y_x to produce the desired corrected Middle Dial reading; thus, 0.984 637 2 + 0.000 004 78 = 0.984 641 98.

(7) By suitable shifts and use of X and Short-Cut Bars, adjust Upper Dial reading until Middle Dial reads as close as possible to the above amount (0.984 641 98), using only one more figure than contained in d'.

Approximate "n" (0.330 036) appears in Upper Dial and "0.984 641 981.." appears in Middle Dial.

It is now necessary to note whether this alteration of "n" from its approximate value of 0.33 will alter the amount in Step 6. In this case it does not. If it did so, it would be necessary to revise the correction of 47.8 and slightly alter the Upper Dial reading to reflect the effect of this revision upon the Middle Dial reading.

The desired x is 2.330 04

No more places can be taken in the change of x (0.330 04) than occur in d' and if the latter begins with 1, 2, or 3, take one less. There will still be an uncertainty in the final figure, because of rounding and possibly also because of the existence of the 3rd difference which was disregarded (See Note below).

CHECK

Using Marchant Method MM439E1, Example IIa, we have:

	f_0 0.986 798 8
0.330 04 X (-.006 535 1)	-.002 156 8
	0.984 642 0
-0.055 X (0.000 086 9)	-.000 004 8
	0.984 637 2 = f_x

NOTE

The fact that the tabular values are rounded precludes taking more figures in "n" than exist in d'. However, it is advisable to check to see how large the 3rd difference can be before it affects the interpolate. Inasmuch as the method applies to the Bessel formula, the following relation holds, assuming that the function tabulated is the usual sort with progressively smaller differences of higher order.

$$d''' \text{ should not exceed } \frac{60\, d'_1}{10^a}$$

in which "a" is the number of figures taken in the change of argument "n".

In this case "n" is taken to 5 figures, hence d''' will affect the value only if it is greater than

$$\frac{60 \times 0.006\ 535\ 1}{10^5} = .000\ 003\ 9$$

It is actually 1/13 of this, so it does not affect the amount. If we could depend upon the tabulated values as being exact instead of rounded, the change of argument could be taken to 6 figures; i.e., 0.330 036.

1,026 6436 8

INVERSE CURVILINEAR INTERPOLATION

THE COMRIE "TWO-CALCULATOR" METHOD

REMARKS: *The method described herein was introduced by Dr. L. J. Comrie at the British Nautical Almanac Office in 1934. As applied to lever-set calculators, the method was described in the British Nautical Almanac for 1937. The principles that govern this method are subject to numerous variations. Practice with the method will disclose them to the skilled computer if he will observe the hints given in the Notes which follow the description of the method.*

OUTLINE: The Bessel interpolation formula for cases in which 4th differences do not exceed 1000 may be expressed in the form

$$(1) \quad f_n = f_0 + nd'_{1/2} + B''(M''_0 + M''_1) + B'''\, d'''_{1/2}$$

in which $(M''_0 + M''_1)$ represents $(d''_0 + d''_1) - 0.184\ (d''''_0 + d''''_1)$ according to the Comrie Throw-Back, as described in Marchant Method MM-189, and with nomenclature as described in MM-189a.

For purposes of inverse interpolation, this formula is put in the form

$$(2) \quad f_n - B''(M''_0 + M''_1) - B'''\, d'''_{1/2} = f_0 + nd'_{1/2}$$

From this, it will be seen that all values are known except "n," B" and B'''.

Inasmuch as B" and B''' vary according to "n," it is seen that if each side of the above equation is evaluated so that they are equal, and if in doing so the B" and B''' values of the left-hand side correspond to the "n" of the right-hand side, the value of "n" that satisfies this condition will be the desired "n" for the inverse interpolation.

Because of the dependence of B" and B''' upon "n", the former cannot be had until "n" is known. However, if we start with an approximate "n" and apply corresponding approximate B" and B''' to the left-hand side, and then revise B" and B''' and their consequent effects on the left-hand side to correspond to the necessary variation of "n" of the right-hand side to produce equality, we are able by this converging process to establish a final value of "n" that satisfies the conditions stated in the previous paragraph.

It will be seen in the following example, how this may be done by the use of two Marchants, side by side. The left-hand Marchant develops the value of the left-hand side of equation (2), and the right-hand Marchant develops the right-hand side of the equation. The desired "n" appears in the Upper Dial of the right-hand Marchant upon completion of the work.

The exact process of solving the example used herein applies to cases in which 4th differences do not exceed 1000. If they are greater than this amount, proceed as in Note E.

EXAMPLE: Given: the function with differences tabulated as on next page.

(over)

		Differences 1st	2nd	3rd	4th
0.2	$+0.91817\,(f_0)$		$+1248\,(d''_0)$		$+100\,(d''''_0)$
		$-2070\,(d'_{1/2})$		$-199\,(d'''_{1/2})$	
0.3	$+0.89747\,(f_1)$		$+1049\,(d''_1)$		$+68\,(d''''_1)$

$$M''_0 + M''_1 = 0.02297 - (0.184 \times 0.00168) = +0.02266.$$

Find: x when y = 0.91546. As y has 5 places, x will not be obtained to more than 5 places at most, or "n" taken to 4 significant figures. Influences that do not affect 5th place in "n" will, therefore, be disregarded.

OPERATIONS: Decimals on each Marchant: Upper Dial 6, Middle Dial 11, Keyboard Dial 5. Upper Green Shift Key down on left-hand Marchant (LHM) and Non-Shift Key down on right-hand Marchant (RHM). It is assumed that some form of M model Marchant is used, though the method is easily adapted to the D models. Inasmuch as values of y decrease as x increases, move Manual Counter Control on RHM toward operator.

(1) On RHM, set up f_0 (0.91817) in Keyboard Dial and, with carriage in 6th position, touch Add Bar. Clear Upper Dial.

(2) On RHM set up $d'_{1/2}$ (0.02070) in Keyboard Dial and reverse multiply by such two-figure amount as will cause Middle Dial to read as close as possible to f_n (0.91546).

First approx. "n" (0.13) appears in Upper Dial.
Corresponding approx. f_n (0.915479) appears in Middle Dial.
From curve or table, corresponding B" is -0.0283 and B''' is +0.0070.

(3) On LHM set up f_n (0.91546) and, with carriage in 6th position, touch Add Bar. Clear Upper Dial.

(4) On LHM set up $d'''_{1/2}$ (0.00199) and multiply by B''' (0.0070) in such manner as will form product $B'''\,d'''_{1/2}$ and simultaneously deduct it from f_n. Inasmuch as $d'''_{1/2}$ is minus and B''' is plus, their product is minus, so because deduction is required, a direct multiplication is made (in case of a "reverse" multiplication move Manual Counter Control toward the operator). See Note C.

First approx. $f_n - B'''\,d'''_{1/2}$ (0.91547393) appears in Middle Dial.

(5) On LHM clear Upper and Keyboard Dials, set up $M''_0 + M''_1$ (0.02266) in Keyboard Dial and multiply by B" (0.0283) in such a way as to form the product $B''(M''_0 + M''_1)$ and simultaneously deduct it from Middle Dial amount. Inasmuch as $M''_0 + M''_1$ is plus, and B" is minus, their product is minus so because deduction is required, a direct multiplication is made.

First approx. of left side of Eq. 2 (0.916 115 208) appears in Middle Dial.

(6) On RHM, without clearing any dials, multiply by such amount as will show an Upper Dial reading in three figures (n), and Middle Dial reading as close as possible to that of LHM, as developed in Step 5.

Second approx. "n" (0.099) appears in Upper Dial.
Corresponding $f_0 + nd'_{1/2}$ (0.916 120 70) appears in Middle Dial.
Corresponding B" is -0.0223 and B''' is +0.0060.

(7) On LHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads the B" value referred to in last line of Step (6) (0.0223), thus reflecting the effect of decrease of B" from first approximation to second approximation.

Middle Dial reads 0.915 979 248.

(8) On RHM, without clearing any dials, multiply by such amount as will show an Upper Dial reading to five places (n) and Middle Dial as close as possible to that of LHM as developed in Step 7.

Third approx. "n" (0.10583) appears in Upper Dial.
Corresponding $f_0 + nd'_{\frac{1}{2}}$ (0.915 979 319) appears in Middle Dial.
Corresponding B" is - 0.0237 and B"' is +0.0062.

(9) On LHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads the new B" (0.0237).

Middle Dial reads 0.916 010 972

(10) On RHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads as close as possible to that of LHM as developed in Step 9.

Fourth approx. "n" (0.10430) appears in Upper Dial.
Corresponding $f_0 + nd'_{\frac{1}{2}}$ (0.916 010 99) appears in Middle Dial.
Corresponding B" is - 0.0234 and B"' is +0.0062.

(11) On LHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads the new B" (0.0234).

Middle Dial reads 0.916 004 174.

(12) On RHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads as close as possible to that of LHM as developed in Step 11.

Fifth approx. "n" (0.10463) appears in Upper Dial (See Note A).
Corresponding $f_0 + nd'_{\frac{1}{2}}$ (0.916 004 159) appears in Middle Dial.
"n" is taken to four places as 0.1046; or, x = 0.21046.

It will now be noted that the corresponding B" does not change from that of Step 10, and similarly for B"'. Before the work may be regarded as complete, however, it is necessary to see whether or not the alteration of B"' from its value in Step 4 (0.0070) to its value in Step 10 (0.0062) will affect 5th place of adjusted f_n in LHM. Inasmuch as the adjustment is 0.0008 x 0.00199, it will be seen that 5th place is not affected. If it were affected, it would be necessary at this point to clear Upper and Keyboard Dials of LHM, set up 0.00199 and multiply by the difference between the value of B"' used in Step 4 and its final value, and then proceed as in Step 12 on RHM, then clear Upper and Keyboard Dials on LHM, set up 0.02266, and multiply by the <u>difference</u> between the last B" used (0.0234) and its new value found after these adjustments are made. RHM must then be adjusted to produce what will doubtless be a final "n". The necessity of doing this greatly slows the process so to prevent the possibility of its happening it is important to see that the original B"' used in Step 4 is so close to the final value that no further adjustment of it is necessary. This subject is fully discussed in Note C.

(over)

NOTES

(A) It will be observed that the tabulated function has a rejected 5th difference of $d''''_1 - d''''_0 = -0.00032$. It is important to know if this will affect accuracy of "n" by more than ½ in 4th place, and if so, how many places may be retained.

The largest 5th difference that may be retained without its affecting 4th place of "n" by more than ½ is shown by the following formula:

(3) $\frac{500\ d'_{1/2}}{10^a}$ in which "a" is number of decimals in "n" not to be affected.

Applying it to this case, we have.

$$\frac{500 \times 0.02070}{10^4} = 0.0010 \text{ absolute value.}$$

Inasmuch as this is greater than the absolute 5th difference of 0.00032, it is evident that "n" may be taken to 4 places, or n = 0.1046, or x = 0.21046.

The coefficients of Eq. (3) applying to rejected differences, other than the 5th, are

3rd, 60; Mean 4th, 20; Mean 6th, 100.

(B) According to the conventions of the mathematics of inverse interpolation, it is assumed that the values of y are exact; that is to say, their final digits do not represent rounded figures. In practice, however, such amounts are rounded and at times this fact materially affects the number of figures that may be retained in "n". The rule is to take the number of places of the <u>change in x</u> to no more than the number of <u>significant</u> figures in the first difference of the y, and if the latter begins with 1, 2, or 3 to take one less place. For example, in this case there are four significant figures in the $d'_{1/2}$ and it begins with 2. The number of figures of the <u>change in x</u> that may be taken is, therefore, 3. Inasmuch as the <u>change</u> of argument is "n" (0.1046), this may be taken only as 0.105 from which, <u>if rounding errors are to be considered</u>, x should not be indicated to more places than x = 0.2105.

(C) The reason that the first multiplication in LHM is by B''' instead of B'' is because in most cases it is not necessary to make subsequent adjustments of the effect of B''' though adjustments of B'' are nearly always required. The work is greatly simplified if reasonable assurance is had at this point that no further adjustment need be made because of the convergence of B''' to its final value. For example, in this case d''' is 0.00199 and B''' is 0.0070 for the first approximation of "n". Inspection of the Chart of values of B''' shows that a wide variation of "n" (from .10 to .20) may be made with a variation of B''' of only 0.001. Inasmuch as such a variation would cause only 0.000002 in the adjusted f_n in Middle Dial, it will be seen that quite wide variation of "n" would not affect fifth place of the adjusted f_n. The computer, therefore, has reasonable assurance that f_n will not require further adjustment (as referred to in Note at end of Step 12) and, therefore, the first approximation of B''' (0.0070) may be used at this step.

However, if the function has a large third difference so the above relationship does not hold, it is better to make an estimate of about what the final B''' is likely to be and to use such estimated value as the multiplier in Step 4, rather than to use the value which corresponds to the first approximation of "n". Such an estimated B''' could be obtained in this case, if desired, as follows: With aid of slide rule $(M''_0 + M''_1)$ is multiplied by first approx B'' and found to be about -0.00064. By the reasoning stated in Step 5, a modified "n" in RHM may be ob-

tained by adjusting Upper Dial of RHM until its Middle Dial equals about 0.91545 plus 0.00064, or 0.91612, yet not using more than two places in the approx. "n". Performing this operation on RHM shows approx. "n" of 0.10 and reference to table shows corresponding B''' to be 0.006, which value would be used in Step 4. Steps 5 and 6 would then be performed though the values found would differ somewhat from those stated herein, because with respect to RHM a great part of the adjustment of Step 6 has already been done in the intermediate step discussed above.

(D) In cases where d'''' is so small as not to be taken into account (apply Note A to 4th differences), method may be somewhat simplified by omitting the "throw-back" from 4th difference to produce $M''_0 + M''_1$, thus using 2nd differences as written. However, the modification of 2nd differences by the "throw-back" is such an easy thing to do that its omission is not recommended, because in some instances doing so improves the precision in border-line cases of accumulated products.

(E) In cases where d'''' exceeds 1000, the left-hand side of the equation uses 2nd and 4th differences instead of a modified 2nd difference. The extra term is B'''' x $(d''''_0 + d''''_1)$. Additional steps are required on LHM to compute the effect of this term and its subsequent adjustments. In certain instances of this sort it will be found impracticable to follow the process exactly, owing to slow convergence. This is likely to occur in cases where the x is to have many more places than exist in the tabulated y's, or where the differences of succeeding orders do not decrease materially from those of next lower order. In such cases, obtaining an approximate "n" using a portion of the figures of the y's gives the region in which the desired x is located. Sub-tabulating the y's to tenths will then enable a new difference array to be set up to which the usual methods may be applied.

In cases of large 4th differences, the 5th and higher differences may become important. There are means of throwing back these differences to those of lower order, so that the process need not be complicated by coefficients B^V and higher. Details as to this "throw-back" will be supplied upon application.

(F) If tables of B'', B''', and B'''' are not available, the chart on reverse side hereof provides approximate values.

(G) It is often convenient to check the work by direct interpolation; thus,

$$f_n = f_0 + nd'_{½} + B''(M''_0 + M''_1) + B'''\, d'''_{½}, \text{ as follows}$$

$$f_0 = 0.91817$$

$$nd'_{½} = 0.10463 \times (-0.02070) = -0.002166$$

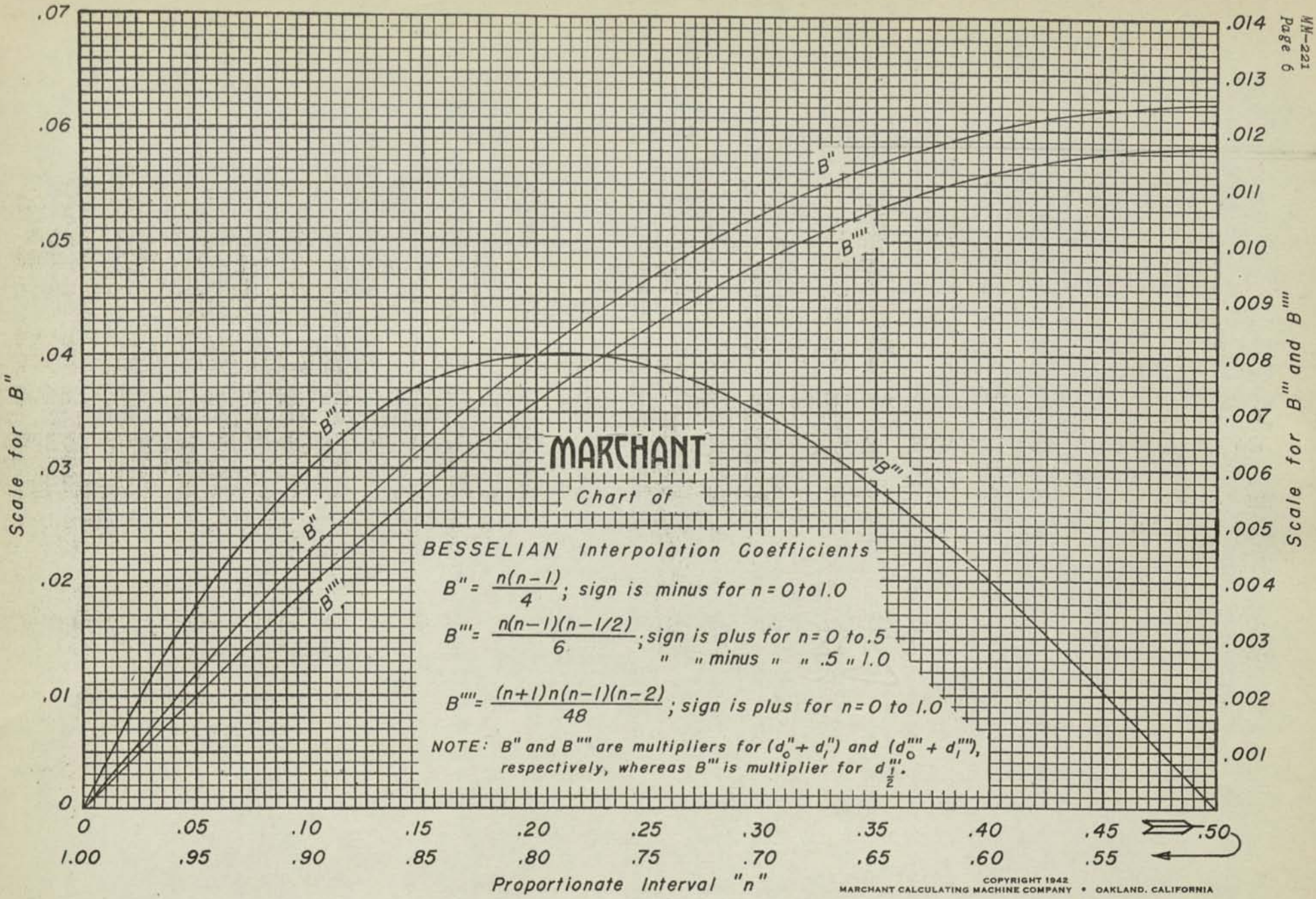
$$B''(M''_0 + M''_1) = -0.0234 \times 0.02266 = -0.000530$$

$$B'''\, d''''_{½} = 0.0062 \times (-0.00199) = \underline{-0.000012}$$

$$f_n = 0.91546$$

In practice, this check would be performed by accumulative multiplication.

1 026 6436 8



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INVERSE CURVILINEAR INTERPOLATION BY "DIVIDED DIFFERENCES"

REMARKS: *When it is desired to determine the argument (independent variable) that corresponds to some function (dependent variable) which lies between values for which there are corresponding tabulated arguments, recourse must be had to Inverse Interpolation. If it is desired to take into account the fact that the function lies on a smooth curve connecting its tabulated values, the process is designated Inverse Curvilinear Interpolation.*

The method herein, basically due to LaGrange, is suitable for infrequent calculations. If extensive work is to be done, such as when transferring tables from one set of coordinates to another, the Comrie Two-Calculator Method is much more suitable. The method herein is also useful for obtaining a real root of a function which is known only by its tabular values or which is difficult to determine from its analytic form.

The method is LIMITED to cases in which the first differences of the function do not change in sign during the intervals from which data are obtained to compute the intermediate functions used in the process. From this it follows that the method is applicable only to cases in which the tabulated values are either CONTINUALLY INCREASING or CONTINUALLY DECREASING within the intervals from which data are obtained to compute the intermediate functions.

If the tabulated values are those of an analytic function of usual form having continuous derivatives up to but not including the order of differences which tend to disappear, then the argument so obtained will correspond to the specified value of the function.

EXAMPLE: Find X when Y = 0.32999 in the table below. (For other purposes to be described later, the table is shown with its differences, as computed by Marchant Method MM-419).

Argument X	Function Y	Differences 1st (d')	2nd (d'')	3rd (d''')	4th (d'''')
1.1	0.45360				3
		-9124	-454		
1.2	0.36236			92	
		-9486	-362		3
1.3	0.26750			95	
		-9753	-267		1
1.4	0.16997			96	
		-9924	-171		2
1.5	0.07073			98	
		-9997	- 73		1
1.6	-0.02924				

By inspection of differences, it will be observed that they are continuous and that the first differences (d') do not change sign.

(over)

OUTLINE OF CALCULATION

The array of differences on the preceding page shows that 5th differences are substantially zero. For precise work, the 4th differences should be taken into account. For reasons outlined on Page 4, they should not be disregarded as would be the case with direct interpolation. The ordinates to be considered in this computation, therefore, are those contained between the left ends of the two diagonal lines; i.e., from 1.2 to 1.6 inclusive. It will be noted that the column of 4th differences tends to vary, probably because of forcing errors from rounding of the functions. For purposes of a correction after completion of the work, the 4th differences are plotted and a "true amount" of 1.3 at argument 1.4 is taken.

RETABULATION WITH DIVIDED DIFFERENCES: The argument and function columns are interchanged below, though to prevent confusion of nomenclature they are left with their original designations. Certain "divided difference" functions f_1, f_2, etc., are included in this array, the values of which are obtained in the next step of this outline. The "y" column has also been altered to express the difference between the actual "y's" and the one for which the unknown "x" is to be found; i.e., 0.36236 - 0.32999 = 0.03237, 0.26750 - 0.32999 = -0.06249, etc. The argument that is to be found is now a function of the value of the new "y" when it is zero; i.e., f (0).

y	x = f(y)	f_1	f_2	f_3	f_4
		"Divided Differences" inserted from next steps			
.03237 (y_0)	1.2 (fy_0)				
		$-1.054185\,(f_1)_{\frac{1}{2}}$			
-.06249 (y_1)	1.3 (fy_1)		$-0.150006\,(f_2)_1$		
		$-1.025326\,(f_1)_{1\frac{1}{2}}$		$-0.206491\,(f_3)_{1\frac{1}{2}}$	
-.16002 (y_2)	1.4 (fy_2)		$-0.089787\,(f_2)_2$		$0.072495\,(f_4)_2$
		$-1.007658\,(f_1)_{2\frac{1}{2}}$		$-0.178102\,(f_3)_{2\frac{1}{2}}$	
-.25926 (y_3)	1.5 (fy_3)		$-0.036937\,(f_2)_3$		
		$-1.000300\,(f_1)_{3\frac{1}{2}}$			
-.35923 (y_4)	1.6 (fy_4)				

The computed figures were actually made to 8 places, though they are rounded to 6 in the above array. This practice is followed throughout this method and serves to explain why the six-place figures, when substituted, may not produce the exact six-place result that is shown.

CALCULATION OF DIVIDED DIFFERENCES: The values of f_1, f_2, etc., of the above array are computed in accordance with the plan of the diagram below, in which the actual "f" values shown are individually described and computed on the next page.

"y"	"x"	"Divided Difference" Functions			
		f_1	f_2	f_3	f_4
y_0	$f(y_0)$				
		$f(y_0y_1)$			
y_1	$f(y_1)$		$f(y_0y_1y_2)$		
		$f(y_1y_2)$		$f(y_0y_1y_2y_3)$	
y_2	$f(y_2)$		$f(y_1y_2y_3)$		$f(y_0y_1y_2y_3y_4)$
		$f(y_2y_3)$		$f(y_1y_2y_3y_4)$	
y_3	$f(y_3)$		$f(y_2y_3y_4)$		
		$f(y_3y_4)$			
y_4	$f(y_4)$				

The actual computations in accordance with the outline at bottom of preceding page are as follows:

(1) $f(y_0y_1) = \dfrac{f(y_0) - f(y_1)}{y_0 - y_1} = \dfrac{-0.1}{0.09486} = -1.054185$

(2) $f(y_1y_2) = \dfrac{f(y_1) - f(y_2)}{y_1 - y_2} = \dfrac{-0.1}{0.09753} = -1.025326$

(3) $f(y_2y_3) = \dfrac{f(y_2) - f(y_3)}{y_2 - y_3} = \dfrac{-0.1}{0.09924} = -1.007658$

(4) $f(y_3y_4) = \dfrac{f(y_3) - f(y_4)}{y_3 - y_4} = \dfrac{-0.1}{0.09997} = -1.000300$

(5) $f(y_0y_1y_2) = \dfrac{f(y_0y_1) - f(y_1y_2)}{y_0 - y_2} = \dfrac{-0.028860}{0.19239} = -0.150006$

(6) $f(y_1y_2y_3) = \dfrac{f(y_1y_2) - f(y_2y_3)}{y_1 - y_3} = \dfrac{-0.017667}{0.19677} = -0.089787$

(7) $f(y_2y_3y_4) = \dfrac{f(y_2y_3) - f(y_3y_4)}{y_2 - y_4} = \dfrac{-0.007358}{0.19921} = -0.036937$

(8) $f(y_0y_1y_2y_3) = \dfrac{f(y_0y_1y_2) - f(y_1y_2y_3)}{y_0 - y_3} = \dfrac{0.060219}{0.29163} = -0.206491$

(9) $f(y_1y_2y_3y_4) = \dfrac{f(y_1y_2y_3) - f(y_2y_3y_4)}{y_1 - y_4} = \dfrac{-0.052850}{0.29674} = -0.178102$

(10) $f(y_0y_1y_2y_3y_4) = \dfrac{f(y_0y_1y_2y_3) - f(y_1y_2y_3y_4)}{y_0 - y_4} = \dfrac{0.017333}{0.39160} = -0.072495$

CALCULATION OF ARGUMENT: The formula applying is $f(0) = f(y_0) - y_0(f_1)_{\frac{1}{2}} + y_0y_1(f_2)_1 - y_0y_1y_2(f_3)_{1\frac{1}{2}} + y_0y_1y_2y_3(f_4)_2 \ldots\ldots$

Substituting values, we have

(1)	fy_0	*NOTE:*	1.2
(2)	$-y_0(f_1)_{\frac{1}{2}}$	*Take "y" values from left-hand column of middle array on Page 2.*	+ ·034124
(3)	$+y_0y_1(f_2)_1$		+ ·000303
(4)	$-y_0y_1y_2(f_3)_{1\frac{1}{2}}$	*Take "f" values from same array at right.*	+ ·000067
(5)	$+y_0y_1y_2y_3(f_4)_2$		+ ·000002
			1.23449643 (to 8 places)
		4th difference correction *(see below)*	·00000077
			1.23449566

FOURTH DIFFERENCE CORRECTION: As stated at top of Page 2, the 4th difference at argument 1.4 is taken as 1.3 instead of as 2. Inasmuch as the 5th item, above, shows the effect of the 4th difference, this reduction of 35% should reduce item (5) correspondingly, and as it is positive, the interpolate is decreased to 1.23449566. For reasons given at bottom of Page 4, it is not good practice, (provided the function values are rounded, as is usually the case) to take this value to more figures than 1.23450, with an uncertainty of 1 or 2 in the last figure.

(over)

NOTES

NUMBER OF FIGURES TO BE TAKEN: The value obtained is the most precise result that this method makes possible. It contains no uncertainty because of a Remainder Term, which is substantially non-existant. However, it cannot be used to the number of places shown because the function values contain an error from rounding off their final figures. Because of this, the rule is to take the number of places of the change in argument to no more than the number of significant figures in the first difference of function, and if the latter begins with 1, 2, or 3, to take one less place. For example, in this case there are four significant figures in the first difference of the function (Y), which begin with a number greater than 1, 2, or 3. The number of places in the change in argument cannot be taken as more than 4. (The change in argument is the set of figures at right of 1.2; viz., 3449566). These can be taken only to four figures, making the value that can be used as 1.23450 with an uncertainty of 1 or 2 in the last figure.

WHY METHOD LIMITED TO CASES WHERE FIRST DIFFERENCE OF FUNCTION DOES NOT CHANGE SIGN: If it changes sign, the function rises and falls in the interpolation interval, thereby having a tangent point in which a change in Y produces a change in X at an infinitely greater rate. Should the intervals be small, the differences that are divisors in producing f_1's (see middle of Page 2), may produce extraordinarily large f_1's, leading to asymptotic conditions of unreliability. Not all cases where first differences change sign will produce these effects, but it is necessary to stress this as a limitation to the application of the method.

SELECTION OF ARGUMENTS TO BE TAKEN INTO ACCOUNT: A frequently heard criticism of the Divided Difference Method is that it provides no means of knowing how many function values should be taken into account. This point is covered herein by means of the difference array and the diagonal lines mentioned at top of Page 2. It might be thought that the higher order differences which involve small amounts could be disregarded by applying the same rules as for direct interpolation; viz., disregard 2nd, if they are less than 4, 3rd, if less than 8, 4th, if less than 12 (for Descending Difference formulas). This does not apply in the case of the inverse operation unless the corresponding derivative of function with respect to argument is "1" or more. As it is not easy to determine the higher-order derivatives from differences when the variables are interchanged (though mathematical methods render this comparatively simple) it is believed advisable to include in the calculation all orders of difference that appear in the array. This makes it unnecessary to compute a Remainder Term of the interpolation polynomial because the next higher order of difference would be non-existant (for the usual type of functions found in computing practice).

Quite a satisfactory result is obtained in the example used if 4th differences are disregarded. A skilled computer would know that this would be true because he would make a rough estimate of the probable fourth derivative at argument 1.4 by considering it as of about the order of magnitude of the slope of the tangent to the curve of "3rd differences divided by cube of interval." However, application of this method is somewhat uncertain and inasmuch as very little extra labor is required to include all significant differences, the method herein has been developed in that manner.

CHECK OF VALUE FOUND BY DIRECT INTERPOLATION: If by MM-439E1 or otherwise, we interpolate a function between tabular values that corresponds to argument of 1.234496, it will be found to be 0.33000 $\pm$ the exact amount depending upon the formula that is used. However, use of argument 1.23450 produces an interpolate of 0.32999+. It is not possible to say that one is more "correct" than the other, as it is not unusual for the values of the inverse and direct interpolations to vary slightly in an opposite manner.

MARCHANT COMPUTATION

The previous outline appears somewhat formidable, but the actual calculation on the Marchant is simple, requiring surprisingly few steps, as follows:

OPERATIONS: Decimals; Upper Dial 8, Middle Dial 16, Keyboard Dial 8, No. 9 Tab Key depressed. Use any 10 column Marchant. (The method is also adapted to 8 column Marchants.)

(1) Compute the differences in the array at bottom of Page 1, using Marchant Method MM-419, and at the same time compute the differences between any two of the arguments whose differences are not shown on the array; i.e., $y_0 - y_2 = 0.19239$; $y_1 - y_3 = 0.19677$; $y_2 - y_4 = 0.19921$; $y_0 - y_3 = 0.29163$; $y_1 - y_4 = .29674$; $y_0 - y_4 = 0.39160$.

(2) With carriage in 9th position, set up in Keyboard Dial the constant interval of the argument (0.1) and add. Similarly, set up the first divisor $y_0 - y_1$ (0.09486) and divide.

$(f_1)_{\frac{1}{2}}$ (1.05418512) appears in Upper Dial, which is copied to array at middle of Page 2, with minus sign, as it is a quotient of a minus amount divided by a plus amount.

(3) Similarly, obtain all values of f_1, f_2, f_3, and f_4, the factors being given on Page 3. Clear all dials.

(4) Obtain the amounts $y_0y_1, \ldots y_0y_1y_2y_3$ as shown in the formula for computing argument (see bottom of Page 3) as follows: Set up in Keyboard Dial y_0 (0.03237) and multiply by y_1 (0.06249), copying answer y_0y_1 from Middle Dial (-0.00202280); clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial; clear Middle Dial and multiply by y_2 (.16002), copying answer $y_0y_1y_2$ (0.00032369) and in the same manner obtain $y_0y_1y_2y_3$ (-0.00008392) and $y_0y_1y_2y_3y_4$ as (0.00003015), affixing signs according as the individual factors are plus or minus.

(5) Clear dials, shift to 9th position, set up in Keyboard Dial $f(y_0)$ (1.2) and add.

(6) Clear Upper Dial, set up $(f_1)_{\frac{1}{2}}$ (1.05418512) and multiply by y_0 (0.03237) because the first factor is minus, the second is plus, and the product is to be subtracted (see Item 2 of formula at bottom of Page 3), so the product is to be added.

(7) Clear Upper and Keyboard Dials, and similarly multiply $(f_2)_1$ (0.15000561) by y_0y_1, (0.00202280), obtaining the latter value from Step 4, above.

(8) Clear Upper and Keyboard Dials, and similarly multiply $(f_3)_{1\frac{1}{2}}$ (0.20649059) by $y_0y_1y_2$ (0.00032369), obtaining the latter value from Step 4, above.

(9) Clear Upper and Keyboard Dials, and similarly multiply $(f_4)_2$ (0.07249451) by $y_0y_1y_2y_3$ (0.00003015).

Inverse Interpolate for 0.32999 (1.23449643) appears in Middle Dial. After 4th difference correction is made (see Page 3), value is 1.23449566.

NOTE: The number of figures of the above that may be taken as correct, assuming rounding off of functional values, is discussed on Page 4.

Whether or not the multiplications of Steps 6, 7, 8, or 9 are made so as to add the products, as formed, or to subtract them, depends upon the sign of the product in accordance with the signs of its factors, and also whether or not the product is to be added to or subtracted from the previous amount. It is assumed that the computer is familiar with this matter.

THE A. C. AITKEN METHOD OF CURVILINEAR INTERPOLATION WITH EQUAL OR UNEQUAL INTERVALS OF THE ARGUMENT

REMARKS: Straight-line interpolation between two points may be performed as described in Marchant Method MM439A1 according to the following well-known principle: If given two functional values u_a and u_b corresponding to arguments "a" and "b", an intermediate value u_x corresponding to argument "x" between "a" and "b" may readily be obtained by a continuous calculator operation as follows:

$$u_x = \frac{u_a(b - x) - u_b(a - x)}{b - a}$$

This relation also holds if "x" does not lie between "a" and "b"; that is to say, the formula is suitable as a means of straight-line extrapolation.

A. C. Aitken* has published a method of curvilinear interpolation based upon the above relationship. The method first obtains a set of approximations to u_x by applying the above formula successively for pairs of values a b, a c, a d, a e, etc., and then by considering the approximations thus found to be a new set of functional values, the process is again applied to the new set to give a second set of approximations still closer to the desired value, and so on until successive iterations produce no change in the interpolate to the number of places it is desired to retain.

The method is equally suitable for interpolation if the functional values are tabulated at equal or unequal intervals of the argument; i.e., the method is suitable for either direct or inverse interpolation.

The particular field of use for the method appears to be cases in which (1) interpolation must be made to an argument-value bearing fractional relationship to the tabulated arguments of more than three decimals, because there are readily available tables of Bessell, Everett, and LaGrange coefficients** for fractional values differing by .001, or (2) for interpolation when values are tabulated at unequal intervals of the argument, in which case the method serves the same purpose as the Method of Divided Differences (See Marchant Method MM-439J2). Aitken has shown that his process is the same as using the general divided-difference formula to the same order of differences as the number of stages of his process.

Particularly do we recommend that this method be given consideration in cases of interpolation at unequal intervals of the argument. The exceedingly simple manner in which the example of Marchant Method MM-439J2 responds to the Aitken method is shown herein on page 4.

(*) *A. C. Aitken, Edinburgh Math. Soc. Proc., s. 2, v. 3, 1932, p. 56. An excellent synopsis of the method written by J. R. Womersley appears in Math. Tables and Other Aids to Computation, Vol. II, No. 15, p. 112, July 1946, National Research Council, Washington, D. C.*

(**) *cf Marchant Methods MM-439G, 439E1, 439H, 439J1, 228, which relate to various methods of direct and inverse curvilinear interpolation. MM-228 gives 7-place Lagrangean 5-point interpolation coefficients for values of p from 0 to 2, by .001. U. S. Bureau of Standards, Mathematical Tables Project, offers more extensive tables of Lagrangean coefficients, and many of its tables of other functions show supplementary tables of Bessell and Everett coefficients. L. J. Comrie's tables and explanatory information, "Interpolation and Allied Tables," H. M. Stationery Office, York House, Kings Way, London, W.C.2, are indispensable to anyone working in this field.*

CASE 1: INTERPOLATION WITH EQUAL INTERVALS OF THE ARGUMENT

EXAMPLE: The example used is the one given in Aitken's paper. Given the tabulated values "u" below, to find the value of "u" for argument 0.68327. Development of the computation is also shown.

Argument	u	Stage (1)	Stage (2)	Stage (3)	Stage (4)	Parts
-2	5772 1566					-2.68327
-1	5608 8546	53 339732.4				-1.68327
0	5447 8931	.. 371133.4	39 2588.8			-0.68327
1	5289 2109	.. 401967.0	.. 2128.1	227 4.0		0.31673
2	5132 7488	.. 432306.6	.. 1674.9	... 6.6	..3.2	1.31673
3	4978 4499	.. 462107.2	.. 1229.9	... 9.3	..3.2	2.31673

OPERATIONS: Decimals: Upper Dial 8 and 1, Middle Dial 16 and 9, Keyboard Dial 8 and 1.
Use 10-Column Marchant Division-Clear Lever toward operator.

In the following, multiply with reference to the Upper-Dial decimal at 8 and Keyboard-Dial decimal at 1. Set up divisors at 8th Keyboard-Dial decimal and read quotients at 1st Upper-Dial decimal. Use "parts" as multipliers.

(1) Compute the column of "parts" by setting up the argument to which interpolation is to be made (.68327) and subtracting it to produce its complement (..99.31673) in the Middle Dial. Then set up the greatest positive argument (3.0000) in Keyboard Dial and add, producing 3 - .68327 = 2.31673 in Middle Dial. The other values in this case are obtained by inspection, though if there are unequal intervals of the argument, after each addition, the amount added should then be subtracted to restore the negative constant (..99.31673) in the Middle Dial.

(2) Each value for Stage (1) is computed as a continuous cross-multiplication and division process as follows:

$$(5772\ 1566 \times -1.68327) - (5608\ 8546 \times -2.68327) = 5333\ 9732.4$$

This step has no division operation as the argument interval is 1.

The successive computations are thus

$$\frac{(5772\ 1566 \times -0.68327) - (5447\ 8931 \times -2.68327)}{2} = 5337\ 1133.4$$

$$\frac{(5772\ 1566 \times\ 0.31673) - (5289\ 2109 \times -2.68327)}{3} = 5340\ 1987.0$$

and so on to

$$\frac{(5772\ 1566 \times\ 2.31673) - (4978\ 4499 \times -2.68327)}{5} = 5346\ 2107.2$$

(continued on next page)

The well-known rule of signs governs the Marchant entries; that is to say, in the preceding expression, the second multiplier (-2.68327) is not entered in reverse as would be indicated by its minus sign. Instead, it is entered directly as a positive amount because the product (4978 4499 x -2.68327) is to be subtracted. This is equivalent to adding (4978 4499 x 2.68327) to the amount that is already in the Middle Dial.

(3) Values for Stage (2) are similarly computed as follows:

$$(33\ 9732.4 \times -0.68327) - (37\ 1133.4 \times -1.68327) = 39\ 2588.8$$

$$\frac{(33\ 9732.4 \times 0.31673) - (40\ 1987.0 \times -1.68327)}{2} = 39\ 2128.1$$

and so on to

$$\frac{(33\ 9732.4 \times 2.31673) - (46\ 2107.2 \times -1.68327)}{4} = 39\ 1229.9$$

(4) The values for Stage (3) are similarly computed as follows:

$$(2588.8 \times 0.31673) - (2128.1 \times -0.68327) = 2274.0$$

and so on to

$$\frac{(2588.8 \times 2.31673) - (1229.9 \times -0.68327)}{3} = 2279.3$$

(5) The values for Stage (4) are as follows:

$$(4.0 \times 1.31673) - (6.6 \times 0.31673) = 3.2$$

$$\frac{(4.0 \times 2.31673) - (9.3 \times 0.31673)}{2} = 3.2$$

which completes the iteration to 5339 2273.2

NOTES: In Step (5) obvious reduction of length of multipliers would be made because only two significant figures are desired.

It is not necessary to write the individual steps in formula form as outlined above because the computer soon has the relationships in mind so it is only necessary to make the computation and copy the respective values as shown under the heading, "Example."

(over)

CASE 2: INTERPOLATION WITH UNEQUAL INTERVALS OF THE ARGUMENT

If the arguments are tabulated at unequal intervals, the method is applied in a similar way.

EXAMPLE: The example of Marchant Method **MM439J2**, solved therein by divided differences, is easily and quickly solved as below for argument 96.94:

Argument	u	Stage (1)	Stage (2)	Parts
96.70	1.449 8542			-.24
96.90	1.456 0764	1.457 3208		-.04
97.02	1.459 8272	..... .340	...252	.08
97.25	1.467 0530	..... .591	...252	.31

The value of "u" corresponding to 96.94 is thus seen to be 1.4573252.

The process is obvious, but we show a few typical computations, as follows:

From Stage (1)

$$\frac{(1.449\ 8542 \times -.04) - (1.456\ 0764 \times -.24)}{96.90 - 96.70} = 1.457\ 3208$$

$$\frac{(1.449\ 8542 \times .08) - (1.459\ 8272 \times -.24)}{97.02 - 96.70} = 1.457\ 3340$$

From Stage (2)

$$\frac{(208 \times .08) - (340 \times -.04)}{97.02 - 96.90} = 252$$

NOTES:

If the final value is inserted in the column of amounts of any stage in line with the argument (untabulated) to which it corresponds, it will be found that the amounts in that stage converge to the value. This is a helpful broad check on the progress of the work.

As in all interpolation, it is recommended that the points used be those which centrally surround the unknown value to the extent that this is possible. In cases where it cannot be done, as at the end of a table, the convergence is slower and rounding errors magnify themselves. As a consequence, usually more points will have to be used to obtain the desired result unless, of course, the tabulated values exactly represent a polynomial of degree at most one less than the number of points used. When interpolating in the end interval, it makes no difference whether the work progresses away from the unknown value or toward it. The former plan gives progressive values in the various stages that are closer to the final value, but the convergence is correspondingly slower. The final result is the same, within limits imposed by rounding.

In the extreme case of using the method for extrapolation to a point one equal argument-interval outside of the set of points, the method gives the same result as is obtained by differencing the tabulated values and extrapolating by assuming a constant (n-1)th difference, where "n" is the number of tabulated values that have to be used to produce convergence.

1 026 64368

AREA BELOW CURVE IN THE CASE OF FRACTIONALLY-SPACED ORDINATES

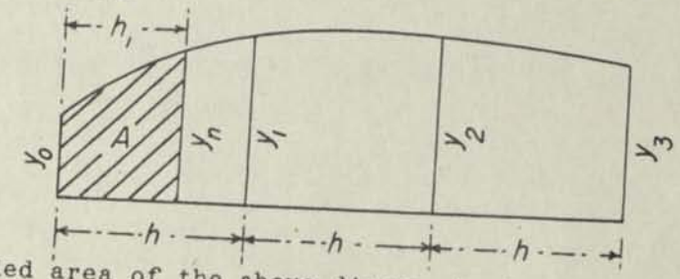
(Also Suitable for Integration of Equations of 3rd Degree, or Less)

REMARKS: *The usual rules for approximate integration (Simpson, Weddle, etc.) require that the base-line be divided into sections by equidistant ordinates. When the heights of these ordinates are known, the area of any or all of the individual sections may be readily computed within limits of error which vary according to the formula used (in Simpson's 1/3 Rule, 3rd differences are taken into account). See Note C on reverse side hereof for limitations.*

It is often desired to know the area of a fractional part of one of the sections bounded by such equidistant ordinates. This problem frequently occurs in computing areas and volumes of irregular shapes in ship construction and elsewhere. The problem is one of approximate integration of $y = f(x)$ when the upper limit of integration is a fraction.

The method herein takes into account 3rd differences, yet it does not require the computation of tables of differences of the ordinates. So far as we are able to determine, the method is an original contribution of the Marchant Calculating Machine Company, though it has its foundation in well-known processes.

OUTLINE:



$$n = \frac{h_1}{h}$$

The shaded area of the above diagram is obtained by the following formula, assuming that 4th differences may be disregarded:

$$\text{Area} = \int_{x_0}^{x_0+nh} y(dx) = h\,(K_0y_0 + K_1y_1 - K_2y_2 + K_3y_3),$$

in which the values of the "K" constants are taken from table below at various values of n. The table shows constants to five decimals. Exact values will be supplied upon application.

n	K_0	K_1	K_2	K_3
0.1	.09116 +	.01418 -	.00685 -	.00151 -
0.2	.16593 +	.05353 +	.02486 +	.0054
0.3	.22616 +	.11351 +	.05051 +	.01084 -
0.4	.2736	.18987 -	.08053 +	.01706 +
0.5	.30990 -	.27864 +	.11198 -	.02344 -
0.6	.3366	.3762	.1422	.0294
0.7	.35516 +	.47918 +	.16885	.03451 -
0.8	.36693 +	.58453 +	.18986 +	.0384
0.9	.37316 +	.68952 -	.20352	.04084 -
1.0	.375	.79167 -	.20833 +	.04166 +

(over)

Values of K's on preceding page that are not followed by + or - are exact.

The attached curve of these coefficients enables their values to be approximately determined at intermediate values of "n".

EXAMPLE: In the diagram on reverse side, the values of y at points 4 ft. apart are $y_0 = 6.342$, $y_1 = 8.502$, $y_2 = 8.680$, and $y_3 = 7.128$. Find area A when y_n is 2.8 ft. from left-hand ordinate ($h_1 = 2.8$, or $n = 2.8/4 = 0.7$).

OPERATIONS: Decimals; Upper Dial 4, Middle Dial 9, Keyboard Dial 5. Any Marchant model.

(1) Set up in Keyboard Dial K_0 for $n = 0.7$ (.35516) and multiply by y_0 (6.342).

(2) Clear Upper and Keyboard Dials only, set up K_1 (.47918) in Keyboard Dial and multiply by y_1 (8.502).

(3) Clear Upper and Keyboard Dials only, move Manual Counter Control toward operator, and similarly reverse multiply K_2 (.16885) by y_2 (8.680).

(4) Clear Upper and Keyboard Dials only, move Manual Counter Control away from operator, and multiply K_3 (.03451) by y_3 (7.128).

Area /h (5.10678) appears in Middle Dial.

(5) Clear Upper and Keyboard Dials only, shift to 5th position, transfer Middle Dial amount (5.10678) to Keyboard Dial, clear Middle Dial, and multiply by h (4.0).

Area desired (20.427 sq. ft.) appears in Middle Dial.

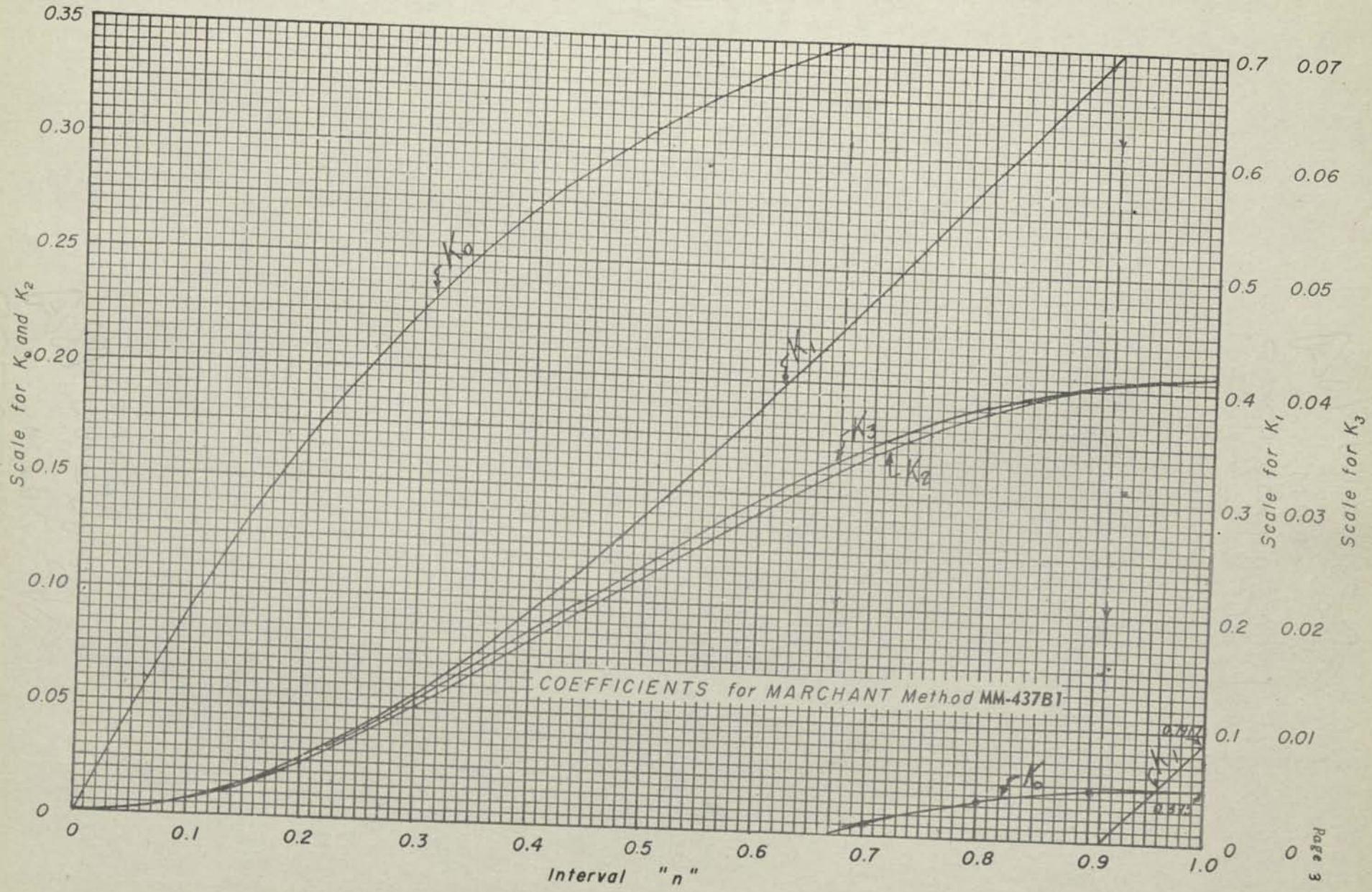
NOTES

A: The ordinates in the above example are taken as four significant figures from the equation $y = .000656\ X^3 - .06981\ X^2 + .80875\ X + 6.342$, the integration of which, between limits of $X = o$ to $X = 2.8$, is 20.427, the same as found by the above process. This illustrates that the method may be used for integrating 3rd degree equations, or any of lesser degree, and of course for the approximate integration of equations of higher degree.

B: If only the ordinates y_0, y_1 and y_2 are available, and it is not convenient to divide the distance between y_0 and y_2 into three equal spaces, the best procedure is to draw a free-hand curve to extend it to a hypothetical y_3, which is then approximately measured.

C: This method obtains the area under a smooth curve of minimum continuous curvature that connects the four points. If the curve has waves in any of the equal sections, or angularities showing abrupt changes of curvature, the method does not apply. Mathematically, this process integrates a continuous single-valued function having constant third derivative.

D: The rounding of the exact coefficients to five figures has been slightly adjusted in order that the algebraic sum of the coefficients of any row will equal "n".


COEFFICIENTS for MARCHANT Method MM-437B1
Scale for K_0 and K_2
Scale for K_1
Scale for K_3
Interval "n"
K_0
K_1
K_2
K_3
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

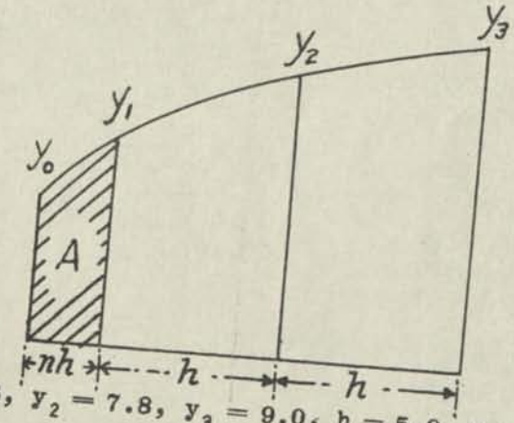
WATKINS METHOD FOR AREA BELOW CURVE WHEN END SECTION HAS DIFFERENT SPACING FROM THAT OF ADJACENT SECTIONS

REMARKS: *Marchant Method 437B1 provides means of solving certain problems involving fractionally-spaced ordinates. However, it does not care for cases in which there is a single area of unequal spacing adjacent to several areas having equal spacing.*

William H. Watkins, Senior Naval Architect and Supervisor of the Scientific and Test Groups, Design Section, Puget Sound Navy Yard, has developed a most satisfactory method of solving such problems by a formula derived from integration of the LaGrange Interpolation Formula of 3rd Degree. The method also has the advantage that the area with unequal spacing may have its limiting ordinates spaced either greater or less than that of adjacent sections having equally-spaced ordinates; furthermore, only two such sections bounded by equally-spaced ordinates are required.

The method assumes that a third-degree curve is passed through the four points; consequently it does not apply if the actual curve has abrupt changes of curvature or points of inflexion.

EXAMPLE:



GIVEN: $y_0 = 4.0$, $y_1 = 5.6$, $y_2 = 7.8$, $y_3 = 9.0$, $h = 5.0$, $n = 0.4$

FIND: Area of shaded part A.

OUTLINE: The formula applying is

$$A = n\,h\,(K_0 y_0 + K_1 y_1 - K_2 y_2 + K_3 y_3)$$

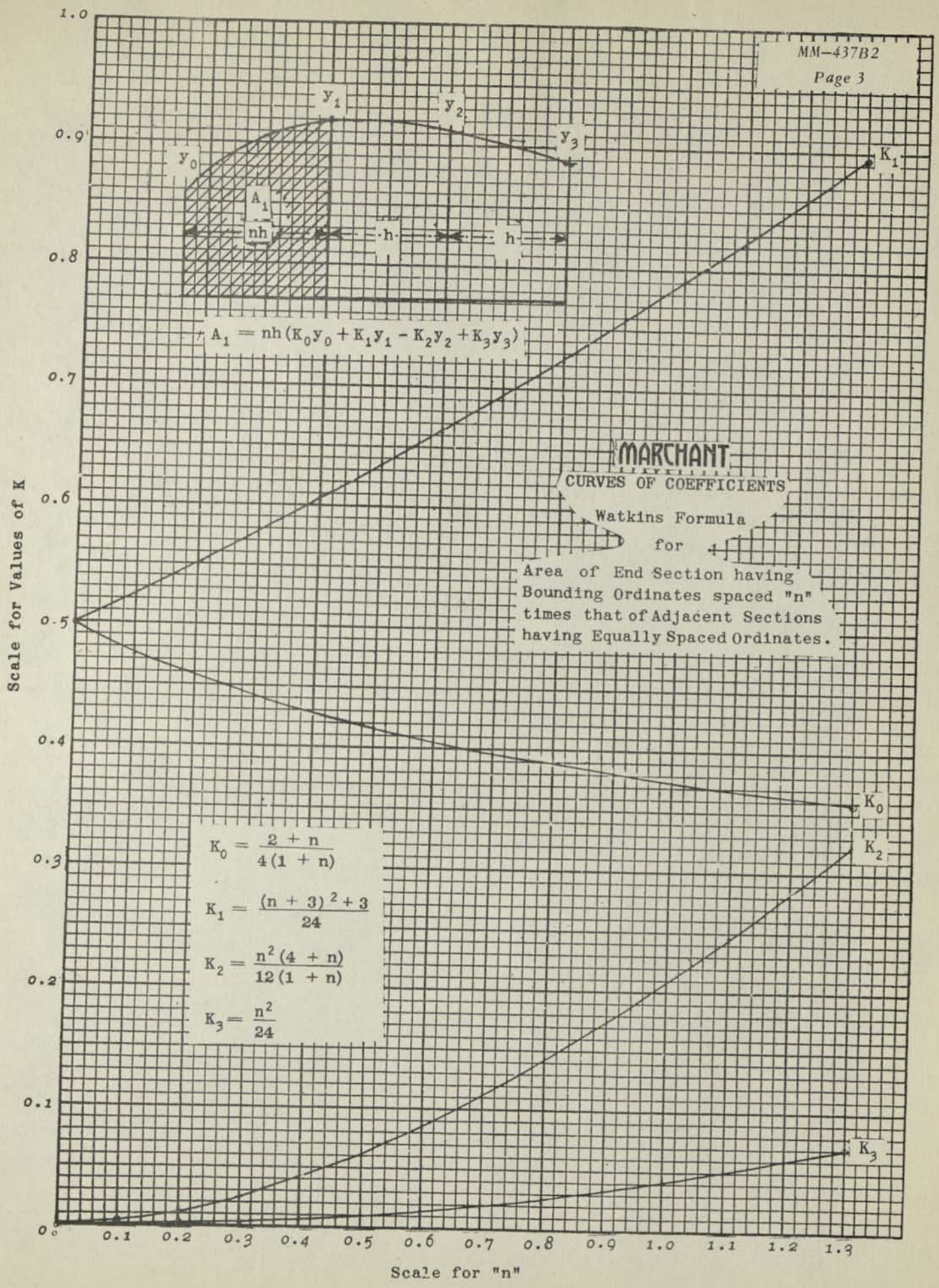
in which the values of K_0, K_1, etc., are obtained from table on reverse side hereof, or from curve on Page 3.

OPERATIONS: By using direct and reverse accumulative multiplication as described in Marchant Method 437B1, solution of the above formula is a continuous calculator process, requiring no copying of intermediate amounts to work sheet. In the above case, the formula with substituted values is

$$A = 0.4 \cdot 5.0\,(0.4286 \cdot 4.0 + 0.6067 \cdot 5.6 - 0.0419 \cdot 7.8 + 0.0067 \cdot 9.0) = 9.691$$

COEFFICIENTS FOR WATKINS METHOD FOR AREA BELOW CURVE
WHEN END SECTION HAS DIFFERENT SPACING FROM THAT OF ADJACENT SECTIONS

n	K_0	K_1	K_2	K_3
0	.500 000	.500 000	0	0
0.05	.488 095	.512 604	-.000 803	.000 104
.10	.477 273	.525 416	.003 106	.000 417
.15	.467 391	.538 437	.006 766	.000 938
.20	.458 333	.551 667	.011 667	.001 667
.25	.450 000	.565 104	.017 708	.002 604
0.30	.442 308	.578 750	-.024 808	.003 750
.35	.435 185	.592 604	.032 893	.005 104
.40	.428 571	.606 667	.041 905	.006 667
.45	.422 413	.620 938	.051 789	.008 438
.50	.416 667	.635 416	.062 500	.010 417
0.55	.411 290	.650 104	-.073 998	.012 604
.60	.406 250	.665 000	.086 250	.015 000
.65	.401 515	.680 104	.099 223	.017 604
.70	.397 058	.695 417	.112 892	.020 417
.75	.392 857	.710 937	.127 232	.023 438
0.80	.388 889	.726 666	-.142 222	.026 667
.85	.385 135	.742 604	.157 843	.030 104
.90	.381 579	.758 750	.174 079	.033 750
.95	.378 205	.775 104	.190 913	.037 604
1.00	.375 000	.791 667	.208 333	.041 666
1.05	.371 951	.808 437	-.226 326	.045 938
.10	.369 047	.825 416	.244 880	.050 417
.15	.366 279	.842 604	.263 987	.055 104
.20	.363 636	.860 000	.283 636	.060 000
.25	.361 111	.877 604	.303 819	.065 104
1.30	.358 696	.895 417	-.324 529	.070 416
.35	.356 383	.913 437	.345 757	.075 937
.40	.354 167	.931 667	.367 500	.081 666
.45	.352 041	.950 104	.389 749	.087 604
.50	.350 000	.968 750	.412 500	.093 750
1.55	.348 039	.987 604	-.435 747	.100 104
.60	.346 154	1.006 667	.459 487	.106 666
.65	.344 340	1.025 937	.483 715	.113 438
.70	.342 593	1.045 416	.508 425	.120 416
.75	.340 909	1.065 104	.533 617	.127 604
1.80	.339 285	1.085 000	-.559 285	.135 000
.85	.337 719	1.105 104	.585 427	.142 604
.90	.336 207	1.125 416	.612 040	.150 417
.95	.334 746	1.145 937	.639 120	.158 437
2.00	.333 333	1.166 667	.666 666	.166 666


y_0
y_1
y_2
y_3
A_1
nh
h
h
$A_1 = nh(K_0y_0 + K_1y_1 - K_2y_2 + K_3y_3)$
MARCHANT
CURVES OF COEFFICIENTS
Watkins Formula
for
Area of End Section having Bounding Ordinates spaced "n" times that of Adjacent Sections having Equally Spaced Ordinates.
$K_0 = \frac{2 + n}{4(1 + n)}$
$K_1 = \frac{(n + 3)^2 + 3}{24}$
$K_2 = \frac{n^2(4 + n)}{12(1 + n)}$
$K_3 = \frac{n^2}{24}$
K_0
K_1
K_2
K_3
Scale for Values of K
Scale for "n"
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3

MOMENT OF INERTIA OF SECTIONS COMPOSED OF RECTANGULAR AREAS

REMARKS: *The method herein comprises nothing unusual except that there is a minimum of dial transfers, and the computation of the Moment of Inertia of each component area is self-proving in that no dial transfers take place in the operation.*

EXAMPLE: It is desired to compute the Moment of Inertia and other related constants of the section shown on Page 3 hereof.

DECIMALS: Orange (fixed) decimals set at 2, with reference to the 2nd Keyboard Dial decimal. White (carriage) decimals set as follows: Upper Dial 4 & 2, Middle Dial 12 & 4, Keyboard Dial 8 and 2. No. 8 Tab Key depressed.

PRELIMINARY: Prepare Work Sheet for data as below. The only given data are the dimensions of section.

Item	Unit of Section	Area (A)	Lever (L)	(L^2)	I of Item
1	22½" x ¾"	16.88	35.63	1269.5	.8
5	6 " x ¼" hole	-1.50	25.00	625.0	-4.5
2	94¾" x ¼"	23.69	-12.63	159.5	17721.3
3	15 x ½"	7.50	-59.75	3570.0	.8
4	18 " x ¼" hole	-4.50	-33.00	1089.0	-121.5
	(1) Total Area	42.07			
	(2) Net Moment (M)		-34.9		
	(3) Total (A x L^2)			46145.	
	(4) Total I of Items				17596.9

Distance from Assumed Axis to Neutral Axis $\bar{X} = \frac{M}{A} = \frac{34.9}{42.07} = .83"$ below

Moment of Inertia of Section:

Total (A x L^2) plus Total I of Items, minus (M x $\bar{X}$)
46,145 + 17596.9 - (34.9 x .83) = 63,712.9 in.^{4}

Radius of Gyration: $\sqrt{I/A} = 38.92"$

(I) COMPUTATION OF AREAS "A"

This is done in the customary way by setting up lengths in Keyboard Dial and multiplying by inches expressed as a decimal, being sure to multiply out to two places at right of decimal of multiplier. Areas will appear in Middle Dial, pointed off by orange decimal. Multipliers will be pointed off by Upper orange decimal.

(II) COMPUTATION OF LEVER "L"

No calculator work, except adding and subtracting, is required. The central point of the unit of section is measured from the Assumed Axis.

(over)

(III) COMPUTATION OF L^2

These will be positive, as the square of negative amounts is positive, so without regard to sign, set up each L in Keyboard Dial and multiply by the same quantity, observing the rule as to entering multipliers mentioned in Part I.

(IV) COMPUTATION OF MOMENT OF INERTIA OF ITEMS

As an example of this computation, let us take the 2nd item (94.75 x .25).

(1) Set up h (94.75) at 2nd Keyboard Dial decimal and multiply at 2nd Upper Dial decimal by same amount (94.75).

(2) Set up h with right-hand digit, reduced by "1", at 8th Keyboard Dial decimal and set up following 9's. (Keyboard reads 94.74999999), clear Upper Dial and multiply at 4th Upper Dial decimal by amount that appears directly below it in Middle Dial (8977.5625).

h^3 (850624.05) appears at 12th Middle Dial decimal.

(3) Clear Keyboard Dial and set up as divisor (12 divided by b($\frac{1}{2}$)), which divisor is usually apparent by inspection. In this case it is 48. Depress Division Key.

Moment of Inertia of Item (17721.3) appears in Upper Dial.

(V) COMPUTATION OF NET MOMENT

(1) Set up first L (35.63) at 2nd Keyboard Dial decimal, and multiply at 2nd Upper Dial decimal by corresponding Area (16.88).

(2) Clear Upper and Keyboard Dials, and set up corresponding amounts for the other items, then repeat Step (1), being sure that negative lever lengths are set up in complementary form and negative areas are entered with Manual Counter Control toward the operator and Reverse Bar depressed. In Item 4 where both factors are negative, it is not necessary to enter either factor in its negative form.

Net Moment .. 99999965.1 appears in Middle Dial.

(3) Duplicate the above set-up in Keyboard Dial directly below its position in Middle Dial and Reverse Multiply by 2.

Net Moment 34.9 appears in Middle Dial.

(VI) COMPUTATION OF TOTAL ($A \times L^2$)

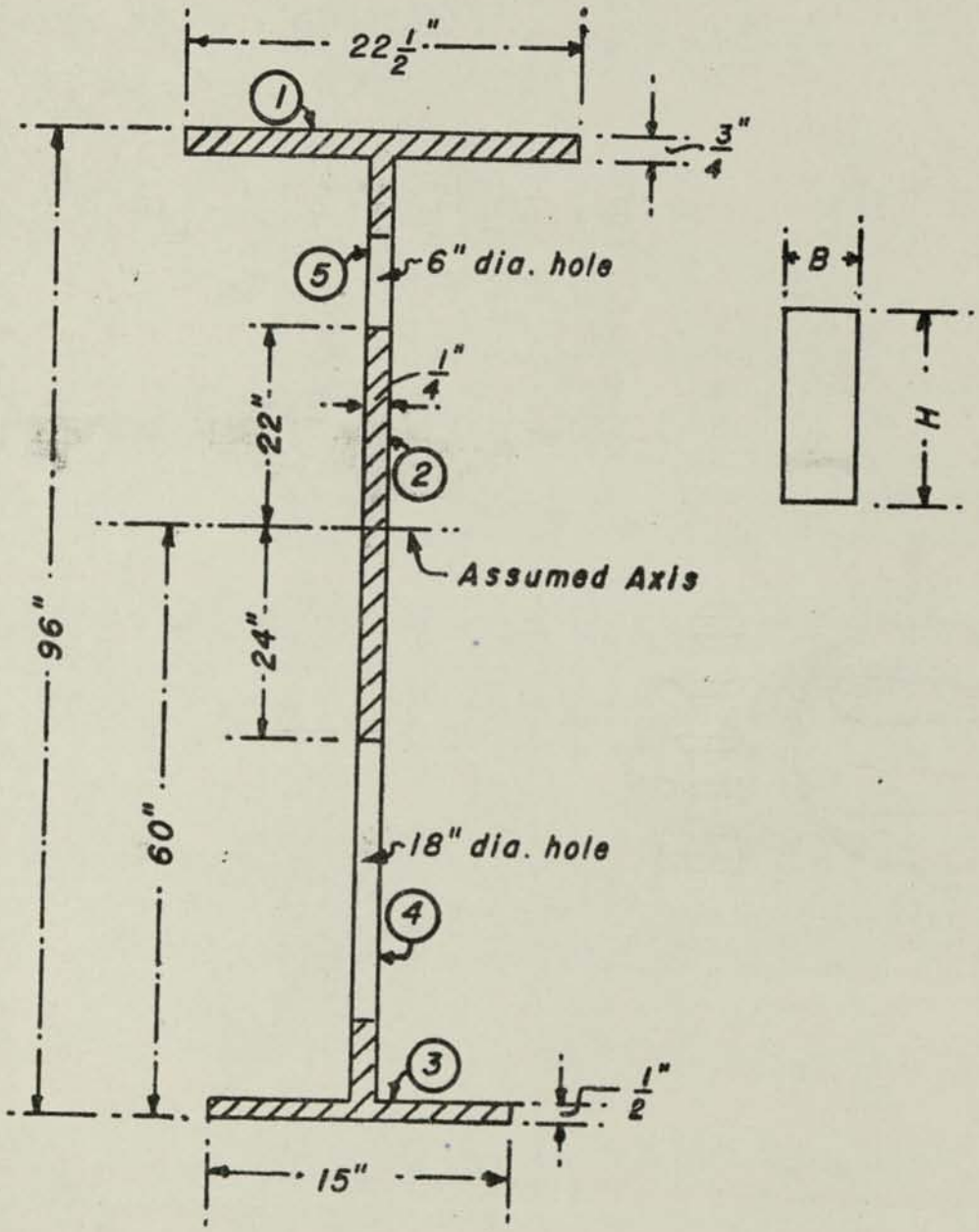
This is done in usual way by setting up each L^2 in Keyboard Dial at 2nd decimal and multiplying by corresponding A's, which when negative are to be multiplied in reverse. Middle Dial is not cleared after each multiplication.

Total ($A \times L^2$) appears in Middle Dial (46.145).

(VII) FINDING CONSTANTS

This is merely the following of the outline of the computation above.

Moment of Inertia	63712.9			
Radius of Gyration	38.92			
Dist. to Extreme Fibre	(above)	36.83	(below)	59.17
Section Modulus	(above	1730.0	(below)	1077.0



MILNE METHOD OF STEP-BY-STEP INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS WHEN STARTING VALUES ARE KNOWN

REMARKS: *The Milne Method is highly regarded because it uses tabular values instead of differences and because its associated Steffensen integration formulas have small repeated coefficients which readily lend themselves to the preparation of tables of factors for use in connection with any particular problem. The method also provides means for estimating the error, provided the order of differences that tend to disappear is known. The example computed herein is the same as the differential equation that was chosen for comparing processes by the Committee on Numerical Integration, National Research Council (Bulletin No. 92). The example has substantially large higher orders of difference so that use of one of the intermediate forms of the Milne Method is required. The simplest exemplification of the method, which is suitable for use when 4th differences tend to disappear, is described in the appended Explanatory Notes, which also discuss other pertinent matters.*

Though this computation appears formidable in review, it is actually extremely simple to apply. The schematic diagrams and work sheets have been developed for the purpose of reducing the computation to simple systematic procedure.

EXAMPLE: Integrate $dy/dx = -xy$ from $x = 0.5$ to $x = 1.0$, with initial value $y = 1$ when $x = 0$, and with starting values as follows:

x		y		$dy/dx = u = -xy$
0	y_{-5}	1.000 000 00	u_{-5}	-0.000 000 00
0.1	y_{-4}	0.995 012 48	u_{-4}	-0.099 501 25
0.2	y_{-3}	0.980 198 67	u_{-3}	-0.196 039 73
0.3	y_{-2}	0.955 997 48	u_{-2}	-0.286 799 24
0.4	y_{-1}	0.923 116 35	u_{-1}	-0.369 246 54
0.5	y_{0}	0.882 496 90	u_{0}	-0.441 248 45

The method of obtaining the above "starting values" or determining how many starting values are needed in any case is beyond the scope of this method. See Marchant-Methods MM-437E1 and 437E2.

The computing plan for this example comprises the use of the 5-term "open-type" formula for integrating ahead and the 5-term "closed-type" formula for back checks, with a final refinement of the entire group of five values by use of the 9-term "closed-type" formula.

OPERATIONS: Decimals; Upper Dial 8, Middle Dial 16, Keyboard Dial 8. Use any 10-column model Marchant.

(1) Compute the factors that are in the three right rows of the Upper and Lower arrays of Page 5 for values from and including $x = 0$ to $x = 0.5$, setting up each value of "u" from the example and multiplying it successively by 11, 14, 26, 7, 32, and 12 insofar as the arrays show that it is necessary to use the factors; for example, it is not necessary to multiply u_{-4} (0.099 501 25) by any multiplier except "11" (for the single entry in the upper array).

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COMPUTING TRIAL VALUES OF "y" AND "u"

(2) The first step in obtaining the trial value of y for $x = 0.6$ (y_{+1}) is to integrate u from u_{-5} to u_{+1} by using its values from u_{-4} to u_0 inclusive (not using the end values of u_{-5} and u_{+1}). This is done by summing the factors of the upper array diagonally, as indicated by the line with the arrows; thus, the sum of these "u" functions, as per formula at upper left of section "0.6" Work Sheet (Page 6) is 1.094 513 75 - 2.744 556 22 + 7.456 780 24 - 5.169 451 56 + 4.853 732 95 = 5.491 019 16, which appears in Middle Dial. It is negative as all of the u's are negative.

(3) Transfer the Middle Dial amount to Keyboard Dial, clear Middle Dial and multiply by Length Factor (.03).

Increment in y from $x = 0$ to $x = 0.6$ (0.164 730 57), which is also negative, appears in Middle Dial.

(4) Transfer the Middle Dial amount (0.164 730 57) to Keyboard Dial, clear Middle Dial, and subtract. Set up y_{-5} (1.000 000 00) and add.

Trial Value y_1 (0.835 269 43) appears in Middle Dial. Copy to Work Sheet (Page 6).

(5) Next, proceed with the calculation of dy/dx when $x = 0.6$ from its equation, both the y and x being known. In this case, the value of $dy/dx = u$ is obtained by multiplying the known "y" by the known "x" (-0.6); thus, transfer Middle Dial amount (0.835 269 43) from Middle Dial to Keyboard Dial, clear Upper Dial and multiply by "0.6".

Trial Value u_1 (0.501 161 66) appears in Middle Dial. Copy to Work Sheet (Page 6).

(6) Transfer the Middle Dial amount (0.501 161 66) to Keyboard Dial and multiply by 11.

Copy Trial Value u_{-1} (0.501 161 66) to Work Sheet (Page 6) and to the Lower Array of Factor Sheet (Page 5).
Copy Middle Dial amount (3.508 131 62) to <u>Lower</u> Array of Factor Sheet.

COMPUTING CHECK VALUE OF "y" AND "u"

(7) The first step in obtaining the Check Value of y_{+1} is to integrate u from u_{-3} to u_{+1}, using these values as well as those that are in-between. This is done by summing the factors of the Lower Array diagonally, as indicated by the line with the arrows; thus, the sum of these "u" functions as per formula at lower left of section "0.6" of Work Sheet (Page 6) is 1.372 278 11 + 9.177 575 68 + 4.430 958 48 + 14.119 950 40 + 3.508 131 62 = 32.608 894 29, which is negative.

(8) Move Upper Dial decimal from 8 to 9 and Keyboard Dial decimal from 8 to 7, set up reciprocal of the common multiplier (225) and divide.

Increment in y from $x = 0.3$ to $x = 0.6$ inclusive (0.144 928 842) appears in Upper Dial, which enter in Work Sheet (Page 6) as negative.

(9) Move Middle Dial decimal to 17 and Keyboard Dial decimal to 8, set up y_{-3} (0.980 198 67) in Keyboard Dial, depress Add Bar, and then depress Subtract Bar.

(10) Set up "1" in 9th column of Keyboard Dial and reverse multiply by Upper Dial amount, except use rounded figure of "2" in 2nd dial instead of "19" as appears in 2nd and 1st dials.

Check value of y_{+1} (0.835 270 25) appears in Middle Dial. Upper Dial shows all ciphers, or all 9's, in every dial, except 1st dial will show effect of rounding. Copy to Work Sheet.

CORRECTING THE CHECK VALUE

(11) The Check Value is usually more nearly correct than the Trial Value, because it is obtained as an integration of only four "sections" (0.3 to 0.6), using five "ordinates." The difference δy between the Trial and Check Values is obtained and given such sign that when it is added to the Check Value their sum equals the Trial Value (see Explanatory Notes); thus,

0.835 270 25 - 0.835 269 43 = 0.000 000 82 recorded as negative

(12) For the conditions of this example (see Explanatory Notes) the Check Value should be corrected by $\delta y/35$ = -0.000 000 02, reducing y_{+1} to 0.835 270 23.

(13) Substitution in the formula for dy/dx is then made (in this case multiplying 0.835 270 23 X - 0.6, producing the Check Value of u_{+1} - 0.501 162 14, which is entered in the Upper Array of the Factor Sheet. The value previously entered in the Lower Array is corrected, as shown, and again multiplied by 11, 14, and 26, to produce the corrected factor in the "11" column and also the values in the other columns. The Lower Array is, likewise, completed by multiplying by 7, 32, and 12.

(14) The above cycle from Steps 2 to 13 is repeated for the remaining values, dropping off values "at the top" as new ones are obtained at the bottom, all as per Work Sheet (Page 6).

In preparing this Work Sheet, it is found convenient to reproduce the formulas used, as shown, and to place the symbol number that identifies the term in the Factor Arrays directly below each term. This provides a "pattern" of calculating that serves to clarify and expedite the process.

The checked y's are tabulated below:

x	y
0.6	0.835 270 23
0.7	0.782 704 55
0.8	0.726 149 06
0.9	0.666 976 83
1.0	0.606 530 69

FINAL REFINEMENT OF VALUES

By the preceding process, there has been obtained the first group of five values beyond the starting values. Before calling the work complete, however, we may make an overall check by using the 9-term closed formula for integrating the differential "u" in the eight sections from x = 0.2 to x = 1.0 inclusive, thus

(over)

x	u	Multiplier
0.2	-0.196 039 73	989
0.3	0.286 799 24	5888
0.4	0.369 246 54	- 928
0.5	0.441 248 45	10496
0.6	0.501 162 14	-4540
0.7	0.547 893 19	10496
0.8	0.580 919 25	- 928
0.9	0.600 279 15	5888
1.0	0.606 530 69	989

$$= -13241.86039 \times \frac{0.8}{28350}$$

$$= -0.373\ 668\ 02$$

which added to y for x = 0.2 produces new y for x = 1.0 of 0.606 530 65.

This is .000 000 04 less than the previously found value, the amount being the accumulation of integrated differences that occurred during the calculation of these values. The difference of "4" should be distributed among these 5 values of y, as below, and new "u" calculated.

		Distributed	New "y"	New "u"
0.6	0.835 270 23	- .8	0.835 270 22	-0.501 162 13
.7	.782 704 55	-1.6	.782 704 53	-0.547 893 17
.8	.726 149 06	-2.4	.726 149 04	-0.580 919 23
.9	.666 976 83	-3.2	.666 976 80	-0.600 279 12
1.0	.606 530 69	-4.0	.606 530 65	-0.606 530 65

These new values of "u" should be substituted in the 9-term formula, above, but to avoid repetition, only the effect of the difference in u's, as found above, and those shown on the upper part of Page 5, is calculated, thus obtaining a final refinement of y_{+5}

x	Difference in "u"
0.6	+.000 000 01
.7	2
.8	2
.9	4
1.0	4

$$\times \begin{matrix} -4540 \\ 10496 \\ -928 \\ 5888 \\ 989 \end{matrix} = .000\ 362\ 16$$

$$\times \frac{0.8}{28350} = +.000\ 000\ 01$$

This increase in "y_{+5}" is, likewise, an accumulation of integrated differences throughout 5 terms, so it is distributed as follows:

x	Correction of y (rounded)	Final "y"
0.6	.000 000 00	0.835 270 22
0.7	0	.782 704 53
0.8	1	.726 149 05
0.9	1	.666 976 81
1.0	.000 000 01	.606 530 66

If additional values beyond x = 1.0 were to be obtained, a new column of values of u corresponding to the above values of y would then be obtained in the customary way.

FACTOR SHEET MILNE-STEFFENSEN 5 POINT (6 ORDINATE) FORMULA

FOR INTEGRATION BY STEP-BY-STEP METHOD

ORIGINAL CALCULATION: h = 0.1 Length Factor: 3 h/10 = .03

x	u_n	u = -xy	-- Column Multipliers -- 11	-14	26
0	u_{-5}	0.000 000 00			
0.1	u_{-4}	-0.099 501 25	1.094 513 75		
0.2	u_{-3}	-0.196 039 73	2.156 437 03	2.744 556 22	
0.3	u_{-2}	-0.286 799 24	3.154 791 64	4.015 189 36	7.456 780 24
0.4	u_{-1}	-0.369 246 54	4.061 711 94	5.169 451 56	9.600 410 04
0.5	u_0	-0.441 248 45	4.853 732 95	6.177 478 30	11.472 459 70
0.6	u_{+1}	-0.501 162 14	5.512 783 54	7.016 269 96	13.030 215 64
0.7	u_{+2}	-0.547 893 19	6.026 825 09	7.670 504 66	14.245 222 94
0.8	u_{+3}	-0.580 919 25	6.390 111 75	8.132 869 50	
0.9	u_{+4}	-0.600 279 15	6.603 070 65		
1.0	u_{+5}	-0.606 530 69			

BACK-CHECK CALCULATION: h = 0.1 Length Factor: 2 h/45 = 1/225

x	u	u = -xy	-- Column Multipliers -- 7	32	12
0.2	u_{-3}	-0.196 039 73	1.372 278 11		
0.3	u_{-2}	-0.286 799 24	2.007 594 68	9.177 575 68	
0.4	u_{-1}	-0.369 246 54	2.584 725 78	11.815 889 28	4.430 958 48
0.5	u_0	-0.441 248 45	3.088 739 15	14.119 950 40	5.294 981 40
0.6	u_{+1}	-0.501 161 ~~66~~ 2 14	3.508 131 ~~62~~ 4 98	16.037 188 48	6.013 945 68
0.7	u_{+2}	-0.547 892 ~~51~~ 3 19	3.835 247 ~~57~~ 52 33	17 532 582 08	6.574 718 28
0.8	u_{+3}	-0.580 918 ~~42~~ 9 25	4.066 428 ~~94~~ 34 75	18.589 416 00	6.971 031 00
0.9	u_{+4}	-0.600 278 ~~21~~ 9.15	4.201 947 ~~47~~ 54 05	19.208 932 80	
1.0	u_{+5}	-0.606 529 ~~74~~ 30 69	4.245 708 18		

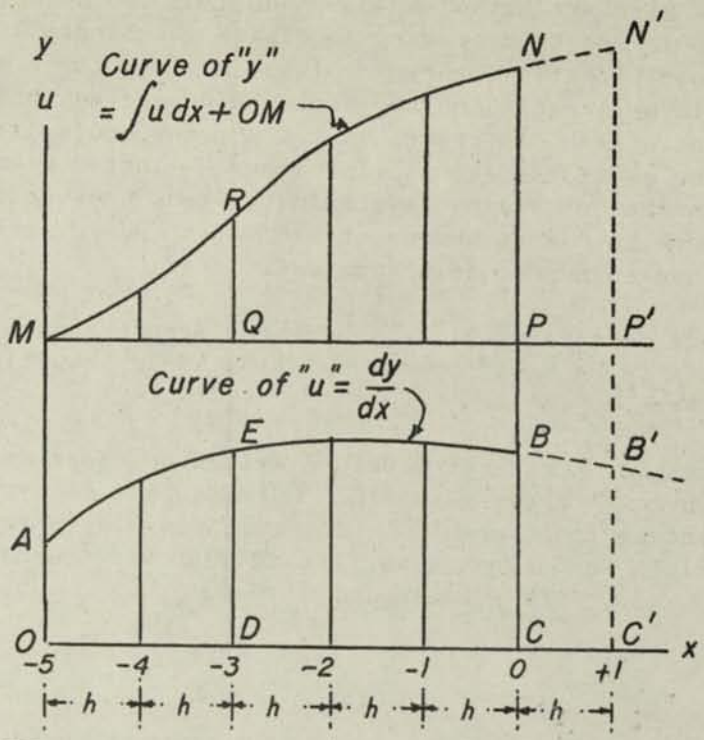
(over)

WORK SHEET MILNE-STEFFENSEN 5 POINT (6 ORDINATE) FORMULA
FOR INTEGRATION BY STEP-BY-STEP METHOD

	FORMULA	TRIAL		CHECK		δy	E
x = 0.6	$-0.03(11u_{-4} - 14u_{-3} + 26u_{-2} - 14u_{-1} + 11u_{0})$	y_{-5}	1.000 000 00	y_{-3}	0.980 198 67		(y/35)
			-0.164 730 57		-0.144 928 42		
		y_{+1}	0.835 269 43		0.835 270 25	-82	-2
			-0.6		-2		
	$-1/225(7u_{-3} + 32u_{-2} + 12u_{-1} + 32u_{0} + 7u_{+1})$	u_{+1}	-0.501 161 66	y_{+1}	0.835 270 23		
					-0.6		
				u_{+1}	-0.501 162 14		
x = 0.7	$-0.03(11u_{-3} - 14u_{-2} + 26u_{-1} - 14u_{0} + 11u_{+1})$	y_{-4}	0.995 012 48	y_{-2}	0.955 997 48		
			-0.212 308 89		-0.173 292 90		
		y_{+2}	0.782 703 59		0.782 704 58	-99	-3
			-0.7		-3		
	$-1/225(7u_{-2} + 32u_{-1} + 12u_{0} + 32u_{+1} + 7u_{+2})$	u_{+2}	-0.547 892 51	y_{+2}	0.782 704 55		
					-0.7		
				u_{+2}	-0.547 893 19		
x = 0.8	$-0.03(11u_{-2} - 14u_{-1} + 26u_{0} - 14u_{+1} + 11u_{+2})$	y_{-3}	0.980 198 67	y_{-1}	0.923 116 35		
			-0.254 050 65		-0.196 967 26		
		y_{+3}	0.726 148 02		0.726 149 09	-107	-3
			-0.8		-3		
	$-1/225(7u_{-1} + 32u_{0} + 12u_{+1} + 32u_{+2} + 7u_{+3})$	u_{+3}	-0.580 918 42	y_{+3}	0.726 149 06		
					-0.8		
				u_{+3}	-0.580 919 25		
x = 0.9	$-0.03(11u_{-1} - 14u_{0} + 26u_{+1} - 14u_{+2} + 11u_{+3})$	y_{-2}	0.955 997 48	y_{0}	0.882 496 90		
			-0.289 021 69		-0.215 520 04		
		y_{+4}	0.666 975 79		0.666 976 86	-107	-3
			-0.9		-3		
	$-1/225(7u_{0} + 32u_{+1} + 12u_{+2} + 32u_{+3} + 7u_{+4})$	u_{+4}	-0.600 278 21	y_{+4}	0.666 976 83		
					-0.9		
				u_{+4}	-0.600 279 15		
x = 1.0	$-0.03(11u_{0} - 14u_{+1} + 26u_{+2} - 14u_{+3} + 11u_{+4})$	y_{-1}	0.923 116 35	y_{+1}	0.835 270 23		
			-0.316 586 61		-0.228 739 51		
		y_{+5}	0.606 529 74		0.606 530 72	-98	-3
			-1.0		-3		
	$-1/225(7u_{+1} + 32u_{+2} + 12u_{+3} + 32u_{+4} + 7u_{+5})$	u_{+5}	-0.606 529 74	y_{+5}	0.606 530 69		
					-1.0		
				u_{+5}	-0.606 530 69		

EXPLANATORY NOTES

The diagram relates to a typical function, not to the one used in the example.



PRINCIPLE OF STEP-BY-STEP SOLUTION OF DIFFERENTIAL EQUATIONS: Let it be assumed that in the case of any given ordinary differential equation of first order $dy/dx = u = f(x,y)$, the values of y and u are known for several "starting" values of x at equally spaced intervals (x_{-5}, x_{-4} ..... x_0). Assume that these values are plotted as the solid curved lines of the diagram in which AEB represents the known values of dy/dx, and MN represents the corresponding known values of y when the initial value is OM. From the relationship shown, it is apparent that the area OABCO is a measure of the increment in y (PN) above the initial value MO, or y = NC; that is to say, any point on the curve MN corresponding to any value of x may be obtained by measuring the area under the differential curve up to that value of x and adding the result so obtained to the initial value of y, OM.

If, now, we have means of determining the area OAB'C'O, when given only the solid-line curve AEB, it is apparent that the area so obtained is likewise a measure of the increment in y, P'N', which when added to the initial value OM would give a new y corresponding to a point x_{+1}. By substituting the new y, so found, in the given differential equation, x of course being known, the new value of dy/dx = u, or the point B' becomes known.

The cycle is now complete, and by like means we can compute other values at the right of NN' and BB', thus continuing the solution to any desired point.

The crux of the solution rests, of course, in the determination of the area OAB'C'O when given only the values of u from A to B. The process is essentially one of extrapolation, and various ways exist for performing this "extension." It is obvious that errors made in computing the first extension steps are cumulative, just as if one were constructing a cantelever bridge, so any process, to be successful, must have means of reducing these errors to a minimum. Also, if possible, it must provide over-all checks to correct a series of values so that the completed structure will have dy/dx = u for every value of x and y.

TRIAL COMPUTATION OF THE MILNE METHOD: The area OAB'C'O is obtained by using only the ordinates of the curve AB at points such as x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0, for example, which values are known as they comprise all but one of the assumed starting values. If these values

(over)

of u are differenced (see Marchant Method MM-419) and it is found that fourth differences are negligible, then it would not be necessary to divide the length AB into five sections (six ordinates). It would be satisfactory to divide it into three sections (four ordinates); the sections would be larger, and the work would proceed that much faster. If it is desired to take account of 7th differences of u (corresponding to 8th differences of y), the curve AB should be sectionalized still further -- into 7 sections (8 ordinates), etc. In any case, the number of ordinates should be even, because the accuracy of the computation based upon any such even number of ordinates is substantially equal to the case if the next higher uneven number of ordinates is used.

Consideration of the above matters provides a hint for proper choice of interval (h) and of the number of ordinates to be used for the solution of any equation. The subject will not be further explored here.

In the example of this method, six starting values are known, corresponding to the solid-line ordinates of both curves. The area OAB'C'O is then obtained by applying the proper Steffenson "open-type" integration formula. In the case diagrammed, the area OAB'C'O, spanning seven ordinates, is found by using five of the known ordinates, excluding the end ordinates for x_1 and x_{-5}. The formula is:

$$(1) \quad \text{Area OAB'C'} = \frac{\text{length OC'}}{20} \left(11u_{-4} - 14u_{-3} + 26u_{-2} - 14u_{-1} + 11u_0\right)$$

which will be recognized as that which was used in the example, because length OC' equals 6h and 6 x 0.1/20 = 0.03.

The general Steffensen formulas for various numbers of terms are given below:

No. of Terms	Coefficients to Central Value					Divisor	Remainder
3	2	-1				3	0.000 31 $F^{(4)}$ (Xi)
5	11	-14	26			20	0.000 001 1 $F^{(6)}$ (Xi)
7	460	-954	2196	-2459		945	0.000 000 002 1 $F^{(8)}$ (Xi)
9	4045	-11690	33340	-55070	67822	9072	0.000 000 000 002 7 $F^{(10)}$ (Xi)

The use of the above coefficients and divisors will be apparent by analogy from what has been previously described. The remainder of the 3-term formula contains the expression $F^{(4)}$ (Xi), which designates the 4th derivative of u (or the 5th derivative of y) with respect to x for some value of x (unknown) that is found within the entire length OC'. This amount makes it possible to determine the maximum error if we know the maximum 4th derivative. The actual error, however, may be less than this, and usually is. The computer rarely needs to pay attention to this matter because the Milne Method offers a simplification of this point, as will be later explained.

The Milne computation of the error is also to be preferred because the above expressions for the Remainder apply to the case when the length OC' is taken as "1". If it has any other value, the above Remainder Coefficients must be multiplied by the length raised to the r th power, in which "r" equals "no. of terms of formula used, plus 2"; i.e., if the length OC' is 6 units long and the 5-term formula is used, the Remainder Coefficient (0.000 001 1) is to be multiplied by 6^7. It therefore equals 0.31.

We have now obtained the area OAB'C'O which, as stated, is a measure of the increment in y. This increment, P'N', added to the initial value OM, gives the new value y_1, corresponding to x_1. It then only remains to substitute the now known y_1 and the known x_1 in the given differential equation and thus obtain u_1, or the point B'.

The cycle is now complete, and if we are satisfied with the accuracy so far obtained, we could proceed to the next step by finding the area under the differential curve from x_{-4} to x_2 and adding the increment in y so found to the value of y_{-4}, thus producing the next value of y_2.

THE MILNE CHECK-BACK: The first refinement of the value of u and y for x_1 is obtained by re-calculating y_1 in a different manner from that previously employed. This second determination of y_1 is obtained by finding the area DEB'C'D of the differential curve between and including the values x_{-3} and x_1 and adding the increment so obtained to y_{-3}. This could be done by Simpson's Rule, if desired, but such a formula would not suffice for this example because it would not take into account 5th differences of u. The formula that was used in the example takes into account such 5th differences; thus,

$$(2) \quad \text{Area DEB'C'D} = \frac{\text{length DC'}}{90} (7u_{-3} + 32u_{-2} + 12u_{-1} + 32u_0 + 7u_{+1})$$

Inasmuch as the length DC' in the example is 4h, the multiplier is 4 X 0.1/90 = 1/225.

The general Newton-Cotes formulas for various numbers of terms are given below:

No. of Terms	Coefficients to Central Value					Divisor	Remainder
3	1	4				6	$-0.000\ 35\ F^{(4)}\ (X_1)$
5	7	32	12			90	$-0.000\ 000\ 52\ F^{(6)}\ (X_1)$
7	41	216	27	272		840	$-0.000\ 000\ 000\ 64\ F^{(8)}\ (X_1)$
9	989	5888	-928	10496	-4540	28350	$-0.000\ 000\ 000\ 000\ 59\ F^{(10)}\ (X_1)$

The use of the above coefficients, divisors and remainders will be apparent by analogy from what has been previously described. However, the Remainder of the above Check formula, when compared with that of the Trial formula, if used in the Milne Method, is smaller than a comparison of coefficients indicates. This is because the Check formula integrates an area with a shorter base line, it being 1/2, 2/3, 3/4, etc., of that of the Trial formula for 3, 5, and 7 terms respectively; and higher powers of this ratio of intervals are involved.

Because the above formula uses the outside ordinates which close the area, it is designated a "Closed Type" formula. The previously described formula that does not use either of the end ordinates is similarly designated an "Open Type" formula.

The formula for 3 terms will be recognized as Simpson's 1/3 Rule.

THE MILNE ERROR CHECK: Inspection of the Remainders, as tabulated for the above Open and Closed-Type formulas, shows that in the case of, say, the 5-term formulas if the seventh derivative of u vanishes; that is to say, the error is only that due to the sixth derivative, and if we also assume that this sixth derivative is positive in the case of each formula, then it is evident that the integral calculated from the open-type formula will be less than its true value because the Remainder is positive. By similar reasoning, it is seen that the closed-type formula produces a value that is greater than its true value. The true value then lies between the values obtained by the two formulas.

Inasmuch as in most cases the differences of any order and the derivatives of the same order are closely proportionate, we may likewise conclude, subject to remotely unusual exceptions, that if the seventh <u>differences</u> of u vanish, the true value of the integral will, likewise, lie between its values when computed by the closed and open-type formulas (of course, the sixth differences that control this error are adjacent differences, rather than identical, because the two 5-term formulas do not use all of the identical five terms, but if these adjacent differences should be of opposite signs we would know that they would be substantially non-existent because of the assumption that seventh differences tend to vanish).

(over)

The above reasoning has been applied to the case where the difference that controls the error (in this case, the 6th) is positive. By similar reasoning, it will be seen that the true value of the integral lies between its "open type" and "closed type" values when the difference that controls the error is negative, and also that this general conclusion applies to any of the formulas, provided that the $n + 2$ difference of any n term formula is assumed to vanish.

By reference to the original expansion from which the remainders were computed, Milne develops the following conclusion for the amount of the error, the sign of which becomes known when it is remembered that the error must be such as to modify the value of the integral when obtained by the closed-type formula so the true value comes between it and the value when obtained by the open-type formula.

> Under the assumptions given, the true value differs from the value as obtained by the closed-type formula by the following fraction of the difference between the values obtained by the open-type and closed-type formulas: 3-term, 1/29; 5-term, 1/35; 7-term, 1/44; 9-term, 1/54. The sign of this difference is such as to make the true value lie between the closed-type and open-type values.

In applying the above principle, it is to be remembered that the correction is to be applied to the value obtained by the closed-term formula.

Sometimes examples are found in texts in which the true value does not lie between the open-term and closed-term values, but if these be examined, it will be found that higher orders of differences exist than permitted by the above assumption. Inasmuch as the choice of the number of starting terms of the solution and of the interval are usually such that differences of u substantially vanish as outlined above, it is seen that the Milne Error Check is as sound as it is easy to apply.

APPLICATION TO DIFFERENTIAL EQUATIONS OF HIGHER ORDER, ETC: Systems of differential equations of the first order may also be solved by this method. Each equation is solved independently for its step in y, but these y values are substituted in the simultaneous equations to give new value of u for each equation.

Differential equations of higher order or systems thereof are reducible to a system of equations of the first order which is then solvable, as above outlined. Milne has developed special means of solving second order equations in which first derivatives are absent, particularly $d^2y/dx^2 = f(x,y)$ and $d^2y/dx^2 + g(x)y = f(x)$.

REFERENCES: The following list will be of assistance to those who wish to study this subject further:

W. E. Milne, *Numerical Integration of Ordinary Differential Equations*, Am. Math. Mo. 33: 455-460 (1926).

W. E. Milne, *On the Numerical Integration of Certain Differential Equations of the Second Order*, Am. Math. Mo. 40: 322-327 (1933).

National Research Council, No. 92, *Numerical Integration of Differential Equations* (Report of A. A. Bennett, W. E. Milne, H. Bateman) (1933).

J. F. Steffensen, *Interpolation*, P. 158-159, 170-177, Williams & Wilkins Co. (1927).

J. L. Scarborough, *Numerical Mathematical Analysis*, P. 280-282, The Johns Hopkins Press, Baltimore (1930).

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MILNE METHOD OF STEP-BY-STEP DOUBLE INTEGRATION OF SECOND ORDER DIFFERENTIAL EQUATIONS IN WHICH FIRST DERIVATIVES ARE ABSENT

(A Supplement to Marchant-Method MM-437E3)

REMARKS: *Dr. W. E. Milne* calls attention to the frequency of problems in dynamics and astronomy that are of the type discussed herein. For their rapid solution, he has published the formulas given below. Application of this method to the Marchant is shown in outline form. A knowledge of MM-437E3 is assumed.*

EXAMPLE: Integrate twice $d^2y/dx^2 = (x^2 - 1)y = u$, with initial value $y = 1$ and $dy/dx = 0$ when $x = 0$, and starting values with differences as below. Seven-place accuracy of "y" is desired.

x	y	u	1st	2nd	3rd	4th	5th
0	1.00000000	-1.00000000					
			1493764				
0.1	0.99501248	-0.98506236		2913400			
			4407164		-217263		
0.2	0.98019867	-0.94099072		2696137		-128177	
			7103301		-345440		23427
0.3	0.95599748	-0.86995771		2350697		-104750	
			9453998		-450190		
0.4	0.92311635	-0.77541773		1900507			
			11354505				
0.5	0.88249690	-0.66187268					

OUTLINE: The published Milne formulas applying to this work are below:

3-TERM FORMULAS, exact if third differences of u are constant (fourth differences vanish) and with known error-factor of $\delta y/13$ if fourth differences are constant (fifth differences vanish).

Open type for integrating ahead:

(1) $$y_{n+1} = y_n + y_{n-2} - y_{n-3} + (h^2/4)\ (5u_n + 2u_{n-1} + 5u_{n-2})$$

Closed type for recalculating by back-check:

(2) $$y_n = 2y_{n-1} - y_{n-2} + (h^2/12)\ (u_n + 10u_{n-1} + u_{n-2})$$

5-TERM FORMULAS, exact if fifth differences of u are constant (sixth differences vanish) and with known error-factor of $\delta y/26$ if sixth differences are constant (seventh differences vanish).

Open type for integrating ahead:

(3) $$y_{n+1} = y_n + y_{n-4} - y_{n-5} + (h^2/48)\ (67u_n - 8u_{n-1} + 122u_{n-2} - 8u_{n-3} + 67u_{n-4})$$

Closed type for recalculating by back-check:

(4) $$y_n = y_{n-1} + y_{n-3} - y_{n-4} + (h^2/240)\ (17u_n + 232u_{n-1} + 222u_{n-2} + 232u_{n-3} + 17u_{n-4})$$

(*) *W. E. Milne, On the Numerical Integration of Certain Differential Equations of the Second Order, Am. Math. Mo. 40:322-327 (1933), also National Research Council, No. 92, Numerical Integration of Differential Equations (Report of A. A. Bennett, W. E. Milne, H. Bateman) (1933).*

(over)

WILL THE INTERVAL CHOSEN FOR STARTING VALUES PROVIDE DESIRED ACCURACY?

The need of making an intelligent guess as to the interval (h) to be used for any desired accuracy of the result occurs in any work of this sort. This matter was only touched upon in Marchant Method MM-437E3, so an analysis of this case may be helpful in suggesting an approach to the problem.

(a) Inasmuch as the six tabulated values of "u" provide only a 5th difference, is it possible to infer what the higher differences might be? We can do so only in very broad terms, and inasmuch as reasoning with regard to these matters is often required, the subject is gone into at length.

(b) It is most unlikely that the function "u" has a constant 5th or 6th difference that would enable the 5-term formula to be used with results accurate to the number of places of the tabulated "u". We know this because a 7th or 8th degree power series (which would have constant 7th or 8th differences of f (x)) could not have its second derivative "u" in the algebraic form of "u". It is probable that "u" has higher differences and most certainly has a 6th and 7th difference.

(c) Inferences as to the size of 6th and 7th differences are made from the following. The descending differences show the following ratios:

Ratio	1st	difference	to	function		.015
"	2nd	"	"	1st	diff.	.195
"	3rd	"	"	2nd	"	.075
"	4th	"	"	3rd	"	.590
"	5th	"	"	4th	"	.181

The ascending differences similarly show the following ratios:

Ratio	1st	difference	to	function		.122
"	2nd	"	"	1st	diff.	.245
"	3rd	"	"	2nd	"	.146
"	4th	"	"	3rd	"	.400
"	5th	"	"	4th	"	.223

Plotting these ratios against orders of difference shows that these fluctuating differences show a trend, and that the amplitude of the fluctuations decreases as "x" increases. This is proved by the fact that the ascending differences have smaller amplitudes than the descending. If the trend of the tops and bottoms of these "swings" is plotted and extended to 6th and 7th orders of difference, it appears that the ratio of 6th ascending difference to 5th is about .47 and of 7th to 6th is about .33. Applying these ratios, it appears to be a good guess that the ascending 6th difference will not exceed .00011 and the ascending 7th difference will not exceed .00004.

(d) The Milne 5-Term Formulas supply an exact "y" if 6th differences in "u" vanish and use of the Milne Error Factors provide means of obtaining an exact "y" if 7th differences of "u" vanish. In functions of the type considered herein, however, neither 6th or 7th differences may be expected to vanish. A good guess as to the ascending 6th difference is made in the above as that it is 0.00011 at $x = 0.5$. What error in "y" may be expected from this difference?

(e) To estimate this, it is necessary to study the error formulas in the light of the derivatives of tabulated u's. These are:

(5) Error of (1) is $(17/240)\ h^6 u^{(4)}(x_p)$

(6) " " (2) " $(-1/240)\ h^6 u^{(4)}(x_q)$

(7) " " (3) " $(7870/120960)\ h^8 u^{(6)}(x_p)$

(8) " " (4) " $(-318/120960)\ h^8 u^{(6)}(x_q)$

In which $u^{(4)}(x_p)$ is the fourth derivative of u with respect to x when x is some value (x_p) that is within the range of all x's that determine the 4th central difference of u corresponding to the value of x for which the error is desired. For example, if it is desired to estimate the error of y at, say x_{-1}, the fourth central difference corresponding to x_{-1} is determined by the 5 values of x from x_{-3} to x_{+1} inclusive. The upper bound of error at point x_{-1} will thus be controlled by the greatest 4th derivative of u for <u>any</u> of the x's from x_{-3} to x_{+1} inclusive.

The expression $u^{(6)}(x_p)$ is similarly defined, and, likewise, for those with subscript q.

(f) As it is difficult to obtain the derivatives of u because y is involved implicitly, the derivatives may be estimated from the difference array of tabulated u, as follows:

(9) $u^{(4)}(x) = (1/h^4)(d''''_0 - (1/6)\,d^{VI}{}_0 + (7/240)\,d^{VIII}{}_0 \ldots)$

(10) $u^{(6)}(x) = (1/h^6)(d^{VI}{}_0 - (1/4)\ d^{VIII}{}_0 + (13/240)\ d^{X} \ldots.)$

in which d'''', d^{VI}, etc., are 4th, 6th, etc., central differences of u corresponding to any tabulated x, using the nomenclature of **Marchant Method MM-439E2.**

(g) It is one of the beauties of the Milne Method that by correcting the result of the check formulas (2) or (4) by means of the Milne Error Correction $\delta y/13$ or $\delta y/26$, respectively, the error is eliminated if 5th and 7th differences, respectively, vanish. When there are differences higher than the 4th and 6th, respectively, however, the Milne Error Correction does not eliminate the entire error because the "x" value for the open-type formula is not necessarily the same as that for the corresponding closed-type formula. The derivatives, therefore, may differ so there is a small residual error that is not eliminated by applying the Milne Error correction to (2) or (4). Just what this is, in any case, has not been explored by us, and it appears that it would be difficult to determine because there is no means of knowing which of the values of x in the range of values is to be taken for determining the derivative that controls the error.

A working rule, which has as its support only the fact that it is satisfactory in the cases for which it has been used, is that if there are differences higher than the 4th and 6th when using the 3-term and 5-term formulas, it is satisfactory to regard the Milne Error Correction as eliminating the effect of the 4th or 6th difference, respectively, but not affecting the error due to differences of higher order. When the derivatives are expressed in terms of differences as in (9) and (10), this means that the error in (2) after applying the Milne Error Correction is perhaps of the order of

(11) $(-1/240)\ h^2\ (-(1/6)\,d^{VI}{}_0 + (7/240)\ d^{VIII}{}_0 \ldots.)$

(over)

and, similarly, in (4) it perhaps may be taken as

(12) $(-318/120960)\ h^2\ (-(1/4)\,d^{VIII}{}_0+(13/240)\ d^X\ \ldots.)$*

the upper bound in either case being the largest value in the range of "x" values corresponding to the u's that determine d'''_0 in (9) or $d^{VI}{}_0$ in (10).

Applying this analysis to the problem at hand, it is noted that if when applying (1) and then applying the Milne Error Factor of $\delta y/13$, and the largest 6th difference of u is 0.00011, as above estimated, we have for the error by applying (11)

$$e = (-1/240)\ (.01)\,(-.00002) = .000000001 \text{ approx.}$$

(h) It will thus be observed that if only one value of y is desired beyond starting values, the 3-term formulas (1) and (2) would probably supply a result accurate to 8 places. However, the propagation of this error as additional values are computed results in an accumulation of errors. Exploration of this propagation of error in any case is not a difficult mathematical task, but even if we ignore the effect on the terms involving u, it is seen that each new y involves the addition of two previous y's and the subtraction of but one. When the computation has proceeded so that the starting values no longer enter the determination, the error in the example considered herein is at least 55e if we use (2) after 10 values subsequent to starting values. This assumes that all errors have same sign. We have no right to assume otherwise, based upon the available data at this point.

From this, it is seen that use of the 3-term formulas should be satisfactory to establish 7-place accuracy if only a few values are required, but the 5-term formulas should be used if 7-place accuracy is desired in the vicinity, roughly, of values beyond the 5th new value.

(i) By this reasoning, we decide to use the 5-term formula under the assumption that values up to and beyond those for x = 1.5 are desired. It now remains to be determined what interval should be used.

By reasoning similar to that in (c), we guess that a rough ascending 8th difference of tabulated u's is .000026, from which error of (4) after applying Milne Error Factor of $\delta y/26$, by using (12), is

$$(318/120960)\ (.01)\,(-.000026/4) = .00000000017 \text{ approx.}$$

When applied for ten values, it will be, roughly, .000000009, which assures 7-place accuracy with interval h = 0.1.

(j) We, therefore, conclude that the best procedure is to use interval h = 0.1 and the 5-term formulas (3) and (4). This will doubtless provide 8-place accuracy in the range up to x = 1.5, instead of the 7-place accuracy desired. Any reasonable attempt to cut the work to obtain 7-place accuracy up to x = 1.5 and no more, such as by using an odd interval, is not warranted.

(k) The next five values are determined by the method of MM-437E3, according to work sheets attached. Calculations are carried to 8 places, with final rounding to seven, developing the following values (Only the first computation is shown completely. The others are obvious as the subscripts advance by 1):

(*) If 5th differences of u from (2) or 7th differences of u from (4) are constant, (9) and (10) are not effective in producing a usable (11) and (12). In such cases, estimate derivatives from formulas similar to (9) and (10) obtained from ascending or descending differences, which contain both odd and even orders of difference. These lead to formulas similar to (11) and (12), which contain only 5th and 7th differences, respectively.

x	Symbol	y to 8 places	Final y
0.6	1	0.83527021	0.8352702
.7	2	.78270454	0.7827045
.8	3	.72614903	0.7261490
.9	4	.66697680	0.6669768
1.0	5	.60653065	0.6065307

DISCUSSION

The computation shows that after applying the Milne Error Correction, the change was not enough to affect last place of the 8-place answer obtained from (4) for any of the values to $x = 1.0$. This might have been predicted and would indicate that if we wished to stop the calculation at $x = 1.0$, it might have been possible to use (3) without any back-check. We did not do so, as the supposition was that at least ten values beyond starting values would be obtained.

The actual differences of u, insofar as we can tabulate them, show them to be less than estimated; thus,

	As predicted	Highest Actual
6th diff. "u"	.00011	.000065
8th diff. "u"	.000026	.000012

The trend of these differences, however, shows the likelihood that there are larger ones than the above "highest actual" values that are in the complete range of values of x, for it must be realized that tabulation of the ten values of u supplies only four 6th differences and two 8th differences.

The reader is cautioned not to accept the fact that if the Milne Error Correction does not produce any alteration in the right-hand digit of y as obtained from (4), it is evidence that the result is thereby accurate to the number of places of y so computed. This is only true if 7th differences vanish if (4) is used. If such difference does not vanish, as stated in (g), there is a residual error in each y believed to be expressed by (12) which accumulates and compounds as the number of values of computed y's increases. In the present case, it is unlikely that 8th place would be affected by the <u>Error Correction</u>, even for five more values, but by considerations outlined in (h) and (i), we see that only 7-place accuracy of y can reasonably be expected.

The function used in this example is, of course, the second derivative of

$$y = e^{-\frac{x^2}{2}}$$

which is much used in Quantum Mechanics. A study of its 6th and 8th derivatives, after giving effect to eliminating the effect of 6th and 8th differences in those derivatives, as in (g), provides reasonable support for the conclusions reached above by the somewhat rough predictions made from the first five tabulated values.

(over)

FACTOR SHEET: MILNE 5-POINT (6 ORDINATE) FORMULA FOR DOUBLE INTEGRATION BY STEP-BY-STEP METHOD

ORIGINAL CALCULATION: $h = 0.1$ Length Factor: $h^2/48 = 0.01/48$

			- Column Multipliers -		
x	u	$u = (x^2 - 1)y$	67	-8	122
0	n = -5	-1.00000000			
0.1	-4	-0.98506236	-65.999178	+7.527926	
0.2	-3	-0.94099072	63.046378	6.959662	-106.134841
0.3	-2	-0.86995771	58.287167	6.203342	94.600963
0.4	-1	-0.77541773	51.952988	5.294981	80.748467
0.5	0	-0.66187268	44.345470	4.276583	65.217897
0.6	+1	-0.53457293	35.816386	3.193435	48.699877
0.7	+2	-0.39917932	26.745014	2.091309	31.892467
0.8	+3	-0.26141366	17.514715	1.013805	15.460522
0.9	+4	-0.12672559	8.490615	0	0
1.0	+5	0	0		

BACK-CHECK CALCULATION: $h = 0.1$ Length Factor: $h^2/240 = 0.01/240$

			17	232	222
0.2	-3	-0.94099072	-15.996842		
0.3	-2	-0.86995771	14.789281	-201.830189	
0.4	-1	-0.77541773	13.182101	179.896913	-172.142736
0.5	0	-0.66187268	11.251836	153.554462	146.935735
0.6	+1	-0.5345729~~1~~3	9.0877~~3~~~~0~~40	124.020920	118.675190
0.7	+2	-0.3991793~~0~~2	6.786048	92.609602	88.617809
0.8	+3	-0.26141366	4.444032	60.647969	58.033833
0.9	+4	-0.1267250~~0~~59	2.154335	29.400337	28.133081
1.0	+5	0	0	0	0

WORK SHEET: MILNE 5-POINT (6 ORDINATE) FORMULA FOR DOUBLE INTEGRATION BY STEP-BY-STEP METHOD

Example: $d^2y/dx^2 = (x^2 - 1)y$

FORMULA	TRIAL		CHECK	
$x = 0.6$	y_0	0.88249690	y_0	0.88249690
	y_{-4}	0.99501248	y_{-2}	0.95599748
	$-y_{-5}$	1.00000000	$-y_{-3}$	0.98019867
		0.87750938		0.85829571
0.01/48 (67u - 8u +122u - 8u +67u)		-0.04223921		
-4 -3 -2 -1 0	y_{+1}	0.83527017		
	$-u_{+1}$	0.53457291		
.01/240 (17u +232u +222u +232u +17u)				-0.02302550
-3 -2 -1 0 1	Final		y_{+1}	0.83527021
$\delta y = 4$ E = .00000004/26			$-u_{+1}$	0.53457293

The remainder of the Work Sheet is similar except that all subscripts are advanced by 1, with values entered accordingly.

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7, 9, AND 11-TERM CLOSED-TYPE FORMULAS FOR MILNE METHOD OF STEP-BY-STEP DOUBLE INTEGRATION OF SECOND-ORDER DIFFERENTIAL EQUATIONS IN WHICH FIRST DERIVATIVES ARE ABSENT

(A supplement to Marchant-Method MM-437E4)

REMARKS: *In applying MM-437E4, it is sometimes desired to use an over-all check-formula of the closed type having more than five terms in order to refine the final "y" beyond what is possible by applying the 5-term closed-type formula given in MM-437E4. The coefficients for 7, 9, and 11-term formulas herein were computed by J. E. Kilpatrick from certain values of related coefficients given in a paper by H. E. Salzer, Jour. Math. and Phys., Nov. 1945, p. 135. Only symmetric odd-point formulas are shown because, as W. E. Milne points out, their accuracy is equal to that of the next larger even-point formulas.*

Of the various kinds of n-point formulas that may be obtained, one has been chosen in each case which minimizes the effect of u_n because this value is ordinarily the least correct of the set, it being usually obtained as a result of applying a predictor or extrapolation formula to the preceding set of points. The relative smallness of the coefficient of u_n will be noted.

FORMULAS:

7-TERM CLOSED-TYPE FORMULA for back check:

$$y_n = y_{n-1} + y_{n-5} - y_{n-6} + \frac{h^2}{60480}(k_0u_n + k_1u_{n-1} + k_2u_{n-2} + k_3u_{n-3} + k_4u_{n-4} + k_5u_{n-5} + k_6u_{n-6})$$

in which	or, divided by 60480, and rounded to 9 places
$k_0 = k_6 = 3935$	0.065 062 831
$k_1 = k_5 = 60810$	1.005 456 349
$k_2 = k_4 = 48945$	0.809 275 794
$k_3 = 75020$	1.240 410 052 *

9-TERM CLOSED-TYPE FORMULA for back check:

$$y_n = y_{n-1} + y_{n-7} - y_{n-8} + \frac{h^2}{518400}(k_0u_n + k_1u_{n-1} + k_2u_{n-2} + k_3u_{n-3} + k_4u_{n-4} + k_5u_{n-5} + k_6u_{n-6} + k_7u_{n-7} + k_8u_{n-8})$$

in which	or, divided by 518400, and rounded to 9 places
$k_0 = k_8 = 31511$	0.060 785 108
$k_1 = k_7 = 540332$	1.042 307 099
$k_2 = k_6 = 346268$	0.667 955 247
$k_3 = k_5 = 807764$	1.558 186 728
$k_4 = 177050$	0.341 531 636

(over)

11-TERM CLOSED-TYPE FORMULA for back check:

$$y_n = y_{n-1} + y_{n-9} - y_{n-10} + h^2(c_0u_n + c_1u_{n-1} + c_2u_{n-2} + c_3u_{n-3} + c_4u_{n-4} + c_5u_{n-5} + c_6u_{n-6} + c_7u_{n-7} + c_8u_{n-8} + c_9u_{n-9} + c_{10}u_{n-10})$$

in which, rounded to 9 places

$c_0 = c_{10} = 0.057\ 408\ 478$

$c_1 = c_9 = 1.078\ 075\ 434$

$c_2 = c_8 = 0.495\ 712\ 877$

$c_3 = c_7 = 2.056\ 290\ 134$ *

$c_4 = c_6 = -0.620\ 995\ 333$

$c_5 = 2.867\ 016\ 820$

These 11-point coefficients are not shown in exact form because of their unwieldy character when so expressed; i.e., the common divisor is 958 003 200, and the numerator for, say, c_5 is 2 746 611 288. With a 10-column calculator available, it is thus easier to use the decimal equivalents than the exact fractions.

NOTE

The asterisks (*) affixed to certain coefficients of the 7 and 11-term sets of coefficients indicate an adjustment of normal rounding to cause the sum of the decimal equivalents of an n-term formula exactly to equal n-2.

1 026 6436 8

STARTING VALUES FOR MILNE-METHOD INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

THE METHOD OF MILNE

REMARKS: *Marchant Method 437E3 describes the Milne Method of integrating differential equations in the form $dy/dx \equiv y' = u(x,y)$ after "starting" values are known. The method submitted herewith is that proposed by Dr. W. E. Milne* for obtaining these starting values for the three-term formulas of MM437E3. An Appendix gives similar formulas for obtaining starting values for continuance by the five-term Milne formulas. A note refers to a method for obtaining starting values for 2nd order equations of the type integrated in Marchant Method 437E4.*

The method herein requires but one differentiation of the differential equation, instead of numerous differentiations, as is necessary in the Method of Taylor's Series (see Marchant Method 437E2). The method is exact if 5th differences of the tabulated y's vanish and, if there are higher differences, it produces results within an upper bound of error that may be computed. Such computation, however, must ordinarily be deferred until sufficient values subsequent to the "starting" values have been found to enable a difference array to show values from which the 5th derivative of y with respect to the x's of the starting values may be estimated.

The computer will doubtless compare the within method with that of Runge-Kutta, which is often proposed as a simplified means of obtaining starting values for further integration by the Milne (or other) quadrature processes. Data supplied us by courtesy of Dr. Milne are conclusive that the Runge-Kutta Method is not to be recommended for such work because it is not exact, even in the case of some differential equations the solutions of which are polynomials of 2nd degree. Furthermore, there appears to be no explicit usable expression for its error. It so happens that the Runge-Kutta Method gives excellent results when integrating the particular equation used as an example herein, but this is only a coincidence. On certain other typical equations, it produces starting values which are substantially worthless.

OUTLINE: Given the differential equation $dy/dx \equiv y' = u\ (x,y)$, with initial value of $y = y_0$ when $x = x_0$; y' is differentiated with respect to x at $x = x_0$, producing $d^2y/dx^2 \equiv y''_0 = v_0(x,y)$. Interval $h = x_1 - x_0$, $x_2 - x_1$, etc.

Trial values are obtained as follows:

$$u_1 = u_0 + hv_0 \quad \text{and} \quad u_{-1} = u_0 - hv_0$$

Using these trial values in (1) and (2) below, first approximations of y_1 and y_{-1} are obtained as follows:

$$(1) \quad y_1 = y_0 + \frac{h}{24}(7u_1 + 16u_0 + u_{-1}) + \frac{h^2 v_0}{4} + \text{Remainder}$$

$$(2) \quad y_{-1} = y_0 - \frac{h}{24}(u_1 + 16u_0 + 7u_{-1}) + \frac{h^2 v_0}{4} + \text{Remainder}$$

(*) *W. E. Milne, American Mathematical Monthly, Jan. 1941, "Note on the Numerical Integration of Differential Equations."*

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The Remainder in (1) is $-\frac{1}{180}h^5y^{(5)}$ and in (2) is $+\frac{1}{180}h^5y^{(5)}$

in which $y^{(5)}$ is the 5th derivative of y with respect to some value x, unknown, which lies between x_{-1} and x_1. As stated under "Remarks," the Remainders are not considered when obtaining the starting values, but inasmuch as they may be approximated at a later stage of the computation, they may subsequently be used to refine the starting values for a recomputation up to that point, if desired, or to show how many places of the starting values are reliable.

The first approximations of y_1 and y_{-1} are then substituted in the differential equation to obtain second approximations of u_1 and u_{-1}. These are again used in (1) and (2) to obtain second approximations of y_1 and y_{-1}, which are again substituted in the differential equation to obtain third approximations of u_1 and u_{-1} respectively, and so on until two successive values of y are alike to the number of places to be retained.

The first approximation of y_2 is then obtained by substituting the already-known values in (3) below:

(3) $y_2 = y_0 + \frac{2h}{3}(5u_1 - u_0 - u_{-1}) - 2h^2v_0 + \text{Remainder}$

in which the Remainder is $+\frac{7}{45}h^5y^{(5)}$ defined in a manner similar to the Remainders of (1) and (2).

The y_2 thus found is substituted in the differential equation, thus obtaining a first approximation of u_2 which is then substituted in (4) below, which will be recognized as the 3-Term Closed-Type Formula used in Marchant Method 437E3 for check-back. It is Simpson's 1/3 Rule.

(4) $y_2 = y_0 + \frac{h}{3}(u_2 + 4u_1 + u_0) + \text{Remainder}$

in which the Remainder is $-\frac{1}{90}h^5y^{(5)}$ defined in a manner similar to the previous Remainders.

This (4) produces a second approximation of y_2, after which the corresponding second approximation of u_2 is found by substituting in the differential equation, and so on, until two successive values of y_2 from (4) are alike to the number of places to be retained.

The desired starting values are; thus,

y_{-1}, y_0, y_1 and y_2

which by using the notation of MM-437E3 become, respectively,

y_{-3}, y_{-2}, y_{-1}, and y_0 with their corresponding derivatives

u_{-3}, u_{-2}, u_{-1}, and u_0

Integration then proceeds according to the procedure outlined in Marchant Method 437E3, but as the latter uses the 5-term formulas in its example, it may be helpful to record the 3-term formulas, as below:

The "predictor" is (5) and the check-back is (6), below:

(5) $y_1 = y_{-3} + \frac{4h}{3}(2u_{-2} - u_{-1} + 2u_0)$

(6) $y_1 = y_{-1} + \frac{h}{3}(u_{-1} + 4u_0 + u_1)$. This is same as (4), with notation of *MM-437E3*.

No Remainders are specified in the above because the Milne-Method error factor of 1/29 of the difference between the values of y obtained by these formulas, when applied as a correction to the y found from (6), supplies an exact y if 6th differences of tabulated y's vanish. Marchant Method *437E3* refers to a vanishing 5th difference of u, which corresponds to a 6th difference of y.

EXAMPLE: Given $dy/dx = y' \equiv -xy$, with initial values of $y = 1$ when $x = 0$. Compute starting values of y to seven places corresponding to x at -0.1, 0.1, and 0.2. Then continue the integration by the regular Milne process of Marchant Method *437E3* until sufficient values are obtained to enable an estimate to be made of the number of computed places of the starting values that are reliable.

Inasmuch as the Marchant calculator application to work of this character is fully covered in *MM-437E3*, or consists of solving simple arithmetical formulas, the example will merely be outlined in numerical form, as follows:

(a) Differentiating $dy/dx = -xy$ with respect to x, we have $d^2y/dx^2 \equiv y'' = -x\frac{dy}{dx} - \frac{dx}{dx}y$ which equals $(x^2 - 1)y$, which is the function "v" referred to in (1), (2) and (3). By substituting initial values, we have $v_0 = -1$.

(b) First trial of $u_1 = u_0 + hv_0 = 0 + 0.1(-1) = -0.1$

" " " $u_{-1} = u_0 - hv_0 = 0 - 0.1(-1) = 0.1$.

(c_1) First substitution in (1)

$$y = 1 + \frac{0.1}{24}(7(-0.1) + 0.1) + \frac{(0.1)^2(-1)}{4} = 0.995$$

from which second trial $u_1 = -xy = -(0.1)(0.995) = -0.0995$

(d_1) First substitution in (2)

$$y_{-1} = 1 - \frac{0.1}{24}(-0.1 + 7(0.1)) - 0.0025 = 0.995$$

from which second trial $u_{-1} = -xy = (-0.1)(0.995) = 0.0995$

(c_2) Second substitution in (1)

$$y_1 = 1 + \frac{0.1}{24}(7(-0.0995) + 0.0995) - 0.0025 = 0.9950125$$

from which third trial u_1 is -0.09950125

(d_2) Second substitution in (2)

It is noted that there is symmetry about y_0, whence $y_{-1} = 0.9950125$ and $u_{-1} = 0.09950125$

It is also noted that third substitutions would not change y to 7 places, so they are retained as above.

(e) Substitution in (3)

$$y_2 = 1 + \frac{0.2}{3}(5(-0.09950125) - 0.09950125) - 2(0.01)(-1) = 0.9801995$$

from which $u_2 = -0.2\ y_2 = -0.1960399$

(over)

(f1) First substitution in (4).

$$y_2 = 1 + \frac{0.1}{3}(-0.1960399 + 4(-0.09950125)) = 0.9801985$$

from which $u_2 = -0.1960397$

Inspection shows that substitution of the above u_2 in (4) would affect y_2 by only 1/15 in 7th place, so y_2 from (f_1) may be accepted.

(g) The four starting values to 7 places are:

Notation (a) to (f)		Notation for Continuing by *MM-437E3*
y_{-1}	0.9950125	y_{-3}
y_0	1.0000000	y_{-2}
y_1	0.9950125	y_{-1}
y_2	0.9801985	y_0

(h) Continuing the integration by Milne Method of *MM-437E3*, per work sheet (Page 7), we have, with differences up to and including the 7th, using notation of *MM-439E1:*

n	x	y	d'	d"	d'''	d''''	d^{v}	d^{vi}	d^{vii}
-3	-0.1	0.9950125		-98265		2908		-169	
			49875		-1485		62		45
-2	0	1.0000000		-99750		2970		-124	
			-49875		1485		-62		-45
-1	0.1	0.9950125		-98265		2908		-169	
			-148140		4393		-231		75
0	0.2	0.9801985		-93872		2677		-94	
			-242012		7070		-325		-52
1	0.3	0.9559973		-86802		2352		-146	
			-328814		9422		-471		
2	0.4	0.9231159		-77380		1881			
			-406194		11303				
3	0.5	0.8824965		-66077					
			-472271						
4	0.6	0.8352694							

NOTE: Values above diagonal obtained as subsequently described in (j).

(i) From the above array, some idea of the value of $y^{(5)}$ for the points $x = -0.1$ to 0.2 can be had which, by substituting in the Remainders of (1), (2) and (4), will provide a reasonably satisfactory answer to the question of the reliability of the starting values found. The usual way of estimating $y^{(5)}$ from an array such as the above, considering only amounts below the diagonal line, is to substitute the <u>descending</u> differences in the Newton-Gregory formula for 5th derivatives, as follows (Also see Note 4, Page 11):

(7) $$y^{(5)} = \frac{1}{h^5}\left(\Delta^{v} - \frac{5}{2}\Delta^{vi} + \frac{25}{6}\Delta^{vii} - \frac{35}{6}\Delta^{viii} \ldots\ldots\right)$$

in which Δ^{v}, corresponding to $x = -0.1$ is -0.0000231 and Δ^{vii} corresponding to $x = -0.1$ is -0.0000052, and similarly for other values of x.

To apply this formula at $x = 0.2$ will require that more values of y be computed, even if we drop terms beyond Δ^{vi}. Before doing so, however, let us apply it to x at $x = -0.1$, and it is found that the usable terms for 6th and 7th differences are so large as compared with their predecessors that there is no apparent convergence within the limit of 7th difference. Obviously, the formula is not suitable for this case. It is given herewith, however, because in many cases it will prove satisfactory.

(j) In cases where (7) cannot be used, additional values of y should be computed in the range beyond $x = -0.1$ in order that central differences may be used in the Newton-Stirling formula for 5th derivatives, which converges rapidly, as below:

(8) $$y^{(5)} = \frac{1}{h^5}\left(d^{v} - d^{vii}/3 \ldots\right)$$

in which d^{v} and d^{vii} are 5th and 7th central differences that are opposite the x values. Inasmuch as these "straddle" the lines of the x values, they are obtained by taking arithmetic means of the tabulated values on either side.

In the present case, the computation of additional y's in the range beyond $x = -0.1$ may be avoided because there is symmetry of the values of y around y_0. This is because y_{-1} equals y_1, as by reference to (3) and (4) it is noted that integrating backwards from y_1 produces the same y's as will integrating forwards from y_{-1}. The array of differences may thus be inverted around a horizontal line drawn through x_0. Care must be taken to place proper signs on this inverted array, as they will not necessarily be the same as those used for the corresponding differences in the lower half of the array.

The missing differences may then be inserted, thus completing the portion of the array that is in italics. It is noted that 6th and 7th differences show irregularities due to slight errors in y (which we seek) and which errors become cumulative in an opposite sense by the inversion of the array. Though irregular, the differences are all we have at this point and must serve as the basis for an estimate, the reliability of which will have to be considered in the light of the effect of this fluctuation on total error.

(k) For applying (8), it is necessary to list the mean 5th and 7th central differences, as below, after which substitution in (8) produces the estimated values of $y^{(5)}$, also shown.

x	mean 5th diff.	mean 7th diff.	$y^{(5)}$
-0.1	(62 + 231)/2 = 147	similarly is -15	1.52
0	(62 + (-62))/2 = 0	" " 0	0
0.1	(-62 + (-231))/2 = -147	" " 15	-1.52
0.2	(-231 + (-325))/2 = -278	" " 12	-2.82

(m) In obtaining the values of y_{-1} and y_1 from (1) and (2) respectively, the largest 5th derivative, based upon the above assumptions, is ± 1.5, and for y_2 it is -2.8. The maximum Remainder and an upper bound of error is; thus,

for y_{-1} $\frac{1}{180}h^5 y^{(5)} = \frac{(.00001)(\pm 1.5)}{180} = \pm 0.0000001$

for y_1 $-\frac{1}{180}h^5 y^{(5)} = -\frac{(.00001)(\pm 1.5)}{180} = \pm 0.0000001$

for y_2 $-\frac{1}{90}h^5 y^{(5)} = -\frac{(.00001)(-2.8)}{90} = 0.0000003$

(n) Now, the Remainders estimated in (m) are as likely to be dependent upon the least 5th derivatives in the range of x values that determine the y's of (1), (2) and (4) as upon the greatest. The uncertainty of the computed values of y_{-1} and y_1 is thus ± 0.0000001, and of y_2 it is between 0 and 0.0000003. The most likely Remainders are thus the means of these extremes. Applying them, we have

over.

For	Computed	Most Likely Remainder	Most Likely y	Uncertainty
y_{-1}	0.9950125	0	0.9950125	± 0.0000001
y_1	0.9950125	0	0.9950125	± 0.0000001
y_2	0.9801985	0.0000002	0.9801987	± 0.0000001

(o) It is therefore concluded that six places are correct in that the computed error in 7th place (even of y_2 after taking into account uncertainties of u's from previous values) does not exceed 5. However, the computer would be entirely justified in using seven places for the starting values as the values given are better than they would be if rounded to 6th place. Taking the new value of y_2 at 0.9801987, the computer could revise the work so far done, if desired. As it stands on the work sheet, only 6-place accuracy is to be expected. From (n) all values of y for those above y_2, at least, should be increased by 2 in 7th place, even if no recomputation is made.

(p) Inasmuch as this function may be integrated analytically, the following comparison shows the closeness of agreement.

x	Values from (n) y	Computed $y^{(5)}$	Actual (to 7 places) y	Actual $y^{(5)}$
-0.1	0.9950125	1.52	0.9950125	1.48
0	1.000000 (initial)	0	1.0000000	0
0.1	0.9950125	-1.52	0.9950125	-1.48
0.2	0.9801987	-2.82	0.9801987	-2.86

If the rough correction indicated in (o) is made; i.e., merely increasing work sheet values above y_2 by 2 in 7th place, the values to and including those at x = 0.6 are correct to 6 places, and up to and including those at x = 0.3 are correct to 7 places.

FACTOR SHEET*: MILNE 3-TERM FORMULA FOR INTEGRATION OF dy/dx = u

h = 0.1 Length Factor for prediction formula 4h/3 = 0.133 ..... For check-back h/3 = 0.0333 .....

n	x	y	u	----Absolute Values---- 2u	4u
-3	-0.1	0.9950125	0.0995025		
-2	0	1.0000000	0.	0	
-1	0.1	0.9950125	-0.09950125	0.1990025	
0	0.2	0.9801985	-0.1960397	0.3920794	0.7841588
		/59973	/7992		
1	0.3	0.95~~60021~~	-0.286~~8006~~	0.5735984	1.1471968
		/159	/464		
2	0.4	0.9231~~252~~	-0.3692~~501~~	0.7384928	1.4769856
		/4965	/483		
3	0.5	0.882~~5094~~	-0.4412~~547~~	0.8824966	1.7649932
		/694	/616		
4	0.6	0.8352~~654~~	-0.5011~~712~~		

NOTE: Prediction by (5) uses lines that terminate in "2u" and check by (6) uses lines terminating in "4u".

WORK SHEET: MILNE 3-TERM FORMULA FOR INTEGRATION OF dy/dx = u

	Trial		Check	
For x = 0.3				
0.133...(2u - u + 2u)	y_{-3}	0.9950125	y_{-1}	0.9950125
		-0.0390104		-0.0390154
-2 -1 0	y_1	0.9560021	y_1	0.9559971
0.033...(u + 4u + u)		-0.3	E	2
-1 0 1	u_1	-0.2868006		0.9559973
δy = 50 E = .0000050/29				-0.3
			u_1	-0.2867992
For x = 0.4				
0.133...(2u - u + 2u)	y_{-2}	1.0000000	y_0	0.9801985
		-0.0768748		-0.0570829
-1 0 1	y_2	0.9231252	y_2	0.9231156
0.033...(u + 4u + u)		-0.4	E	3
0 1 2	u_2	-0.3692501		0.9231159
δy = 96 E = .0000096/29				-0.4
			u_2	-0.3692464

and so forth, the subscripts advancing by 1, with values entered accordingly.

(*) *This form is useful when 3-term formulas are used. It is a condensation of the plan used in MM-437E3 for the case of the 5-term formulas.*

(over)

APPENDIX

Most numerical integrations by the Milne Method are made by aid of the 3-term formulas because in ordinary design computations the accuracy that they provide is adequate without having to reduce h to such a point that the number of increments in y must be excessive to cover a given range in x.

Another reason for preferring the 3-term formulas is that with the least amount of work they provide exact results when the function obtained as a result of the integration is a polynomial of 4th degree or less, and by use of the Milne Error Correction (see MM-437E3) this exactness is extended to cover polynomials of 5th degree for the values above those that are used to start the integration.

Furthermore, by the procedure outlined herein the starting values from the 3-term formulas may likewise be found exactly, provided the polynomial is of 5th degree, as in such a case there is no uncertainty about the value of x corresponding to the derivative to be used in the Remainders of (1), (2) and (4). The derivative in such a case is a constant.

It is, therefore, of no avail to use 5-term formulas if the integral function is known to be or later found to be a polynomial of 5th degree or less. Use of the 5-term formulas, in such a case, provides no additional accuracy, and their use is only a waste of time.

If, however, the function that results from the integration is a polynomial of higher degree than the 5th, as evidenced by its having differences of higher order than the 5th, when tabulated to the number of places required, the computer either may use the 3-term formulas and by continuing the difference array check the upper bound of error and reduce h when the bound exceeds that allowable, or he may use the 5-term formulas.

It is important to note that if the computer elects to continue to use the 3-term formulas it is not sufficient to determine the error bound at one point of the table. Such error may be within the prescribed limit at that point, but the existence of differences of higher order than the 5th will cause the error to change as additional values are computed; i.e., the differences (and derivatives) of 5th order will increase or decrease, and with them the bound of error.

If in the development of the work a continuing difference array shows that differences of 6th and higher order exist (when y is tabulated to the number of places to be retained), it is probable that the work would be expedited by changing to the 5-term formulas, rather than to decrease the interval and continue with the 3-term formulas. The extra work of computing an increment in y when using 5-term formulas, when modern calculating machines are available, is ordinarily far less than that caused by increasing the number of increments to be computed by reducing h and continuing with the 3-term formulas.

The advantage of the 5-term formulas is that for continuing integration after starting values have been found, they provide exact results if the integral function is a polynomial of 7th degree, assuming that the Milne Error Correction is used. No use of the 3-term formulas, regardless of how small the interval h, will provide this accuracy in general, though the error in any certain range of x values may be made as small as desired by suitably reducing the interval h.

To sum up: It is believed advisable to start all new work with the 3-term formu[illegible] using the 5-term formulas, if at all, for an over-all check. If the difference array developed from the 3-term values tabulated to the number of places to be retained shows substantial differences of 6th order and higher, continuance of the work by use of 5-term formulas is advisable. Similarly, if the nature of the function that is being integrated is known, such as is the case when the differential equation differs only in constant coefficients from a function previously integrated, and it is known that the function cannot be represented to the accuracy desired by a polynomial of 5th degree or less, it is advisable to start with the 5-term formulas.

For starting such integration by the 5-term formulas, values of y_{-1}, y_1 and y_2 are computed as herein, using the 3-term formulas. The value of y_{-2} is next obtained on the 3-term basis by use of (3) and (4) applied in a backward direction from y_1, as follows:

(3a) $$y_{-2} = y_0 - \frac{2h}{3}(5u_{-1} - u_0 - u_1) - 2h^2 v_0 + \text{Remainder}$$

and checked by applying (4) similarly reversed, as follows:

(4a) $$y_{-2} = y_0 - \frac{h}{3}(u_{-2} + 4u_{-1} + u_0) + \text{Remainder}$$

The values of y_{-2} to y_2 inclusive are then regarded as first approximations for substitution in 5-term formulas computed by and reproduced through courtesy of Dr. Milne, as follows:

(9) $$y_{-2} = (y_0 - \frac{4h}{15}u_0 - \frac{h^2}{3}v_0) - \frac{h}{180}(3u_2 - 32u_1 + 288u_{-1} + 53u_{-2}) + R$$

in which Remainder (R) is $\frac{32}{7560}h^7y^{(7)}$

(10) $$y_{-1} = (y_0 - \frac{19h}{30}u_0 + \frac{11h^2}{48}v_0) - \frac{h}{2880}(-11u_2 + 144u_1 + 944u_{-1} - 21u_{-2}) + R$$

in which Remainder (R) is $-\frac{5}{7560}h^7y^{(7)}$

(11) $$y_1 = (y_0 + \frac{19h}{30}u_0 + \frac{11h^2}{48}v_0) + \frac{h}{2880}(-21u_2 + 944u_1 + 144u_{-1} - 11u_{-2}) + R$$

in which Remainder (R) is $\frac{5}{7560}h^7y^{(7)}$

(12) $$y_2 = (y_0 + \frac{4h}{15}u_0 - \frac{h^2}{3}v_0) + \frac{h}{180}(53u_2 + 288u_1 - 32u_{-1} + 3u_{-2}) + R$$

in which Remainder (R) is $-\frac{32}{7560}h^7y^{(7)}$

Second approximations of y_{-2} to y_2 inclusive are obtained by applying in succession formulas (9), (10), (11) and (12), not considering Remainders, using the last found value for the u's; i.e., in (10) the u_{-2} corresponding to the y_{-2} found from (9) is used, etc., so that by the time (12) is applied, three of the four terms containing u will have been obtained from (9), (10) and (11).

Third approximations are then obtained by substituting the last found values of u's in (9) to (12) inclusive, and so on, until no further refinement of u alters the y's. In practice, these substitutions ordinarily do not require complete re-calculation; often the amount of alteration caused by the new value may be easily determined by considering only the differences of the new u's from those used in the previous application of the formula.

A predicted value for y_3 is then obtained by

(13) $$y_3 = y_0 + \frac{3h}{320}(401u_2 - 944u_1 + 416u_0 + 496u_{-1} - 49u_{-2}) + \frac{153h^2}{16}v_0 + R$$

in which Remainder (R) is $\frac{1107}{7560}h^7y^{(7)}$

(over)

This is then corrected by applying the 5-term closed-type formula on Page 9 of Marchant Method *437E3*, which may be written as shown below:

$$(14) \quad y_{n+1} = y_{n-3} + \frac{2h}{45}(7u_{n+1} + 32u_n + 12u_{n-1} + 32u_{n-2} + 7u_{n-3}) + R$$

in which Remainder (R) is $-\frac{64}{7560} h^7 y^{(7)}$

The regular process, as outlined in Marchant Method *437E3*, is then continued using as a "predictor" the 5-term open-type formula of Page 8 of *MM-437E3*, which may be written as below:

$$(15) \quad y_{n+1} = y_{n-5} + \frac{3h}{10}(11u_n - 14u_{n-1} + 26u_{n-2} - 14u_{n-3} + 11u_{n-4}) + R$$

in which Remainder (R) is $\frac{2214}{7560} h^7 y^{(7)}$

and (14) is then used to correct such values.

NOTE 1

The question is sometimes asked: "How much must 'h' be reduced when 3-term formulas are used if accuracy is to be obtained equivalent to that supplied by the 5-term formulas?" The answer, to be of any value, can be given only for one value of x and then only when the values of 5th and 7th derivatives (if x is one of the "starting values") are known, or when the values of 6th and 8th derivatives are known if x has a value subsequent to those used for the starting values, and then only after applying the Milne Error Correction. It is thus seen that the answer to the question is not easy to give in any specific case. However, in the case of equality of derivatives (5th and 7th, or 6th and 8th, as the case may be), it can be shown that increments obtained from the 5-term formulas have an error of 1/(13 x 10) of that obtained from the 3-term formulas; that is to say, with equivalence of derivatives, two extra places will be accurate if 5-term formulas are used. If the higher-order derivative is one-half of that of lower order, the error ratio is 1/260, etc.

NOTE 2

Of more practical value is the answer to the question: "How does the error vary with reduction of 'h' provided 3-term formulas are used?" Inasmuch as the error terms contain h^5, the error of any increment in y will be proportional to the 5th power of h, but as more increments are required to cover any interval in x, the error of the complete integration varies according to the 4th power of h. From this, it is seen that halving h reduces the error to 1/16, quartering it reduces it to 1/256, etc. These ratios are subject to the assumption that the derivative that controls the error is the same in the small interval as it is in the large. Because of this limitation, these ratios would be seriously erroneous as the function approached an asymptotic condition.

If 5-term formulas are used, any reduction of "h" causes the error to be reduced so it is in proportion to the 6th power of h.

NOTE 3

The method also may be used as a basis for obtaining starting values of 2nd order differential equations of the type integrated by Marchant Method *437E4*; namely, $d^2y/dx^2 = v(x,y)$. The second order equation requires differentiation once; that is to say, d^3y/dx^3 at $x = x_0$ is obtained. The "v_0" of (1), (2) and (3) then becomes this third derivative. The "u's" of the equations are 2nd derivatives, and the y's of the equations produce the column of first derivatives.

The process is applied again to the column of d^2y/dx^2 values, thus obtaining the final y's. This may be somewhat simplified, however, because y_2 may be obtained directly by applying (4), and y_3 by applying (4) with y_1 as initial value with the subscripts of u's advanced by 1.

Dr. W. E. Milne* has also published a method for numerical integration of equations in the form $d^2y/dx^2 + g(x)y = f(x)$. By setting $f(x) = 0$, it may be used to integrate many equations of the type considered herein.

NOTE 4

The procedure described in Sections "h" to "p" of the computation is necessary to fulfill as rigorously as possible the requirement that the number of places of starting values that are reliable shall be found. In much computing work, the preparation of the difference array and the subsequent estimate of the derivatives which control the error may be avoided because the Milne Error-Corrections ($E = \delta/29$) found when obtaining the y's by applying (5) and (6) give a clue to the error in the starting value.

In the problem considered, by interpolating from the values of the Milne Error-Correction ($E = \delta/29$), an estimate of the errors of the starting values may be obtained, after which the y's found so far may be corrected by the change made in y_2 of the starting values. Applying this process to the within case, we have

Milne Error-Correction (E) for	x = 0.5	0.00000046
	0.4	0.00000033
	0.3	0.00000017

Interpolating to the known "0" error of the initial value, we have

(E) for	x = 0.2	0.00000008
	0.1	0.00000002

However, inasmuch as the Remainder in (2) is one-half of that of (4) and (6) to which the above E's apply, the error in y_1 found in (c_2) may be presumed not to exceed 0.00000001, which is not sufficient to affect 7th place. However, the error of y_2 found in (f_1) is estimated to be the full amount of the estimated E shown above (0.00000008), which is sufficient to affect 7th place, thereby increasing y_2 to 0.9801986.

This is less by "1" in 7th place than the value obtained by the more precise analysis, which value is correct to 7 places, as proved by analytic integration.

Though the Milne Error-Corrections ($E = \delta/29$) used when obtaining the y's for x = 0.3, 0.4, etc., lead to exact y's when 6th differences of y vanish, they do not provide an exact measure of the error of (6) when differences of higher order exist, as in this case. The interpolation plan of this Note corrects to some extent the condition arising because of this fact, but there are elements that prevent its doing so entirely. This explains why the method of this Note does not give quite the precision that is obtained by using a difference array.

(*) *W. E. Milne, American Mathematical Monthly, Feb. 1942 -- The Numerical Integration of $y'' + g(x)y = f(x)$.*

STARTING VALUES FOR MILNE-METHOD INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER, OR OF SECOND ORDER WHEN FIRST DERIVATIVES ARE ABSENT

THE METHOD OF TAYLOR'S SERIES

(Compare also Marchant Method MM-437E1

REMARKS: *Marchant Method MM-437E3 describes the Milne Method of integrating differential equations in the form of $dy/dx \equiv y' = u(x,y)$, and Marchant Method MM-437E4 similarly relates to equations in the form $d^2y/dx^2 \equiv y'' = v(x,y)$. Each requires that a few starting values at equidistant values of x be known. Integration then proceeds from these starting values by quadrature processes.*

This method relates to obtaining starting values for use in MM-437E3 (and 437E4) when the functions y' (or y") may be differentiated with respect to x, and each derivative so found may likewise be differentiated until a sufficient number of derivatives of successively higher order are obtained for substitution in a Taylor's Series that will give the desired y's to the accuracy required. In cases where this successive differentiation is impracticable, see Marchant Method MM-437E1.

It is assumed that a knowledge of Taylor's Series is had. The method herein is subject to the limitations in the use of this series that have to do with convergence, continuity of the successive derivatives with respect to both x and y, etc. However, the use of this series for obtaining the few terms necessary to provide starting values for subsequent Milne-Method integration introduces certain matters which are not discussed in usual texts. Such subjects are covered herein (see particularly Note 2).

OUTLINE: The Taylor's Series in the form (1) below

$$(1) \quad f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f''''(x_0)}{4!}(x-x_0)^4 + R$$

is written in the equivalent form (2), as follows:

$$(2) \quad y = y_0 + y'_0 h + \frac{y''_0}{2}h^2 + \frac{y'''_0}{3!}h^3 + \frac{y''''_0}{4!}h^4 + R \text{ (Remainder)}$$

in which $h = (x-x_0)$, $0 < h < 1$

Inasmuch as the initial condition gives the value of $y = y_0$ when $x = x_0$ in functions of the type $dy/dx = u(x,y)$, and furthermore gives $y' = y'_0$ in the case of functions of the type $d^2y/dx^2 = v(x,y)$, and the successive differentiations give the y''_0, y'''_0, y''''_0 .... etc., everything is known for substituting in (2), thus obtaining y corresponding to the value of x which differs from x_0 by the amount h, except for the Remainder.

If h is chosen sufficiently small and enough terms are taken, the Remainder may be reduced to as small an amount as we please, which provides accuracy control of the method.

Four consecutive starting values of y which correspond to equally-spaced values of x are required to start the Milne-Method solution by the Milne 3-term formulas. Six values are required if the 5-term formulas are used, etc. As one of the required values is the initially known y, three and five new values have to be computed, respectively.

(over)

To aid in the identification of the y's found by Taylor's Series with the starting values of the Milne-Method integrations of MM-437E3 & 437E4, the table below is given. It also shows how h in (2) is obtained. As to this, the Taylor's Series procedure, as outlined herein, uses a variable interval h to obtain the successive y's. The intervals, however, are in multiples of s, which is equal to the "h" of the Milne-Method integrations; that is to say, certain intervals of the Taylor's Series integrations (those which are x_0-s and x_0 +.s, respectively) are the same as the h of the Milne-Method integrations, but one is twice as large when obtaining a Taylor's Series set of starting values for Milne-Method 3 or 5-term formulas, and another is three times as large for obtaining the last starting value for the Milne 5-term formulas. This will be clear from inspection of the following table and the example.

Taylor's Series Notation			Milne-Method Notation of MM-437E3 and 437E4
x	y	Interval h	
Corresponding to 3-term Milne formulas:			
x_{-1}	y_{-1}	$x_0 - s$	y_{-3}
x_0	y_0	x_0 initial values	y_{-2}
x_1	y_1	$x_0 + s$	y_{-1}
x_2	y_2	$x_0 + 2s$	y_0
Corresponding to 5-term Milne formulas:			
x_{-2}	y_{-2}	$x_0 - 2s$	y_{-5}
x_{-1}	y_{-1}	$x_0 - s$	y_{-4}
x_0	y_0	x_0 initial values	y_{-3}
x_1	y_1	$x_0 + s$	y_{-2}
x_2	y_2	$x_0 + 2s$	y_{-1}
x_3	y_3	$x_0 + 3s$	y_0

The foregoing procedure is recommended for obtaining the few starting values for subsequent integration by the Milne (or other) method. This is mentioned because some texts when describing the Taylor's Series procedure suggest proceeding from each x to the next; that is to say, each new y with its corresponding x is taken as a new pair of initial values. Such a plan has the disadvantage that a new set of derivatives must be evaluated for each pair of x,y values. Furthermore, it cannot take full advantage of the simplification of work when obtaining such derivatives that occurs when the initial x is zero. In such a case, terms of each derivative which contain x as a factor drop out and the series reduces to that of Maclaurin.

The example given herein shows the use of the Taylor's Series method for obtaining starting values for integrating by the Milne 3-term formulas of MM-437E3 $dy/dx = -xy$. Similar procedure would be followed if a double integration of $d^2y/dx^2 = (x^2 - 1)y$ of MM-437E4 were to be made. In the latter case, y'_0 is also known as one of the initial conditions so everything is required for substitution in (2), just as is the case when using the method for the first-order equation of MM-437E3.

EXAMPLE: Find starting values for integrating by Milne 3-term formulas $dy/dx \equiv u(x,y) = -xy$ with initial condition that $y = 1$ when $x = 0$, with intervals in multiples of $s = 0.1$, corresponding to Milne-Method integration when $h = 0.1$; the final y_2 is to be accurate to 5 places; i.e., 6th place error is to be less than 5.

We start by obtaining the successive derivatives of $y' \equiv dy/dx = -xy$. Because of the fact that the initial $x = x_0$ is zero, it is unnecessary to find the derivatives in terms of x and y. Time is saved by expressing each succeeding derivative in terms of the former, as follows:

$y' = -xy$	as $x_0 = 0$ and $y_0 = 1$		$y'_0 = 0$
$y'' = -xy' - y$	" " " " " " " "		$y''_0 = -1$
$y''' = -xy'' - 2y'$	" " " " "	y'_0 " 0	$y'''_0 = 0$
$y'''' = -xy''' - 3y''$	" " " " "	$y''_0 = -1$	$y''''_0 = 3$
$y^{v} = -xy'''' - 4y'''$	" " " " "	$y'''_0 = 0$	$y^{v}_0 = 0$

and similarly $y^{vi} = -15$, $y^{vii} = 0$, and $y^{viii} = 105$.

Substituting in (2) the various values of the derivatives at $x = x_0$, as well as the initial y_0, it will be seen that terms having odd powers of h vanish, leaving

(3) $y = 1 - h^2/2 + 3h^4/4! - 15h^6/6! + 105h^8/8! +$ Remainder

As this is an alternating converging series (see Note 1), the Remainder is less than the first term discarded. Thus, if we discard the term containing h^6, the error will have an upper bound, as follows:

When obtaining	y_{-1}	R	will be	less	than		$-15 \times .000001/720 = -.00000002$
"	"	y_1	"	"	"	"	$-15 \times .000001/720 = -.00000002$
"	"	y_2	"	"	"	"	$-15 \times .000064/720 = -.00000133$

Inasmuch as this is less than the permitted error of 0.000005 and it is obvious that fewer terms will not be satisfactory, the desired values of y_{-1} to y_2 are found using terms of (3) up to and including the one containing h^4, as follows:

	h	1st term	2nd term	3rd term	Total
y_{-1}	-0.1	1.00000	$-(-0.1)^2/2 = -0.05$	$3(-0.1)^4/24 = 0.0000125$	0.99501
y_0	0	1.00000	--	--	1.00000
y_1	0.1	1.00000	$-(0.1)^2/2 = -0.05$	$3(0.1)^4/24 = 0.0000125$	0.99501
y_2	0.2	1.00000	$-(0.2)^2/2 = -0.2$	$3(0.2)^4/24 = 0.0002$	0.98020

Inasmuch as this integration may be done analytically, the prediction of the value of the Remainder may be verified. The correct y_2 to 8 places is 0.98019867. The Remainder, R, in (3) is thus, -0.00000133, which is the same as the upper bound that was predicted. However, if more places are taken, the absolute value of R will be found to be minutely less.

NOTE 1: Though knowledge of the principles of Taylor's Series is assumed, it may be helpful to mention that when the series has its terms of the same sign, as distinct from the alternating series of the example, a strictly mathematical computation of the Remainder ordinarily is not easy.

Though an explicit expression for it is sometimes possible, the usual means of expressing the Remainder in (2) is in the form of the first rejected term of the series, with the exception that the derivative is taken not at $x = x_0$, but at some value of x, unknown, which lies between x_0 and $x_0 + h$. To compute the maximum possible value of the Remainder under this condition requires that the derivative of required order be known in terms of x and y (or the y for any x approximated) and also that such value of x (with its corresponding approximated y) be used as will cause the Remainder to be a maximum. Though this provides a true upper bound, it may be much higher than the actual Remainder, so more terms of (2) would be taken than actually would be required if the true Remainder were known.

(over)

Now, when obtaining starting values for subsequent continuance by the Milne Method, it is not important to know the exact value of the Remainder in (2); it is only necessary to know that it does not exceed a certain permissable error in the final y.

The most practicable way of determining this is to compute a few extra terms, plot their values with respect to the order of the term, and, by inspection, obtain some idea of the ratio that the value of each succeeding term bears to its predecessor. By adding the values and projecting them to a limit -- they will usually converge rapidly -- it may be readily determined whether their sum will cause the total error to be less than the permissable error.

NOTE 2: It will have been noted that the starting values for the Milne integration are obtained herein by the expedient of first integrating backwards by the interval $h = -s$ when values for the Milne 3-term formulas are being found, and by the additional backward integration by the interval $h = -2s$ in the case of the 5-term formulas. This differs from customary practice in Taylor's Series integration of always integrating ahead from $x = x_0$. It thus provides an unwanted value (or values) and fails to compute, instead, a value (or values) that can be used.

This objection does not hold if the computer establishes the initial values because in such a case these values would merely be taken greater by s (or 2s) than the starting value that was required to be known.

Regardless of this, the reason for recommending the within procedure; i.e., integrating backwards at the start, is that doing so reduces the coefficients of the last retained term so that desired accuracy usually may be had by using fewer terms of the series, with the further advantage that much differentiating is avoided. The time so saved more than offsets the slight extra time of computing an additional Milne-Method value (or two), for the latter is done very rapidly once the Milne-Method computation is started.

To show the advantage of starting the Taylor's Series integration by backward integration instead of forward, there is given below the error of y_3 if it were obtained by forward integration by Taylor's Series in the conventional manner.

When obtaining	y_1	R	will	be	less	than	(as previously computed)			-0.00000002
"	"	y_2	"	"	"	"	"	"	"	-0.00000133
"	"	y_3	"	"	"	"	"	-15 x .000729/720		-0.00001519

which is 3 times that permitted. As a consequence, in this case, the term containing h^6 in (3) would have to be retained if the conventional plan of integrating forward were followed.

The effect described becomes increasingly significant when obtaining starting values for Milne-Method integration by the 5-term formulas, or larger. In the case of the 5-term formulas, for example, the remainders when computing the necessary y_4 and y_5 are often so great as substantially to invalidate the process. Even in the case of a series which converges as rapidly as (3) and discarding the term containing h^8, it will be found that if the plan outlined herein is followed (using backward integration to obtain y_{-2} and y_{-1} and thence forward successively to y_3), 6-place accuracy of the last value is had, whereas if forward integration from y_0 to y_5, inclusive, is used according to the conventional method, only 3-place accuracy of the final value is obtained.

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Model 10EFA
or
AB10FA

SUMMATIONS IN STATISTICAL METHODS

REMARKS: A LARGE PART OF THE COMPUTATION IN STATISTICAL METHOD PROCEDURES CONSISTS OF SUMMING X, Y, XY, X^2, Y^2. SOME TYPES OF WORK REQUIRE ALL OF THESE, OTHERS ONLY A PART. THE METHODS FOLLOWING SHOULD PROVE HELPFUL IN ORGANIZING SUCH WORK. PARTICULAR ATTENTION HAS BEEN PAID TO PROCESSES OF CALCULATING THAT ARE SELF-CHECKING. BECAUSE THE MARCHANT CALCULATOR HAS DIALS FOR ALL THREE FACTORS OF A MULTIPLICATION, IT IS POSSIBLE TO USE SIMPLIFIED CALCULATING ROUTINES AND CHECKS.

All decimal settings shown are for whole numbers. If the amounts have decimal fractions, adjust decimals correspondingly.

441 A 1 *SUMMATIONS OF X, Y, X^2, Y^2, AND 2XY FOR TWO-DIGIT AMOUNTS*

EXAMPLE:

X	and	Y
95		85
96		91
97		95
75		69
70		65
433		405

OPERATIONS: Decimal setting for 10-column Marchant: Keyboard Dial 7 and 0; Upper Dial 7 and 0; Middle Dial 14, 7 and 0. Tab Keys Nos. 9 and 2 depressed.

(1) Carriage in Position 9. Enter X at left and Y at right of Keyboard Dial, around pre-set decimals. Keyboard Dial shows: 095.0000085.

(2) Enter X and Y in Multiplier Keyboard so that X appears at left and Y at right of Upper Dial, around pre-set decimals:

With Carriage in Position 9, enter X (95.) in Multiplier Keyboard. *Lightly* touch Tab Key "1" to shift Carriage into Position 2. (If your Marchant has "live" tab keys, simply touch Tab Key "2" to position Carriage.)

Enter Y (85.) in Multiplier Keyboard.

Upper Dial now shows: 095.00000085. and
Middle Dial shows: 9025.0016150007225.

Note: Always read Upper Dial entries (X and Y) direct from Keyboard Dial instead of from original work.

(Over)

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(3) Touch TAB Bar twice. Change Keyboard Dial to second values of X and Y (096.0000091.). Reading from Keyboard Dial, multiply by this same number.

(4) Proceed as described above for subsequent X and Y values. Upon completion, Upper Dial shows ΣX, ΣY: 433.0000405.; Middle Dial shows ΣX^2, $\Sigma 2XY$ ΣY^2: 38175.0071502003 3517.

$$\Sigma XY = 71502 \times .5 = 35751$$

CHECK: Add the X's and Y's as they appear on the original work. If the amounts agree with Upper Dial readings, it is substantially proved that the X's and Y's were properly entered in the Keyboard Dial and were properly used as multipliers. This would not necessarily be true if there were two equal and offsetting errors, but it is usually satisfactory to regard the likelihood of such errors as too remote for consideration. It will be observed that this check is not reliable unless the multipliers are read from the Keyboard Dial.

NOTE: AN ELEVENTH UPPER DIAL PROVIDES CAPACITY FOR APPROXIMATELY 200 PAIRS OF AMOUNTS.

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Model 10EFA
or AB10FA

SUMMATIONS OF X^2 AND (UX^2), OR X^3

A frequent summation required in Statistical Method and Least Squares is that of (UX^2) in which U equals 1/10, 1/100 or 1/1000 of X. It will be seen that this is similar to summing X^3 because U is the same as X except for the decimal point. (It is also often desired that summations of X^2 be made as a part of the same operation.)

(A) Summations of X^2 and (UX^2) for two-digit amounts:

Example:

U	X	U	X
.65	65	.74	74
.46	46	.89	89

SETUP: Decimals: Keyboard Dial 8 and 0; Upper Dial 1 ; Middle Dial 9 and 1. Tab Keys Nos. 3 and 5 depressed.

OPERATIONS: (1) With Carriage in Position 3, enter first X (65) in Keyboard Dial at "0" decimal, and multiply by X (65) around 1st Upper Dial decimal (reading X from right of Keyboard Dial, *not* from original work.)

First X^2 (4225) now appears in Middle Dial.

(2) Enter first U (.65) in Keyboard Dial at 8th decimal, with right hand digit reduced by "1", and fill in columns to right with 9's.

Keyboard Dial now reads: .64999999.

(Be sure, in entering U, to read it from right of Keyboard Dial, *not* from work sheet.)

(3) TAB Carriage to Position 5, and multiply by amount appearing in Middle Dial at 1st decimal (4225).

Now first (UX^2) appears in Middle Dial at left (2746.25) and Middle Dial at right shows all ciphers.

(4) Clear-TAB Keyboard Dial only. Repeat steps 1, 2 and 3 for each succeeding pair.

Upon completion,

$\Sigma(UX^2)$ appears in Middle Dial at 9th decimal (14821.54)

$\Sigma(X^2) + \Sigma X$ appears in Upper Dial (20012)

(5) To subtract ΣX from Upper Dial, touch Repeat-Neg.X Keys; multiply by ΣX (274, as separately totaled), leaving remainder in Upper Dial: ΣX^2 (19738).

(Cont.)

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NOTE: If directions have been followed, the work will be practically error-proof. However, there is no check of this except by obtaining $\Sigma(X^2)$ by accumulative multiplication. If it agrees with $\Sigma(X^2)$ as found in step 5, $\Sigma(UX^2)$ is substantially proved, provided the entries in step 2 were actually the same (less 1) as the X at right of the Keyboard Dial.

(B) Summations of X^2 and (UX^2) for Three-Digit Amounts.

This is done in a similar manner to that of example A, except that a 10 column Marchant must be used.

The capacity is about 40 pairs of three-digit scores averaging 500. If the scores tend to concentrate at any other value, the capacity of the process would be affected correspondingly.

(C) Summations of (UX^2) Without Obtaining Summation of X^2.

By clearing Upper Dial after steps 2 and 3, $\Sigma(UX^2)$ will be obtained with a somewhat better proof of accuracy, particularly if the entry at left of Keyboard Dial in step 2 is made by copying it from the Upper Dial and checking it with the amount at right of Keyboard Dial, thus providing a "closing proof."

The fact that after step 3, Middle Dial at right shows all ciphers is proof that multiplication was actually made by X^2.

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OUTLINE OF SUMMATION-WORK FOR LINEAR MULTIPLE CORRELATION

REMARKS: It is assumed that the computer is familiar with the customary method of obtaining the regression equations of multiple linear correlation in the form

$$X_1 = a + b_2X_2 + b_3X_3$$

or

$$X_1 = a + b_2X_2 + b_3X_3 + b_4X_4$$

for two and three variables, respectively. The calculating work principally comprises obtaining the required summations that become the coefficients of the terms of the normal equations.

These normal equations are then solved for the respective a and b values, preferably by the Crout Method (see Marchant Methods MM-434-B1 and MM-434-B2). It is recommended that the original scores be reduced to two-digit amounts in cases where they are larger. This may be done by either subtracting a fixed amount, or by dividing by 10, 100, etc., and rounding to a two-digit amount. The coefficients obtained by use of these modified scores are easily converted to ones suitable for use with unmodified scores.

It is also recommended that the original scores be used instead of their deviations from the point of averages. Though this adds slightly to the work of solving the normal equations, the time is believed to be more than offset by the elimination of obtaining deviations from the point of averages, as well as the time of converting the resulting regression equation into one that is usable with raw scores.

OUTLINE FOR CASE OF TWO INDEPENDENT VARIABLES:

The normal equations are

$$\text{I} \quad \Sigma(X_1) = aN + b_2\Sigma(X_2) + b_3\Sigma(X_3)$$

$$\text{II} \quad \Sigma(X_1X_2) = a\Sigma(X_2) + b_2\Sigma(X_2^2) + b_3\Sigma(X_2X_3)$$

$$\text{III} \quad \Sigma(X_1X_3) = a\Sigma(X_3) + b_2\Sigma(X_2X_3) + b_3\Sigma(X_3^2)$$

The summations are found as follows:

1st. According to Marchant Method MM-441-A1, find ΣX_2; ΣX_3; $\Sigma(X_2^2)$; $\Sigma(X_3^2)$; $\Sigma(2X_2X_3)$,

2nd. Enter in Keyboard Dial each X_2 at left and X_3 at right, and multiply by the corresponding X_1, producing by accumulation

$$\Sigma(X_1X_2) \quad \text{and} \quad \Sigma(X_1X_3)$$

The first step is self-checking if the procedure of Marchant Method MM-441-A1 is followed, so it need not be repeated for proof. The second step should be repeated. Sub-totals taken every 20 scores will aid in the proof.

(over)

OUTLINE FOR CASE OF THREE INDEPENDENT VARIABLES:

The normal equations are

I $\Sigma(X_1) = aN + b_2\Sigma(X_2) + b_3\Sigma(X_3) + b_4\Sigma(X_4)$

II $\Sigma(X_1X_2) = a\Sigma(X_2) + b_2\Sigma(X_2^2) + b_3\Sigma(X_2X_3) + b_4\Sigma(X_2X_4)$

III $\Sigma(X_1X_3) = a\Sigma(X_3) + b_2\Sigma(X_2X_3) + b_3\Sigma(X_3^2) + b_4\Sigma(X_3X_4)$

IV $\Sigma(X_1X_4) = a\Sigma(X_4) + b_2\Sigma(X_2X_4) + b_3\Sigma(X_3X_4) + b_4\Sigma(X_4^2)$

The summations are found as follows:

1st. According to Marchant Method MM-441-A1 ΣX_2; ΣX_3; $\Sigma(X_2^2)$; $\Sigma(X_3^2)$; $\Sigma(2X_2X_3)$,

2nd. Similarly find ΣX_1; ΣX_4; $\Sigma(X_1^2)$; $\Sigma(X_4^2)$; $\Sigma(2X_1X_4)$

3rd. Enter in Keyboard Dial each X_2 *and* X_3 and multiply by the corresponding X_1, producing

$$\Sigma(X_1X_2) \text{ and } \Sigma(X_1X_3)$$

4th. Enter in Keyboard Dial each X_2 and X_3 and multiply by the corresponding X_4, producing

$$\Sigma(X_2X_4) \text{ and } \Sigma(X_3X_4)$$

The first and second operations are self-checking if the procedure of Marchant Method MM-441-A1 is followed, so they need not be repeated for proof. The third and fourth operations, however, should be repeated.

1 026 64368

SUMMATION OF X, XY AND XY^2

REMARKS: In certain cases of statistical computing, it is often desired to find Σx*, Σxy and Σxy^2, when given a large number of pairs of factors "x" and "y". This method is suitable for cases in which the factors have no more than two digits each.

EXAMPLE: Given:

x	y
2	7
3	12
31	43
14	17
5	22

Find:

$\Sigma x = 55$

$\Sigma xy = 1731$

$\Sigma xy^2 = 64315$

OPERATIONS: Decimals: Keyboard Dial 5 and 0; Upper Dial 5 and 0; Middle Dial 10, 5 and 0.

(1) Enter first "y" (7) in Keyboard Dial at 5th decimal, and multiply by first "x" (2) so it appears in Upper Dial around "0" decimal.

(2) Decrease right-hand figure of "y" in Keyboard Dial by "1", and enter all 9's in columns at right so that Keyboard Dial reads 6.99999.

Position Carriage and multiply around 5th Upper Dial decimal by "xy" (14) which appears directly below in Middle Dial around 5th decimal.

The Middle Dial should now show all ciphers at right of 10th decimal.

xy^2 (98) appears at 10th Middle Dial decimal, and
xy (14) appears at 5th Upper Dial decimal, but the amounts need not be separately noted.

(3) Clear keyboard Dial only, and proceed as in Steps 1 and 2 for the remaining pairs of values.

Σxy^2 appears at left of Middle Dial.
Σxy appears at left of Upper Dial.
Σx appears at right of Upper Dial.

NOTE: If Σx at right of Upper Dial equals the sum of the x values when they are separately added, it is substantially a proof that all x values have been correctly entered as multipliers. Any error of entry of xy as multiplier in Step 2, or the improper filling-in of 9's, is signalized by Middle Dial failing to clear after Step 2.

It will thus be seen that this process provides first-run accuracy control, except for values of "y". However, the likelihood of an error in setting "y" is remote because it appears in the Keyboard Dial and it is also separately noted in that dial when its right-hand figure is reduced by 1.

(*) The symbol indicates "the summation of."

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SUMMATION OF FACTORS OF THE TYPE OF $\frac{A\ B}{K}$ *WHEN A, B, AND K ARE VARIABLE*

EXAMPLE: $\frac{546.32 \times 39.75}{176.32} + \frac{392.56 \times 78.62}{194.55} + \frac{425.87 \times 55.83}{187.72} = 408.46$

SETUP: Decimals: Keyboard Dial 2; Upper Dial 3; Middle Dial 5. Tab Key No. 5 depressed.

OPERATIONS:

(1) Enter in Keyboard Dial the first A (546.32), and multiply by the first B (39.75), around pre-set decimals.

(2) Clear-TAB Keyboard Dial only. Enter the first K (176.32) in Keyboard Dial as divisor. Touch Stop and Division Keys so that Upper Dial does not clear.

The first quotient, plus the first B, appears in Upper Dial (1962.91), but it is not separately noted.

(3) Enter in Keyboard Dial the second A (392.56). Enter second B (78.62) in Multiplier Keyboard.

(4) Clear-TAB Keyboard Dial only. Enter second K (194.55) as divisor. Touch Stop and Division Keys so that Upper Dial does not clear.

The sum of the quotients, as well as of the first and second B's appear in Upper Dial (400.171), but it is not separately noted.

(5) Repeat steps 3 and 4 for the third expression.
The sum of the quotients, as well as of the three B's appear in Upper Dial (582.659), but it is not separately noted.

(6) Clear Keyboard Dial. From a separately made summation, the B's = 174.20. (Sum the B's with Carriage shifted to extreme right, if desired.)

With Carriage in Position 6, touch Repeat-Neg.X Keys; negatively multiply by the sum of the B's.

The desired sum of the quotients (408.459) appears in Upper Dial.

NOTE: The above shows calculation with positive amounts. Similar calculations when there are mixed positive and negative amounts are easily handled. In such cases, perform all multiplications as if the factors were positive. If inspection shows that the quotient is negative, move Manual Counter Control toward the operator before touching Division Key. The subtraction of B's in step 6 should then be the sum of all of the B's considered as positive amounts, even though some of them may be negative.

Model EFA

CORRELATION COEFFICIENTS

CALCULATION OF CORRELATION COEFFICIENTS IS A COMMON PROBLEM OF STATISTICIANS, SCIENTISTS, AND MATHEMATICIANS. CORRELATION IS USED TO DETERMINE THE EXTENT OF RELATIONSHIP THAT ONE SET OF DATA HAS TO ANOTHER. THE "CORRELATION COEFFICIENT" IS THE NUMERICAL INDICATION OF THE DEGREE OF RELATIONSHIP. THUS, IT CAN BE DETERMINED WHETHER IT IS JUST A CHANCE RELATIONSHIP, OR WHETHER THE RELATIONSHIP CAN BE TRACED TO SOME MUTUAL CAUSE, OR FACTOR.

The PEARSON CORRELATION COEFFICIENT r_{xy} is briefly discussed here.

Example: To illustrate, assume the problem of finding the Pearson Correlation Coefficient r_{xy} between the grades of a group of students in both Science and Mathematics . . . . .

The Direct Form of the equation will be used thus:

$$r_{xy} = \frac{(N \cdot \text{Sigma } XY) - (\text{Sigma } X \cdot \text{Sigma } Y)}{\sqrt{\left[N\ (\text{Sigma } X^2) - (\text{Sigma } X)^2\right]\left[N\ (\text{Sigma } Y^2) - (\text{Sigma } Y)^2\right]}}$$

In the above r_{xy} equals the Correlation Coefficient.

"Sigma" indicates "the summation of" all the X, X^2, Y, Y^2, or XY as the case may be.

N equals the number of pairs of observations (5).

The grades are:

STUDENT	X...Science	Y...Mathematics
A	95	85
B	96	91
C	97	95
D	75	69
E	70	65
	433	405

The calculation is simply:

r above the line: (5 x 35751) – (433 x 405)
xy below the line: The square root of:

(5 x 38165) - (433 x 433) x (5 x 33517) - (405 x 405)

Then divide the figure above the line by the figure below the line.

(over)

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Using MARCHANT METHOD , MM-441 A1, the following summations are found:

Sigma X	433	Sigma X^2	38175
Sigma Y	405	Sigma Y^2	33517
Sigma XY	35751	N	5

Substituting in the equation:

$$r_{xy} = \frac{(5 \times 35751) - (433 \times 405)}{\sqrt{\left[(5 \times 38175) - (433)^2\right]\left[(5 \times 33517) - (405)^2\right]}}$$

OPERATIONS:

Set decimals: Upper Dial 4, Middle Dial 4, Keyboard Dial 0. Tab Keys 5 & 7 depressed.

(1) With carriage in 5th position, enter Sigma Y^2 (33517) in Keyboard Dial at extreme right. Multiply by N (5) around 4th Upper Dials decimal. Clear-TAB Upper and Keyboard Dials. Tab to 7th position. Enter Sigma Y (405) in Keyboard Dial at 0 decimal, and with Manual Counter Control toward operator negatively multiply by Sigma Y (405). Amount in lower right bracket of equation (3560) appears in Middle Dial around 4th decimal. Copy to scratch pad. Clear-TAB all dials. Restore Manual Counter Control to normal position.

(2) Enter Sigma X^2 (38175) in Keyboard Dial at 0 decimal and multiply by N (5). Clear-TAB Upper and Keyboard Dials. Tab to 7th position. Enter Sigma X (433) in Keyboard Dial at 0 decimal and with Manual Counter Control toward operator , negatively multiply by Sigma X (433). Difference (3386) appears in Middle Dial . Clear-TAB Upper and Keyboard Dials. Restore Manual Counter Control to normal position. Enter Middle Dial amount (3386) in Keyboard Dial and clear Middle Dial . With carriage in 8th position, multiply by scratch pad figure (3560). Product (12054160) appears in Middle Dial .

(3) Using Marchant Table #82, find the Square Root of Middle Dial amount.

$\sqrt{12054160} = 3472$ Copy 3472 to scratch pad. Clear dials.

(4) Enter Sigma XY (35751) in Keyboard Dial at 0 decimal and multiply by N (5). Clear-TAB Upper and Keyboard Dials. Tab to 7th position. Enter Sigma X (433) in Keyboard Dial at 0 decimal. With Manual Counter Control toward operator, negatively multiply by Sigma Y (405). Clear-TAB Upper and Keyboard Dials. Restore Manual Counter Control to normal position.

(5) Enter divisor of the equation as found above (3472) in Keyboard Dial at 0 decimal. Touch Division Key. r_{xy} (.976) appears in Upper Dial around 4th decimal.

THE FACT THAT THE PEARSON PRODUCT-MOMENT CORRELATION COEFFICIENT (.976) IS CLOSE TO "1" INDICATES THAT THERE IS A VERY CLOSE PROPORTIONALITY OR RELATIONSHIP BETWEEN GRADES IN SCIENCE AND GRADES IN MATHEMATICS FOR THE GROUP UNDER CONSIDERATION.

1 026 6436 8

CHARLIER METHOD FOR CHECKING SUMMATIONS OF X, X^2, Y, Y^2, $2XY$

REMARKS: This method is described in "Statistical Methods" by F. C. Mills. Henry Holt & Co., 1924, p. 157.

EXAMPLE: Observations and Summations as found in Marchant-Method 444 A.

The equation for the Charlier Check is:

Sigma (X + 1) (Y + 1) equals Sigma XY + Sigma X + Sigma Y + N

N equals the number of pairs of observations.

Hence, if we accumulate the products of (each X value + 1) by (each Y value + 1), we should be able to check that against the sum of (Sigma XY + Sigma X + Sigma Y + N). Referring to the observations and summations as found in Marchant-Method 444 A, we have:

(95 + 1) (85 + 1) + (96 + 1) (91 + 1) + etc.

equals 35751 + 433 + 45 + 5

OPERATIONS: (1) Tab Key No. 2 depressed. Carriage in Position 2.

Enter first value of X + 1 (96) in Keyboard Dial at extreme right. Multiply by first value of Y + 1 (86). Clear-TAB Keyboard and Upper Dials only.

(2) Enter second value of X + 1 (97) in Keyboard Dial at extreme right. Multiply by second value of Y + 1 (92). Clear-TAB Keyboard and Upper Dials only.

(3) Repeat above steps for subsequent values of X + 1 and Y + 1. Sigma (X + 1) (Y + 1), (36594), appears in Middle Dial. Clear Keyboard Dial only.

(4) Enter Sigma XY (35751) in Keyboard Dial at extreme right. Touch Subtract Bar. Continue subtraction of Sigma X (433), Sigma Y (405) and N (5).

Middle Dial shows ciphers.

Therefore, the summations are correct. Only Sigma X^2 and Sigma Y^2 are left unchecked, and the accuracy of Sigma X, Sigma Y, and Sigma XY practically insures the accuracy of the squared terms.

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STANDARD DEVIATIONS

ACCORDING TO MARCHANT METHOD MM-444A

Usually the calculation of the Standard Deviation is a means to an end; viz., the calculation of r_{xy} by the Method of Mean and Deviation. However, as r_{xy} may be more easily calculated by the formula at the bottom of the first page of Marchant Method MM-444A, it is not customary to calculate the Standard Deviations when only the Correlation Coefficient is required.

Sometimes, however, the Standard Deviations are required for their individual significance, and if so they are easily obtained from the summations of MM-444A, thus

The Standard Deviation of the X's,

(1) $$d_x \text{ equals } \sqrt{\frac{\text{Sigma } X^2}{N} - M_x{}^2}$$

and that of the Y's,

(2) $$d_y \text{ equals } \sqrt{\frac{\text{Sigma } Y^2}{N} - M_y{}^2}$$

in which

(3) M_x is the mean of all the X's; equals $\frac{\text{Sigma } X}{N}$

(4) and, M_y is the mean of all the Y's; equals $\frac{\text{Sigma } Y}{N}$

When given the above quantities, r_{xy} may be obtained by the Mean and Deviation formula, thus

(5) $$r_{xy} \text{ equals } \frac{\frac{\text{Sigma (X times Y)}}{N} - M_x M_y}{d_x d_y}$$

This formula is not as well adapted to machine calculation as is that of MM-444A as it requires the preliminary calculation of the Standard Deviations and the Means.

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STATISTICAL METHOD

MEAN AND STANDARD DEVIATION - DATA GROUPED BY EQUAL CLASS INTERVALS

REMARKS: "Standard Deviation" is an index figure widely used in statistical analysis. It is the square root of the quotient obtained by dividing the sum of the squares of the deviation of each of a number of amounts from its arithmetic mean by the number of such amounts; thus

$$\sigma = \sqrt{\frac{\Sigma(x^2)}{n}}$$

in which σ - Standard Deviation (the Greek small letter Sigma)
x = individual deviations from arithmetic mean
N = total number of items

The Standard Deviation is computed with respect to either the X or Y axis.

This method relates to obtaining the Standard Deviation with respect to either the X or Y axis when the data are grouped by "equal" classes; that is to say, the data are sorted into groups, the X (or Y) spread of all of the classes being the same, and each class being successively of an equal higher amount than its predecessor. This is by far the most usual form in which problems involving Standard Deviation appear.

EXAMPLE: 221 average grades of students vary between marks of 59.0 and 93.8. They are grouped for convenience of calculating into class intervals of 2 points each, ranging from 58.0 to 100.0. The number of students whose average marks fall in each class interval is shown by tallies on the attached sheet. The total in each group, or "frequency", is shown in column f. The number of the class interval, reading from the bottom, is shown in column d. This column also designates the interval, or "deviation", of each group from the Arbitrary Origin (A.O.) of 59, which is the mid-point of the lowest group. The columns at the right are, respectively,

d^2 equals the square of amount in column d

d_1 equals d increased by "1"

d_1^2 equals the square of amount in column d_1

For any given pattern of calculating, sheets are prepared in advance and mimeographed or printed with all amounts in columns d, d^2, d_1 and d_1^2 completed. The only part filled in by the operator is the tally marks and their totalizing in the column headed "f".

OUTLINE: In this method the actual deviations from the arithmetic mean are not obtained. The final result, however, is the Standard Deviation exactly as defined.

The Charlier Check of the summations is a part of the process.

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(Cont.)

PART I: SUMMATIONS OF f, fd, f(d²), fd, f(d₁²)

OPERATIONS: **Set decimals:** Keyboard Dial 8 and 0; Upper Dial 6; Middle Dial 14 and 6. Tab Key No. 8 depressed.

In all of these summations, the multipliers (f) are entered as if they were two-digit amounts; e.g., "3" is entered as "03", etc.

(1) With carriage in Position 8, and without Keyboard Dial entries, multiply by the frequency (01) corresponding to the group at Arbitrary Origin (59).

NOTE: This is done to record that there was 1 tally in class 60-58. As the midpoint is the Arbitrary Origin there is no deviation, so the operation is one of multiplying the "0's" of column d and d^2 by the f of the group. Touch TAB Bar.

(2) Enter in Keyboard Dial at 8th and "0" decimals respectively the amounts that are in columns d and d^2 for group 62-60 (1.0 and 1.0) and multiply by corresponding frequency (03). Touch TAB Bar.

(3) Change Keyboard Dial to read the next greater amounts of d and d^2 (2.0 and 4.0) and multipl y by corresponding frequency (15). Touch TAB Bar.

(4) Change Keyboard Dial to read the next greater amounts of d and d^2 (3.0 and 9.0) and multiply by corresponding frequency (19). Touch TAB Bar.

(5) Proceed as in Step 4 for all other groups.

Sigma f (221) appears in Upper Dial.

Sigma fd (1429.) and Sigma $f(d^2)$ (11265.) appear at left and right of Middle Dial.

(6) Clear all dials and with carriage in Position 8 enter d_1 and d_1^2 (1.0 and 1.0) at 8th and "0" Keyboard Dial decimals respectively and multiply by corresponding frequency (01).

(7) Change Keyboard Dial to read the next greater amounts of d_1 and d_1^2 (2.0 and 4.0) and multiply by corresponding frequency (03).

(8) Proceed as in Step 7 for all other groups.

Sigma f (221) appears in Upper Dial.

Sigma fd_1 (1650) and Sigma $f(d_1^2)$ (14344) appear at left and right of Middle Dial.

(Cont.)

PART II: THE CHARLIER CHECK

In order to prove the correctness of the summations they are checked as follows:

(a) Σfd (1429) + Σf(221) = Σfd_1 (1650)

(b) $\Sigma f(d^2)$ (11265) + 2 Σfd (2858) + Σf (221) = $\Sigma f(d_1{}^2)$ (14344)

Any failure to reconcile will require a re-run of the summations.

(1) With carriage in Position 1, enter in Keyboard Dial at "0" decimal Sigma f (221) and add.

(2) Similarly enter Sigma fd (1429) and multiply by "1".
Sigma fd_1 (1650) appears in Middle Dial.

(3) Clear Upper and Middle Dials and multiply by 2.

(4) Change Keyboard to read Sigma $f(d^2)$ (11265) and add.

(5) Enter Sigma f (221) and add.
Sigma $f(d_1{}^2)$ (14344) appears in Middle Dial.

PART III: STANDARD DEVIATION IN CLASS INTERVAL UNITS

Much work of this type merely requires that the Standard Deviation be obtained in terms of Class Intervals; that is to say, each Class Interval (in this case 2) is regarded as the unit of measurement. Obviously, the Standard Deviation in terms of the units from which the Class Intervals are obtained equals the "Standard Deviation in terms of Class Intervals" multiplied by the "Number of Units in a Class Interval".

The formula is:

Standard Deviation in terms of Class Intervals $= \dfrac{\sqrt{(\text{Sigma f})\ \ (\text{Sigma f}(d)^2) - (\text{Sigma fd})^2}}{\text{Sigma f}}$

or, in this example, $= \dfrac{\sqrt{221 \times 11265 - 1429^2}}{221}$

Set decimals: Keyboard Dial 8, 5 and 2; Upper Dial 5 and 1; Middle Dial 13, 10 and 7.

(1) Enter in Keyboard Dial at 2nd decimal Sigma $f(d^2)$ (11265) and multiply at 6th Upper Dial decimal by (Sigma f) (221).

(2) Clear Upper and Keyboard Dials. Move Manual Counter Control toward the operator. Enter in Keyboard Dial Sigma fd (1429) at 2nd decimal and negatively multiply by Sigma fd (1429) at 6th Upper Dial decimal. Move Manual Counter Control away from the operator.
The amount under radical sign (447524) appears in Middle Dial at 7th decimal.

(3) Refer to Marchant Table of Square Root Divisors, No. 82: $\sqrt{447524}$ = 668.97+

(4) Enter 668.97 in Middle Dial at 10th decimal.

(5) Shift carriage to Position 6. Change Keyboard Dial to read Sigma f (221) at 5th decimal and divide.

(6) Standard Deviation in Class Interval Units is 3.0270 (see Note C).

(Cont.)

PART IV: STANDARD DEVIATION IN UNITS FROM WHICH CLASS INTERVALS ARE DERIVED

This is obviously the result of Part III, Step 6, multiplied by the number of units per Class Interval, or, in this case,

2 x 3.0270 or 6.0540 (See Note C)

PART V: ARITHMETIC MEAN

This is $\frac{\text{Sigma fd}}{\text{Sigma f}}$ x Number of Units per Class Interval + A.O., in which A.O. is "Arbitrary Origin", or, in this case,

$\left(\frac{1429}{221} \times 2\right) + 59$, or 71.9321 (see Note E).

NOTES

A. The work has been subdivided into parts in order to clarify the individual parts of the work.

B. If it is desired to obtain directly the Standard Deviation in Units from which the Class Intervals are derived (Part IV), the divisor in Step 6 of Part III can be Sigma f divided by the number of units that comprise a class interval.

C. The method is based upon the assumption that all scores are concentrated at the mid point of each class interval. It is probable, however, that they are continuously distributed throughout the interval. The inherent error from this assumption causes the Standard Deviation to be slightly less than the values shown in Parts III and IV. For this reason it is recommended that the Standard Deviations be rounded downward to not more than 3 significant figures:, thus,

For Part III to 3.02
For Part IV to 6.05

Most textbooks dealing with this subject show how to correct the Standard Deviations obtained by this method for continuous distribution in an interval and tapering frequencies.

D. "Sigma f" is also referred to as "N" in many formulas. It is the total number of individual scores.

E. After the division, the addition of 59 may be made in the Upper Dial by multiplying by 59. This causes Upper Dial to provide a direct reading of 71.9321

F. The distribution shown herewith would be described as
Mean 71.932 with Standard Deviation of 6.05

WORK SHEET FOR MARCHANT METHOD
MEAN AND STANDARD DEVIATION

Data Grouped by Equal Class Intervals

Computer _____
Checker _____
Class Interval 2
Arbitrary Origin 59

Name of Report _____
Description _____
Mean 71.932
Standard Deviation 6.05

GROUP RANGE	SCORE TALLIES	f	d	d^2	d_1	d_1^2
100 - 98			20	400	21	441
98 - 96			19	361	20	400
96 - 94			18	324	19	361
94 - 92	/	1	17	289	18	324
92 - 90		0	16	256	17	289
90 - 88	////	4	15	225	16	256
88 - 86	///	3	14	196	15	225
86 - 84	/	1	13	169	14	196
84 - 82	////	4	12	144	13	169
82 - 80	~~////~~ /	6	11	121	12	144
80 - 78	~~////~~ ~~////~~ /	11	10	100	11	121
78 - 76	~~////~~ ~~////~~ ~~////~~ ~~////~~ /	21	9	81	10	100
76 - 74	~~////~~ ~~////~~ ~~////~~ ////	10	8	64	9	81
74 - 72	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ///	28	7	49	8	64
72 - 70	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~	40	6	36	7	49
70 - 68	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ /	26	5	25	6	36
68 - 66	~~////~~ ~~////~~ ~~////~~ ////	19	4	16	5	25
66 - 64	~~////~~ ~~////~~ ~~////~~ ////	19	3	9	4	16
64 - 62	~~////~~ ~~////~~ ~~////~~	15	2	4	3	9
62 - 60	///	3	1	1	2	4
60 - 58	/	1	0	0	1	1

Σf or N 221

NOTE: All items that are printed on the above form appear in regular type. Only the tallies and the italicized figures are entered by the computer.

STATISTICAL METHOD
LINEAR "LEAST SQUARES" LINE OF REGRESSION AND COEFFICIENT OF REGRESSION

REMARKS *It is often desired to know the equation of the straight line that best represents a scatter diagram according to the principle of "least squares;" e.g., the line is such that the sum of the squares of the deviations of observed values of Y from the corresponding values of Y on the line will be a minimum. If this line be represented by the equation $Y = a + bX$; the coefficient "b" represents the slope of the line or defines its angle with reference to the X axis ($\tan^{-1} b$) and "a" represents the Y intercept.*

The coefficient "b" of this least squares line of regression is sometimes referred to as the "Coefficient of Regression."

EXAMPLE Given the observed values of X and Y of Marchant Method MMLLLLA, find the linear Coefficient of Regression and also the equation of the line assuming the regression to be linear. To avoid reference to MMLLLLA, it is noted here that

Sigma X equals 433	Sigma X^2 equals 38175
Sigma Y equals 405	Sigma Y^2 equals 33517
Sigma XY equals 35751	N equals 5

OUTLINE The equation for the line of regression is determined by the solution for "a" and "b" in the two equations below:

(I) Sigma Y = Na + b Sigma X
(II) Sigma XY = a Sigma X + b Sigma X^2

Solving these equations for "b," and transforming so the equation is best suited for machine calculation, we have

$$\text{(III)} \quad b = \frac{N \text{ Sigma } XY - \text{Sigma } X \text{ Sigma } Y}{N \text{ Sigma } X^2 - (\text{Sigma } X)^2}$$

$$\text{(IV)} \quad a = \frac{\text{Sigma } Y - b \text{ Sigma } X}{N}$$

OPERATIONS Decimals: Upper Dial 5. Middle Dial 10, Keyboard Dial 5.
Use Marchant 10 column Model

To Compute "b"

(1) Set up in Keyboard Dial Sigma (X^2) (38175) and multiply by (5).

(2) Mov Manual Counter Control toward operator, clear Upper and Keyboard Dials, set up in Keyboard Dial Sigma X (433) and reverse multiply by Sigma X (433).

Middle Dial shows N Sigma X^2 - (Sigma X)^2 (3386) Copy to report.

(over)

(3) Move Manual Counter Control away from operator. Clear dials and set up in Keyboard Dial Sigma XY (35751) and multiply by N (5).

(4) Move Manual Counter Control toward operator, clear Upper and Keyboard Dials, set up in Keyboard Dial Sigma X (433) and reverse multiply by Sigma Y (405).

Middle Dial shows N Sigma XY – Sigma X Sigma Y (3390).

(5) Move Manual Counter Control away from operator, clear Keyboard and Upper Dials, set up in Keyboard Dial the amount copied to report from Step 2 (3386) and divide.

Coefficient of Linear Regression "b" (1.0012) appears in Upper Dial.

To Compute "a"

(a) Set up in Keyboard Dial Sigma Y (405) and multiply by "1."

(b) Clear Keyboard and Upper Dials, set up in Keyboard Dial Sigma X (433), move Manual Counter Control toward the operator and reverse multiply by "b" (1.0012) If the amount in Middle Dial were positive, procedure would be as in Step "1c" below.

(1c) Move Manual Counter Control away from operator, clear Upper and Keyboard dials, set up N (5) in Keyboard Dial and divide.

Coefficient "a," the Y intercept, appears in Upper Dial.

In this case the amount in Middle Dial is a negative number, i.e., (.. 99971.4804 . This amount may b evaluated as a positive number

and then divided by N, or advantage may be taken of the ability of the Marchant to divide a negative number dir ctly; thus, as in this case, proceeding from Step "b":
Move Manual Counter Control away from the operator.

(2c) The Middle Dial after Step "b" reads (...99971.4804). Clear Upper and Keyboard Dials, set up N (5) in Keyboard Dial. shift to 7th position (so the left digit of the divisor is direc'ly below the first digit to the right of the 9's), multiply by anything that clears the 9's (in this case multiplying by "1" will do so).

d Move Manual Counter Control toward the operator and divide.* Coefficient "a" (5.704) appears in Upper Dial, but it is to be written as –5.704, because the Upper Dial was producing negative quotients while "a" was developed.

The equation for the Linear "Least Squares" Line of Regression is thus:

$$Y = -5.704 + 1.0012X$$

(*) *Depress Division Key in the manner that prevents Upper Dial clearance.*

1 026 6436 8

HANSEN-AHLBERG METHOD FOR OBTAINING PARABOLIC TRENDS

Remarks: STATISTICIANS WHO EXTEND SECOND-DEGREE CURVES MAY READILY DO SO BY THE PROCEDURE HEREIN. THIS APPLICATION TO CASES OF STATISTICAL TRENDS WAS BROUGHT TO OUR ATTENTION BY MR. RAYMOND AHLBERG, STATISTICIAN, DENVER, COLO. SIMILAR PROCEDURES HAVE BEEN USED FOR INTERPOLATION BY INTEGRATION OF CONSTANT DIFFERENCES (SEE MARCHANT METHOD MM-439 G.)

Outline: The second degree curve is characterized by having a constant second difference. Advantage is taken of this as the basis for the method. The curves may have any of several forms. Examples are:

(1) $Y = a + bX + cX^2$ (2) $\text{Log } Y = a + bX + cX^2$

(3) $Y = a + b/X + c/X^2$ (4) $Y = 1/(a + bX + cX^2)$

The example herein is in the form of (1).

If (2) applies, it is only necessary to obtain anti-logs of the log Y's that appear in the Marchant.

If (3) applies, it is put in the form $X^2Y = aX^2 + bX + c$. The Marchant then gives the values of X^2Y, which when divided by the X^2's gives Y.

If (4) applies, it is put in the form $1/Y = a + bX + cX^2$. The Marchant then gives values of 1/Y, the reciprocals of which are the desired Y's.

Example: Given $Y = 7.2131 - .5114X + .3044X^2$, obtained from a least squares analysis. It is desired to tabulate the trend by intervals of 0.1 from X = 2.0

Preliminary: Compute four adjacent values of Y in the neighborhood of X = 2.0 and tabulate with differences, a check of correctness being the constancy of the second difference; thus,

X	Y	1st Diff.	2nd diff.
1.8	7.278836		
1.9	7.340324	.061488	
2.0	7.407900	.067576	.006088
2.1	7.481564	.073664	.006088

Decimals: Upper Dial 1, Middle Dial 10 & 6, Keyboard Dial 10 & 6. Non-Shift Key depressed.

(1) By suitable means, obtain initial entries as follows: Upper Dial 2.0; Middle Dial 7.408 at 10th decimal and .073664 at 6th decimal; Keyboard Dial .074 (rounded .073664) at 10th decimal and constant 2nd difference .006088 at 6th decimal. These are starting values and are always set up in this pattern when signs of both differences are plus (see Note B).

(2) With carriage in 1st position, touch No. 1 Multiplier Key. Y for 2.1 (7.482) appears at left of Middle Dial and the new 1st difference (.079752) appears at right. The Keyboard Dial at 10th Decimal is then changed so it reads .080 (the rounded value of .079752). See Note A.

(3) Touch Multiplier Key "1". Y for 2.2 (7.562) appears at left of Middle Dial and the new 1st difference (.085840) appears at right. The Keyboard Dial at 10th Decimal is then changed so it reads .086 (the rounded value of .085840).

(4) Repeat Step 3 for succeeding values, the Upper Dial showing values of X.

(SEE OTHER SIDE FOR NOTES.)

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Note A: Constant 2nd diff. should be set up as nearly exact as possible. Rounding that space limitations require should be in 1st diffs. and Y's.

Note B: If Y's increase but 2nd diff. is negative, set it in complementary form, bridge with 9's and proceed as herein.

If Y's decrease but 2nd diff. is positive, have Manual Counter Control toward operator, and depress "Neg. X" key prior to touching No. 1 Multiplier Key.

If Y's decrease and 2nd diff. is negative, invert the table; i.e., start from the smallest Y. Then, proceed exactly as outlined in the above method except have Manual Counter Control toward the operator.

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PRECISION OF MEASUREMENTS -- I

FROM A SERIES OF SEPARATE MEASUREMENTS OF EQUAL IMPORTANCE AND WEIGHT, TO FIND THE MOST PROBABLE VALUE, ITS PROBABLE ERROR, AND THE PROBABLE ERROR OF AN INDIVIDUAL MEASUREMENT

REMARKS: *In precise surveying, scientific laboratories, observatories ... wherever accurate measurements are made ... it is customary to indicate the value obtained from such measurements in terms of an average value plus or minus its Probable Error.* It is also often required that the P.E. of a single measurement be known. These P.E.'s are usually computed by formulas which require that the deviation of each measurement from their average be listed. The Method herein is better adapted to calculator computations as it does not require finding and recording the individual deviations.*

DEFINITION: The average of a series of measurements written as the one herein (3.5164 ± 0.0013 P.E.) does not indicate that the true value lies between 3.5151 and 3.5177, nor does it indicate that the value 3.5164 is probably in error by 0.0013. It indicates that so far as normal uncontrollable errors are concerned the true value is just as likely to lie between 3.5151 and 3.5177 as it is to lie outside of these values; that is to say, its probability of being within the limits is 50%. Knowing the P.E., however, and assuming that the distribution of uncontrollable errors is normal, the probability that the true value lies within any assumed limits is known. For example, if it is desired to know the probability that the true value lies between limits twice as great (in this case, 3.5164 ± 0.0026), reference to the Normal Probability table shows that it is 82%. It is thus seen that the P.E. is a guide to the precision of the measurement always assuming, of course, that the uncontrollable errors follow a normal distribution.

EXAMPLE: The following ten (n = 10) measurements were made. Find the most probable value, its P.E., and the P.E. of any single measurement. It is assumed that each measurement is made under equal conditions of accuracy control and that each is not an average of a group of measurements:

3.524, 3.512, 3.518, 3.505, 3.523, 3.516, 3.518, 3.509, 3.521, 3.518

OPERATIONS: Decimals; Upper Dial 8 & 5, Middle Dial 11 & 8, Keyboard Dial 6 & 3. No. 4 Tab Key down. The method is adaptable to any 8 or 10 column Marchant.

SUMMATIONS AND MOST PROBABLE VALUE (THE ARITHMETIC MEAN)

(1) Set up "1" at 6th Keyboard Dial decimal and, with carriage in 4th position, set up first measurement above base of 3.500 (M_1 = .024) at 3rd Keyboard Dial decimal and multiply at 5th Upper Dial decimal by same amount (.024), depressing Keys of Single-Row Keyboard (2-4). Always read amount from Keyboard Dial instead of from work; then compare Upper Dial reading with work.**

(2) Change Keyboard Dial at right to read next measurement above base (M_2 = .012), clear-return Upper Dial only, and similarly multiply by same amount (.012) continuing in this manner for all M's.

Summation of M (ΣM) (.164) appears at 11th Middle Dial decimal.**
" " M^2 (ΣM^2) (.003024) appears at 8th Middle Dial decimal.
Most Probable Value = Base + (ΣM)/n = 3.5164.

(*) In statistical work, it is customary to use the term Standard Deviation, which is 1.4826 times the Probable Error.

(**) If upon completion, the summation of the M's (0.164) that appears at left of Middle Dial is checked by independent addition of the M's, a double-check of the computation is had. It will be observed that if the method is exactly followed, this checks the entries of both factors of each squaring.

PROBABLE ERROR OF ARITHMETIC MEAN

This is obtained by applying the following formula:

$$\text{P.E. of Arithmetic Mean} = \frac{0.6745}{n}\sqrt{\frac{n\Sigma(M^2) - (\Sigma M)^2}{n-1}}$$

(3) Clear all dials, set up $\Sigma(M^2)$ (.003024) at 6th Keyboard Dial decimal and multiply at 5th Upper Dial decimal by n(10).

$n\Sigma(M^2)$ = .03024 appears in Middle Dial, but it need not be noted.

(4) Clear Upper and Keyboard Dials, set up M(.164) at 6th Keyboard Dial decimal, and with Manual Counter Control toward operator, reverse multiply at 5th Upper Dial decimal by same amount (.164) and move Manual Counter Control away from operator.

$n\Sigma(M^2) - (\Sigma M)^2$ = .003344 appears in Middle Dial, but it need not be noted.

(5) Clear Upper and Keyboard Dials, set up n-1 (9.) at 3rd Keyboard Dial decimal and divide.

$\frac{n\Sigma(M^2) - (\Sigma M)^2}{n-1}$ = .00037156 appears at 8th Upper Dial decimal.
(Right-hand figure rounded to 6 by reference to remainder, "5/9".)

(6) Extract Square Root of .00037156 according to method of Marchant Table 82; thus, perform the following.

(a) With carriage in 8th position, set up at extreme left of Keyboard Dial the significant figures of number whose root is desired (37156) and add.
(b) From Table 82, set up at extreme left the Column B number (between 3.60 and 3.72: 366) and add.
(c) Set up at extreme left, divisor (Column C: 38263) and divide.

Figures of root (19276) appear in Upper Dial, written as .0193.

(7) Clear all dials, set up Constant (.6745) at 6th Keyboard Dial decimal and multiply at 5th Upper Dial decimal by root from Step 6 (.0193).

(8) Clear Upper and Keyboard Dials, set up n(10) at 6th Keyboard Dial decimal and divide.

P.E. of Arithmetic Mean (.0013) appears at 5th Upper Dial decimal.

Measurement is 3.5164 ± .0013 P.E.

PROBABLE ERROR OF ANY SINGLE MEASUREMENT

(9) Clear all dials, set up $\sqrt{n}$ (3.162) at 3rd Keyboard Dial decimal and multiply at 5th Upper Dial decimal by P.E. of Arithmetic Mean (.0013).

P.E. of Single Measurement (.0041) appears at 8th Middle Dial decimal.

NOTE: If only the Probable Error of a Single Measurement is required, it may be obtained directly as follows:

$$\textit{P.E. of Single Measurement} = 0.6745\sqrt{\frac{n\Sigma(M^2) - (\Sigma M)^2}{n(n-1)}}$$

the Marchant solution of which is similar to that for P.E. of Arithmetic Mean.

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