MARCHANT METHODS

CONVERSION OF DECIMAL EQUIVALENTS TO COMMON FRACTIONS

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Remarks: Marchant Method MM-131 covers the solution of problems of this type when the common fraction has a large numerator and denominator as in machine designing. However, it is often desired to obtain the common fraction which corresponds to a decimal equivalent when it is known that the numerator and denominator are comparatively small. Such cases occur in connection with distribution of oil-well income, royalties, and the like.

Operations: Decimals: Upper Dial 0, Middle Dial 5, Keyboard Dial 5. Use any Marchant model.

Outline: Set up in Keyboard Dial the known decimal equivalent and multiply successively by "1". After each multiplication, note the reading of the Middle Dial, stopping the process whenever there is shown at right of decimal an amount which is very close to 10000, 20000, 25000, 33333, 50000, 99999, or 00000. In cases where the process is stopped with Middle Dial at right of decimal reading close to 10000, 20000, 25000, or 33333, increase Upper Dial respectively to 10, 5, 4, or 3 times its reading. The numerator of the desired fraction then appears in Middle Dial; the denominator appears in Upper Dial.

Example A: What common fraction has the decimal equivalent .53846?

(1) Set up in Keyboard Dial the given decimal equivalent (.53846) and multiply successively by "1". When Middle Dial reads "6.99998," the Upper Dial reads "13".

The desired common fraction is 7/13.

Example B: What common fraction has the decimal equivalent .5625?

(1) Set up in Keyboard Dial the given decimal equivalent (.5625) and multiply successively by "1". When Middle Dial reads "2.25000" the Upper Dial reads "4".

(2) By reference to the Outline (see above), it is noted that the Upper Dial should be increased to four times its present reading. This is done by multiplying by 12, (or on D model by building up Upper Dial until it reads 16), at which point the Middle Dial reads "9.00000."

The desired common fraction is 9/16.

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DIRECT INTERPOLATION AND SUB-TABULATION
(IF FOURTH DIFFERENCES DO NOT EXCEED 1000)

EXPLANATORY APPENDIX TO MARCHANT METHOD MM-189

A NOTE ON OBTAINING 4TH DIFFERENCES FOR USE WITH "COMRIE THROW-BACK" IN EXAMPLE IV

Reference was made in the second paragraph of the "Remarks" section, Page 1, of Marchant Method MM-189, to the fact that in sub-tabulation it is not necessary to obtain third and fourth differences, except at infrequent intervals, and then only in order to obtain their general range as a guide to the selection of method or as a means of obtaining the 4th difference correction of Example IV.

Inasmuch as a 4th difference must be known before the "4th difference correction" can be determined, it might appear that the statement is inconsistent, because obviously 4th differences will normally vary somewhat from interval to interval.

Actually, however, in ordinary computing practice, it will be found that the 4th difference correction generally can be obtained without the necessity of completely tabulating the 3rd and 4th differences. This is because the large majority of functions which are tabulated to the number of places used in ordinary computing, - rarely more than 7 places - will have no great variation in 4th differences; that is to say, a small 5th difference.

By following the procedure below, the tabulation of 4th differences for every interval may usually be avoided.

A plan that does this is to obtain 4th differences at about every fifth or tenth interval and observe their trend, plotting them graphically and obtaining the 4th differences for the intermediate intervals from the curve so drawn.

This will ordinarily give quite as accurate a 4th difference as would actual differencing at each interval, because the graphical method eliminates the error forced into the 4th difference due to rounding up of the right-hand digit of the tabulated function. Such round-ups can affect the right-hand digit of the 4th difference by as much as 8.

In considering the precision of this approximation, it is to be noted that in the computation of the interpolates only "0.184 x 4th difference" is used. This is additional justification for the procedure of eliminating 4th differencing for every interval when the work falls within the class of Example IV.

MARCHANT METHODS

TO RAISE A DECIMAL FRACTION TO AN ODD POWER

REMARKS: *The interesting method submitted herein was developed by Miss Emily T. Hannan. It is made available to other computers by courtesy of the company by which she is employed, which has requested that its name be withheld. Though the method is simple, it exemplifies a unique Marchant operation; viz., the entry of a multiplier, a portion of which is negative, and a portion positive. Marchant is the only calculator that permits automatic multiplication of this type.*

EXAMPLE: To find $.9754^{.285714}$

OPERATIONS: Decimals; Upper Dial 7, Middle Dial 13, Keyboard Dial 6. Use any model of Marchant. If M models are used, have Upper Green Shift Key depressed.

(1) Set up in Keyboard Dial the exponent (.285714) and, with carriage in 8th position, reverse multiply by the absolute value of the characteristic (-1) and then multiply by the decimal portion (mantissa) of the logarithm of .9754 (.9891828).

<u>Mantissa</u> of logarithm of $.9754^{.285714}$ (.9969094) appears in Middle Dial at right of decimal.

<u>Characteristic</u> of logarithm is indicated by the figure at left of Middle Dial decimal. In this case, it is "9", the characteristic is thus $\bar{1}$., and the entire logarithm is thus $\bar{1}.9969094$.

NOTE: If figure at left of Middle Dial decimal is 8, 7, etc., the characteristic is correspondingly $\bar{2}$, $\bar{3}$, etc.

(2) A table of logarithms shows that the amount which has logarithm of $\bar{1}.9969094$ is .99291.

OTHER SAMPLE CALCULATIONS

Function	Log of Whole No.	Log of Function	Function
$.09754^{.285714}$	$\bar{2}.9891828$	$\bar{1}.7111954$	.514275
$.00009754^{.285714}$	$\bar{5}.9891828$	$\bar{2}.8540534$	.071458

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MARCHANT — METHODS

MILNE METHOD OF STEP-BY-STEP INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS WHEN STARTING VALUES ARE KNOWN

REMARKS: *The Milne Method is highly regarded because it uses tabular values instead of differences and because its associated Steffensen integration formulas have small repeated coefficients which readily lend themselves to the preparation of tables of factors for use in connection with any particular problem. The method also provides means for estimating the error, provided the order of differences that tend to disappear is known. The example computed herein is the same as the differential equation that was chosen for comparing processes by the Committee on Numerical Integration, National Research Council (Bulletin No. 92). The example has substantially large higher orders of difference so that use of one of the intermediate forms of the Milne Method is required. The simplest exemplification of the method, which is suitable for use when 4th differences tend to disappear, is described in the appended Explanatory Notes,-which also discuss other pertinent matters.*

Though this computation appears formidable in review, it is actually extremely simple to apply. The schematic diagrams and work sheets have been developed for the purpose of reducing the computation to simple systematic procedure.

EXAMPLE: Integrate $dy/dx = -xy$ from $x = 0.5$ to $x = 1.0$, with initial value $y = 1$ when $x = 0$, and with starting values as follows:

x		y		$dy/dx = u = -xy$
0	y_{-5}	1.000 000 00	u_{-5}	-0.000 000 00
0.1	y_{-4}	0.995 012 48	u_{-4}	-0.099 501 25
0.2	y_{-3}	0.980 198 67	u_{-3}	-0.196 039 73
0.3	y_{-2}	0.955 997 48	u_{-2}	-0.286 799 24
0.4	y_{-1}	0.923 116 35	u_{-1}	-0.369 246 54
0.5	y_{0}	0.882 496 90	u_{0}	-0.441 248 45

The method of obtaining the above "starting values" or determining how many starting values are needed in any case is beyond the scope of this method. The subject is briefly touched upon in the Explanatory Notes.

The computing plan for this example comprises the use of the 5-term "open-type" formula for integrating ahead and the 5-term "closed-type" formula for back-checks, with a final refinement of the entire group of five values by use of the 9-term "closed-type" formula.

OPERATIONS: Decimals; Upper Dial 8, Middle Dial 16, Keyboard Dial 8. Use any 10-column "M" model with Upper Green Shift Key down.

(1) Compute the factors that are in the three right rows of the Upper and Lower Arrays of Page 5 for values from and including $x = 0$ to $x = 0.5$, setting up each value of "u" from the example and multiplying it successively by 11, 14, 26, 7, 32, and 12 insofar as the arrays show that it is necessary to use the factors; for example, it is not necessary to multiply u_{-4} (0.099 501 25) by any multiplier except "11" (for the single entry in the upper array).

(over)

COMPUTING TRIAL VALUES OF "y" AND "u"

(2) The first step in obtaining the trial value of y for x = 0.6 (y_{+1}) is to integrate u from u_{-5} to u_{+1} by using its values from u_{-4} to u_0 inclusive (not using the end values of u_{-5} and u_{+1}). This is done by summing the factors of the upper array diagonally, as indicated by the line with the arrows; thus, the sum of these "u" functions, as per formula at upper left of section "0.6" Work Sheet (Page 6) is 1.094 513 75 − 2.744 556 22 + 7.456 780 24 − 5.169 451 56 + 4.853 732 95 = 5.491 019 16, which appears in Middle Dial. It is negative as all of the u's are negative.

(3) Transfer the Middle Dial amount to Keyboard Dial, clear Middle Dial and multiply by Length Factor (.03).

Increment in y from x = 0 to x = 0.6 (0.164 730 57), which is also negative, appears in Middle Dial.

(4) Transfer the Middle Dial amount (0.164 730 57) to Keyboard Dial, clear Middle Dial, and subtract. Set up y_{-5} (1.000 000 00) and add.

Trial Value y_1 (0.835 269 43) appears in Middle Dial. Copy to Work Sheet (Page 6).

(5) Next, proceed with the calculation of dy/dx when x = 0.6 from its equation, both the y and x being known. In this case, the value of dy/dx = u is obtained by multiplying the known "y" by the known "x" (−0.6); thus, Transfer Middle Dial amount (0.835 269 43) from Middle Dial to Keyboard Dial, clear Upper Dial and multiply by "0.6."

Trial Value u_1 (0.501 161 66) appears in Middle Dial. Copy to Work Sheet (Page 6).

(6) Transfer the Middle Dial amount (0.501 161 66) to Keyboard Dial and multiply by 11.

Copy Trial Value u_{-1} (0.501 161 66) to Work Sheet (Page 6) and to the Lower Array of Factor Sheet (Page 5).
Copy Middle Dial amount (3.508 131 62) to Lower Array of Factor Sheet.

COMPUTING CHECK VALUE OF "y" AND "u"

(7) The first step in obtaining the Check Value of y_{+1} is to integrate u from u_{-3} to u_{+1}, using these values as well as those that are in-between. This is done by summing the factors of the Lower Array diagonally, as indicated by the line with the arrows; thus, the sum of these "u" functions as per formula at lower left of section "0.6" of Work Sheet (Page 6) is 1.372 278 11 + 9.177 575 68 + 4.430 958 48 + 14.119 950 40 + 3.508 131 62 = 32.608 894 29, which is negative.

(8) Move Upper Dial decimal from 3 to 9 and Keyboard Dial decimal from 8 to 7, set up reciprocal of the common multiplier (225) and divide.

Increment in y from x = 0.3 to x = 0.6 inclusive (0.144 928 842) appears in Upper Dial, which enter in Work Sheet (Page 6) as negative.

(9) Move Middle Dial decimal to 17 and Keyboard Dial decimal to 8, set up y_{-3} (0.980 198 67) in Keyboard Dial, depress Add Bar, and then depress Subtract Bar.

(10) Set up "1" in 9th column of Keyboard Dial and reverse multiply by Upper Dial amount, except use rounded figure of "2" in 2nd dial instead of "19" as appears in 2nd and 1st dials.

Check value of y_{+1} (0.835 270 25) appears in Middle Dial. Upper Dial shows all ciphers, or all 9's, in every dial, except 1st dial will show effect of rounding. Copy to Work Sheet.

CORRECTING THE CHECK VALUE

(11) The Check Value is usually more nearly correct than the Trial Value, because it is obtained as an integration of only four "sections" (0.3 to 0.6), using five "ordinates." The difference δy between the Trial and Check Values is obtained and given such sign that when it is added to the Check Value their sum equals the Trial Value (see Explanatory Notes); thus,

0.835 270 25 - 0.835 269 43 = 0.000 000 82 recorded as negative.

(12) For the conditions of this example (see Explanatory Notes) the Check Value should be corrected by $\delta y/35$ = -0.000 000 02, reducing y_{+1} to 0.835 270 23.

(13) Substitution in the formula for dy/dx is then made (in this case multiplying 0.835 270 23 X - 0.6, producing the Check Value of u_{+1} - 0.501 162 14, which is entered in the Upper Array of the Factor Sheet. The value previously entered in the Lower Array is corrected, as shown, and again multiplied by 11, 14, and 26, to produce the corrected factor in the "11" column and also the values in the other columns. The Lower Array is, likewise, completed by multiplying by 7, 32, and 12.

(14) The above cycle from Steps 2 to 13 is repeated for the remaining values, dropping off values "at the top" as new ones are obtained at the bottom, all as per Work Sheet (Page 6).

In preparing this Work Sheet, it is found convenient to reproduce the formulas used, as shown, and to place the symbol number that identifies the term in the Factor Arrays directly below each term. This provides a "pattern" of calculating that serves to clarify and expedite the process.

The checked y's are tabulated below:

x	y
0.6	0.835 270 23
0.7	0.782 704 55
0.8	0.726 149 06
0.9	0.666 976 83
1.0	0.606 530 69

FINAL REFINEMENT OF VALUES

By the preceding process, there has been obtained the first group of five values beyond the starting values. Before calling the work complete, however, we may make an overall check by using the 9-term closed formula for integrating the differential "u" in the eight sections from x = 0.2 to x = 1.0 inclusive, thus

(over)

x	u	[illegible]
0.2	-0.196 039 73	989
0.3	0.286 799 24	5888
0.4	0.369 246 54	-928
0.5	0.441 248 45	10496
0.6	0.501 162 14	-4540
0.7	0.547 893 19	10496
0.8	0.580 919 25	-928
0.9	0.600 279 15	5888
1.0	0.606 530 69	989

$$= -13241.86039 \times \frac{0.8}{28350}$$

$$= -0.373\ 668\ 02$$

which added to y for x = 0.2 produces new y for x = 1.0 of 0.606 530 65.

This is .000 000 04 less than the previously found value, the amount being the accumulation of integrated differences that occurred during the calculation of these values. The difference of "4" should be distributed among these 5 values of y, as below, and new "u" calculated.

		Distributed	New "y"	New "u"
0.6	0.835 270 23	- .8	0.835 270 22	-0.501 162 13
.7	.782 704 55	-1.6	.782 704 53	-0.547 893 17
.8	.726 149 06	-2.4	.726 149 04	-0.580 919 23
.9	.666 976 83	-3.2	.666 976 80	-0.600 279 12
1.0	.606 530 69	-4.0	.606 530 65	-0.606 530 65

These new values of "u" should be substituted in the 9-term formula, above, but to avoid repetition, only the effect of the difference in u's, as found above, and those shown on the upper part of Page 5, is calculated, thus obtaining a final refinement of y_{+5}

x	Difference in "u"	X
0.6	+.000 000 01	-4540
.7	2	10496
.8	2	-928
.9	3	5888
1.0	4	989

$$= .000\ 362\ 16$$

$$\times \frac{0.8}{28350} = +.000\ 000\ 01$$

This increase in "y_{+5}" is, likewise, an accumulation of integrated differences throughout 5 terms, so it is distributed as follows:

x	Correction of y (rounded)	Final "y"
0.6	.000 000 00	0.835 270 22
0.7	0	.782 704 53
0.8	1	.726 149 05
0.9	1	.666 976 81
1.0	.000 000 01	.606 530 66

If additional values beyond x = 1.0 were to be obtained, a new column of values of u corresponding to the above values of y would be obtained in the customary way.

FACTOR SHEET MILNE-STEFFENSEN 5 POINT (6 ORDINATE) FORMULA

FOR INTEGRATION BY STEP-BY-STEP METHOD

ORIGINAL CALCULATION: h = 0.1 Length Factor: 3 h/10 = .03

			-- Column Multipliers --		
x	u_n	u = -xy	11	-14	26
0	u_{-5}	0.000 000 00			
0.1	u_{-4}	-0.099 501 25	1.094 513 75		
0.2	u_{-3}	-0.196 039 73	2.156 437 03	2.744 556 22	
0.3	u_{-2}	-0.286 799 24	3.154 791 64	4.015 189 36	7.456 780 24
0.4	u_{-1}	-0.369 246 54	4.061 711 94	5.169 451 56	9.600 410 04
0.5	u_0	-0.441 248 45	4.853 732 95	6.177 478 30	11.472 459 70
0.6	u_{+1}	-0.501 162 14	5.512 783 54	7.016 269 96	13.030 215 64
0.7	u_{+2}	-0.547 893 19	6.026 825 09	7.670 504 66	14.245 222 94
0.8	u_{+3}	-0.580 919 25	6.390 111 75	8.132 869 50	
0.9	u_{+4}	-0.600 279 15	6.603 070 65		
1.0	u_{+5}	-0.606 530 69			

BACK-CHECK CALCULATION: h = 0.1 Length Factor: 2 h/45 = 1/225

			-- Column Multipliers --		
x	u	u = -xy	7	32	12
0.2	u_{-3}	-0.196 039 73	1.372 278 11		
0.3	u_{-2}	-0.286 799 24	2.007 594 68	9.177 575 68	
0.4	u_{-1}	-0.369 246 54	2.584 725 78	11.815 889 28	4.430 958 48
0.5	u_0	-0.441 248 45	3.088 739 15	14.119 950 40	5.294 981 40
0.6	u_{+1}	2 14 -0.501 161 ~~66~~	4 98 3.508 131 ~~62~~	16.037 188 48	6.013 945 68
0.7	u_{+2}	3 19 -0.547 892 ~~5.~~	52 33 3.835 247 ~~57~~	17 532 582 08	6.574 718 28
0.8	u_{+3}	9 25 -0.580 918 ~~42~~	34 75 4.066 ~~428 94~~	18.589 416 00	6.971 031 00
0.9	u_{+4}	9.15 -0.600 278 ~~21~~	54 05 4.201 947 ~~47~~	19.208 932 80	
1.0	u_{+5}	30 69 -0.606 529 ~~74~~	4.245 708 18		

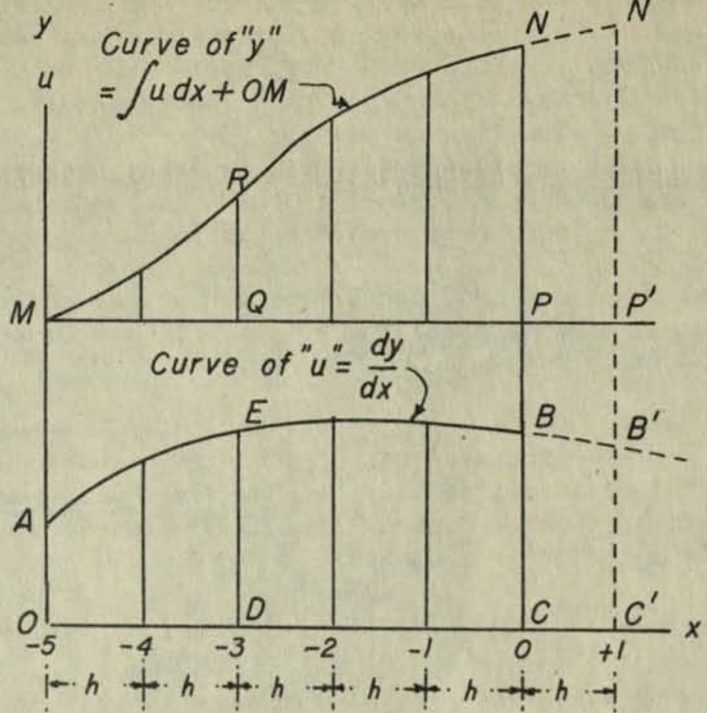
(over)

WORK SHEET MILNE-STEFFENSEN 5 POINT (6 ORDINATE) FORMULA
FOR INTEGRATION BY STEP-BY-STEP METHOD

	FORMULA		TRIAL		CHECK	δy	E (y/35)
		y_{-5}	1.000 000 00	y_{-3}	0.980 198 67		
x = 0.6	$-0.03(11u_{-4} - 14u_{-3} + 26u_{-2} - 14u_{-1} + 11u_{0})$		−0.164 730 57		−0.144 928 42		
		y_{+1}	0.835 269 43		0.835 270 25	−82	−2
			−0.6		−2		
	$-1/225(7u_{-3} + 32u_{-2} + 12u_{-1} + 32u_{0} + 7u_{+1})$	u_{+1}	−0.501 161 66	y_{+1}	0.835 270 23		
					−0.6		
				u_{+1}	−0.501 162 14		
		y_{-4}	0.995 012 48	y_{-2}	0.955 997 48		
x = 0.7	$-0.03(11u_{-3} - 14u_{-2} + 26u_{-1} - 14u_{0} + 11u_{+1})$		−0.212 308 89		−0.173 292 90		
		y_{+2}	0.782 703 59		0.782 704 58	−99	−3
			−0.7		−3		
	$-1/225(7u_{-2} + 32u_{-1} + 12u_{0} + 32u_{+1} + 7u_{+2})$	u_{+2}	−0.547 892 51	y_{+2}	0.782 704 55		
					−0.7		
				u_{+2}	−0.547 893 19		
		y_{-3}	0.980 198 67	y_{-1}	0.923 116 35		
x = 0.8	$-0.03(11u_{-2} - 14u_{-1} + 26u_{0} - 14u_{+1} + 11u_{+2})$		−0.254 050 65		−0.196 967 26		
		y_{+3}	0.726 148 02		0.726 149 09	−107	−3
			−0.8		−3		
	$-1/225(7u_{-1} + 32u_{0} + 12u_{+1} + 32u_{+2} + 7u_{+3})$	u_{+3}	−0.580 918 42	y_{+3}	0.726 149 06		
					−0.8		
				u_{+3}	−0.580 919 25		
		y_{-2}	0.955 997 48	y_{0}	0.882 496 90		
x = 0.9	$-0.03(11u_{-1} - 14u_{0} + 26u_{+1} - 14u_{+2} + 11u_{+3})$		−0.289 021 69		−0.215 520 04		
		y_{+4}	0.666 975 79		0.666 976 86	−107	−3
			−0.9		−3		
	$-1/225(7u_{0} + 32u_{+1} + 12u_{+2} + 32u_{+3} + 7u_{+4})$	u_{+4}	−0.600 278 21	y_{+4}	0.666 976 83		
					−0.9		
				u_{+4}	−0.600 279 15		
		y_{-1}	0.923 116 35	y_{+1}	0.835 270 23		
x = 1.0	$-0.03(11u_{0} - 14u_{+1} + 26u_{+2} - 14u_{+3} + 11u_{+4})$		−0.316 586 61		−0.228 739 51		
		y_{+5}	0.606 529 74		0.606 530 72	−98	−3
			−1.0		−3		
	$-1/225(7u_{+1} + 32u_{+2} + 12u_{+3} + 32u_{+4} + 7u_{+5})$	u_{+5}	−0.606 529 74	y_{+5}	0.606 530 69		
					−1.0		
				u_{+5}	−0.606 530 69		

EXPLANATORY NOTES

The diagram relates to a typical function, not to the one used in the example.



PRINCIPLE OF STEP-BY-STEP SOLUTION OF DIFFERENTIAL EQUATIONS: Let it be assumed that in the case of any given ordinary differential equation of first order $dy/dx = u = f(x,y)$, the values of y and u are known for several "starting" values of x at equally spaced intervals (x_{-5}, x_{-4} ..... x_0). Assume that these values are plotted as the solid curved lines of the diagram in which AEB represents the known values of dy/dx, and MN represents the corresponding known values of y when the initial value is OM. From the relationship shown, it is apparent that the area OABCO is a measure of the increment in y (PN) above the initial value MO, or y= NC; that is to say, any point on the curve MN corresponding to any value of x may be obtained by measuring the area ~~of~~ under the differential curve up to that value of x and adding the result so obtained to the initial value of y, OM.

If, now, we have means of determining the area OAB'C'O, when given only the solid-line curve AEB, it is apparent that the area so obtained is likewise a measure of the increment in y, P'N', which when added to the initial value OM would give a new y corresponding to a point x_{+1}. By substituting the new y, so found, in the given differential equation, x of course being known, the new value of $dy/dx = u$, or the point B' becomes known.

The cycle is now complete, and by like means we can compute other values at the right of NN' and BB', thus continuing the solution to any desired point.

The crux of the solution rests, of course, in the determination of the area OAB'C'O when given only the values of u from A to B. The process is essentially one of extrapolation, and various ways exist for performing this "extension." It is obvious that errors made in computing the first extension steps are cumulative, just as if one were constructing a cantelever bridge, so any process, to be successful, must have means of reducing these errors to a minimum. Also, if possible, it must provide over-all checks to correct a series of values so that the completed structure will have $dy/dx = u$ for every value of x and y.

TRIAL COMPUTATION OF THE MILNE METHOD: The area OAB'C'O is obtained by using only the ordinates of the curve AB at points such as x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0, for example, which values are known as they comprise all but one of the assumed starting values. If these values

(over)

of u are differenced (see Marchant Method MM-100) and it is found that fourth differences are negligible, then it would not be necessary to divide the length AB into five sections (six ordinates). It would be satisfactory to divide it into three sections (four ordinates); the sections would be larger, and the work would proceed that much faster. If it is desired to take account of 7th differences of u (corresponding to 8th differences of y), the curve AB should be sectionalized still further -- into 7 sections (8 ordinates), etc. In any case, the number of ordinates should be even, because the accuracy of the computation based upon any such even number of ordinates is substantially equal to the case if the next higher uneven number of ordinates is used.

Consideration of the above matters provides a hint for proper choice of interval (h) and of the number of ordinates to be used for the solution of any equation. The subject will not be further explored here.

In the example of this method, six starting values are known, corresponding to the solid-line ordinates of both curves. The area OAB'C'O is then obtained by applying the proper Steffensen "open-type" integration formula. In the case diagrammed, the area OAB'C'O, spanning seven ordinates, is found by using five of the known ordinates, excluding the end ordinates for x_1 and x_{-5}. The formula is:

$$\text{Area OAB'C'} = \frac{\text{length OC'}}{20}\,(11u_{-4} - 14u_{-3} + 26u_{-2} - 14u_{-1} + 11u_0)$$

which will be recognized as that which was used in the example, because length OC' equals 6h and 6 X 0.1/20 = 0.03.

The general formula for various numbers of terms is given below:

No. of Terms	Coefficients to Central Value					Divisor	Remainder
3	2	-1				3	0.000 31 $F^{(4)}$ (Xi)
5	11	-14	26			20	0.000 001 1 $F^{(6)}$ (Xi)
7	460	-954	2196	-2459		945	0.000 000 002 1 $F^{(8)}$ (Xi)
9	4045	-11690	33340	-55070	67822	9072	0.000 000 000 002 7 $F^{(10)}$ (Xi)

The use of the above coefficients and divisors will be apparent by analogy from what has been previously described. The remainder of the 3-term formula contains the expression $F^{(4)}$ (Xi), which designates the 4th derivative of u (or the 5th derivative of y) with respect to x for some value of x (unknown) that is found within the entire length OC'. This amount makes it possible to determine the maximum error if we know the maximum 4th derivative. The actual error, however, may be less than this, and usually is. The computer rarely needs to pay attention to this matter because the Milne Method offers a simplification of this point, as will be later explained.

The Milne computation of the error is also to be preferred because the above expressions for the Remainder apply to the case when the length OC' is taken as "1". If it has any other value, the above Remainder Coefficients must be multiplied by the length raised to the r th power, in which "r" equals "no. of terms of formula used, plus 2;" i.e., if the length OC' is 6 units long and the 5-term formula is used, the Remainder Coefficient (0. 000 001 1) is to be multiplied by 6^7. It therefore equals 0.31.

We have now obtained the area OAB'C'O which, as stated, is a measure of the increment in y. This increment, P'N', added to the initial value OM, gives the new value y_1, corresponding to x_1. It then only remains to substitute the now known y_1 and the known x_1 in the given differential equation and thus obtain u_1, or the point B'.

The cycle is now complete, and if we are satisfied with the accuracy so far obtained, we could proceed to the next step by finding the area under the differential curve from x_{-4} to x_2 and adding the increment in y so found to the value of y_{-4}, thus producing the next value of y_2.

THE MILNE CHECK-BACK: The first refinement of the value of u and y for x_1 is obtained by re-calculating y_1 in a different manner from that previously employed. This second determination of y_1 is obtained by finding the area DEB'C'D of the differential curve between and including the values x_{-3} and x_1 and adding the increment so obtained to y_{-3}. This could be done by Simpson's Rule, if desired, but such a formula would not suffice for this example because it would not take into account 5th differences of u. The formula that was used in the example takes into account such 5th differences; thus,

$$\text{Area DEB'C'D} = \frac{\text{length DC'}}{90}\ (7u_{-3} + 32u_{-2} + 12u_{-1} + 32u_0 + 7u_{+1})$$

Inasmuch as the length DC' in the example is 4h, the multiplier is 4 X 0.1/90 = 1/225.

The general formula for various numbers of terms is given below:

No. of Terms	Coefficients to Central Value					Divisor	Remainder
3	1	4				6	$-0.000\ 35\ F^{(4)}$ (Xi)
5	7	32	12			90	$-0.000\ 000\ 52\ F^{(6)}$ (Xi)
7	41	216	27	272		840	$-0.000\ 000\ 000\ 64\ F^{(8)}$ (Xi)
9	989	5888	-928	10496	-4540	28350	$-0.000\ 000\ 000\ 000\ 59\ F^{(10)}$ (Xi)

The use of the above coefficients, divisors and remainder will be apparent by analogy from what has been previously described. However, the Remainder of the above Check formula, when compared with that of the Trial formula, if used in the Milne Method, is smaller than a comparison of coefficients indicates. This is because the Check formula integrates an area with a shorter base line, it being ½; 2/3, ⅝, etc., of that of the Trial formula for 3, 5, and 7 terms respectively; and higher powers of this ratio of intervals are involved.

Because the above formula uses the outside ordinates which close the area, it is designated a "Closed Type" formula. The previously described formula that does not use either of the end ordinates is similarly designated an "Open Type" formula.

The formula for 3 terms will be recognized as Simpson's 1/3 Rule.

THE MILNE ERROR CHECK: Inspection of the Remainders, as tabulated for the above Open and Closed-Type formulas, shows that in the case of, say, the 5-term formulas if the seventh derivative of u vanishes; that is to say, the error is only that due to the sixth derivative, and if we also assume that this sixth derivative is positive in the case of each formula, then it is evident that the integral calculated from the open-type formula will be less than its true value because the Remainder is positive. By similar reasoning, it is seen that the closed-type formula produces a value that is greater than its true value. The true value then lies between the values obtained by the two formulas.

Inasmuch as in most cases the differences of any order and the derivatives of the same order are closely proportionate, we may likewise conclude, subject to remotely unusual exceptions, that if the seventh <u>differences</u> of u vanish, the true value of the integral will, likewise, lie between its values when computed by the closed and open-type formulas (of course, the sixth differences that control this error are adjacent differences, rather

(over)

than identical, because the two 5-term formulas do not use all of the identical five terms, but if these adjacent differences should be of opposite signs we would know that they would be substantially non-existent because of the assumption that seventh differences tend to vanish).

The above reasoning has been applied to the case where the difference that controls the error (in this case, the 6th) is positive. By similar reasoning, it will be seen that the true value of the integral lies between its "open type" and "closed type" values when the difference that controls the error is negative, and also that this general conclusion applies to any of the formulas, provided that the n + 2 difference of any n term formula is assumed to vanish.

By reference to the original expansion from which the remainders were computed, Milne develops the following conclusion for the amount of the error, the sign of which becomes known when it is remembered that the error must be such as to modify the value of the integral when obtained by the closed-type formula so the true value comes between it and the value when obtained by the open-type formula.

> Under the assumptions given, the true value differs from the value as obtained by the closed-type formula by the following fraction of the difference between the values obtained by the open-type and closed-type formulas: 3-term, 1/29; 5-term, 1/35; 7-term, 1/44; 9-term, 1/54. The sign of this difference is such as to make the true value lie between the closed-type and open-type values.

In applying the above principle, it is to be remembered that the correction is to be applied to the value obtained by the closed-term formula.

Sometimes examples are found in texts in which the true value does not lie between the open-term and closed-term values, but if these be examined, it will be found that higher orders of differences exist than permitted by the above assumption. Inasmuch as the choice of the number of starting terms of the solution and of the interval are usually such that differences of u substantially vanish, as outlined above, it is seen that the Milne Error Check is as sound as it is easy to apply.

APPLICATION TO DIFFERENTIAL EQUATIONS OF HIGHER ORDER, ETC: Systems of differential equations of the first order may also be solved by this method. Each equation is solved independently for its step in y, but these y values are substituted in the simultaneous equations to give new value of u for each equation.

Differential equations of higher order or systems thereof are reducible to a system of equations of the first order which is then solvable, as above outlined. Milne has developed special means of solving second order equations in which first derivatives are absent.

REFERENCES: The following list will be of assistance to those who wish to study this subject further:

W. E. Milne, *Numerical Integration of Ordinary Differential Equations*, Am. Math. Mo. 33: 455-460 (1926).
W. E. Milne, *On the Numerical Integration of Certain Differential Equations of the Second Order*, Am. Math. Mo. 40: 322-327 (1933).
National Research Council, No. 92, *Numerical Integration of Differential Equations* (Report of A. A. Bennett, W. E. Milne, H. Bateman) (1933).
J. F. Steffensen, *Interpolation*, P. 158-159, 170-177, Williams & Wilkins Co. (1927).
J. L. Scarborough, *Numerical Mathematical Analysis*, P. 280-282, The Johns Hopkins Press, Baltimore (1930).

MARCHANT METHODS

INVERSE CURVILINEAR INTERPOLATION

SHORT METHOD IN WHICH CONSTANT 2ND DIFFERENCES ARE ASSUMED

REMARKS: *This method is an adaptation to the Marchant of the Comrie Process which is described in the British Nautical Almanac for 1937. It is very satisfactory for inverse interpolation in the ordinary table which usually has tabulated values so close together that differences greater than the second can be ignored. The method requires the use of a pre-calculated table of $B''(d''_0 + d''_1)$, or that slide-rule be used for this auxiliary calculation.*

A supplemental Note provides information as to the size of 3rd difference that may be ignored without its affecting the last place of the interpolate by more than ½.

EXAMPLE: In the following table find "x" which corresponds to y = 0.984 637 2. Differences are also shown which were computed as per Marchant Method MM-100.

x	y	d'	d"	d'''
1.0	0.993 377 5 (f_{-1})			
		-65 787 ($d'_{-\frac{1}{2}}$)		
2.0	0.986 798 8 (f_0)		+ 436 (d''_0)	
		-65 351 ($d'_{\frac{1}{2}}$)		-3 ($d'''_{\frac{1}{2}}$)
3.0	0.980 263 7 (f_1)		+ 433 (d''_1)	
		-64 918 ($d'_{1\frac{1}{2}}$)		
4.0	0.973 771 9 (f_2)			

OPERATIONS: Decimals; Upper Dial 7, Middle Dial 14, Keyboard Dial 7. Use any Marchant model (As Model "D" is very suitable for this work, the problem is exemplified on Model ACR-8 D). As this is a decreasing function, set Manual Counter Control toward the operator. If it were increasing, it would be away from operator.

(1) Set up (f_0) (0.986 798 8) in Keyboard Dial and, with carriage in 8th position, depress Add Bar.

(2) Set up ($d'_{\frac{1}{2}}$) (0.006 535 1) and touch Short-Cut Bar once.

(f_1) (0.980 263 7) appears in Middle Dial as proof.

(3) Depress X Bar once and clear Upper Dial.

(f_0) (0.986 798 8) is restored in Middle Dial.

(4) Shift to 7th position and hold down Short-Cut Bar until Middle Dial amount falls just below (f_x) (0.984 637 2), and then tap X Bar once.

Left figure of approximate interval ratio "n" (0.3) appears in Upper Dial.

(5) Shift to 6th position and hold down Short-Cut Bar until Middle Dial amount falls just below (f_x) (0.984 637 2), and then tap X Bar once.

Left figures of approximate interval ratio "n" (0.33) appear in Upper Dial. Middle Dial shows "0.984 642 217" but the amount need not be separately noted.

(6) Consult Chart of Bessellian Coefficients (see Marchant Method MM-189) and note that for n = 0.33 the coefficient "B" is -0.055 for Direct Interpolation. However, for inverse interpolation the sign is +. The approximate correction to be applied is + 0.055 X (436 + 433) = 47.8, which value is obtained from pre-calculated table of B" ($d''_0 + d''_1$) or by slide rule. This amount (47.8), being positive, is added to y_x to produce the desired corrected Middle Dial reading; thus, 0.984 637 2 + 0.000 004 78 = 0.984 641 98.

(7) By suitable shifts and use of X and Short-Cut Bars, adjust Upper Dial reading until Middle Dial reads as close as possible to the above amount (0.984 641 98), using only one more figure than contained in d'.

Approximate "n" (0.330 036) appears in Upper Dial and "0.984 641 981.." appears in Middle Dial.

It is now necessary to note whether this alteration of "n" from its approximate value of 0.33 will alter the amount in Step 6. In this case it does not. If it did so, it would be necessary to revise the correction of 47.8 and slightly alter the Upper Dial reading to reflect the effect of this revision upon the Middle Dial reading.

The desired x is 2.330 04

No more places can be taken in the change of x (0.330 04) than occur in d' and if the latter begins with 1, 2, or 3, take one less. There will still be an uncertainty in the final figure, because of rounding and possibly also because of the existence of the 3rd difference which was disregarded (See Note below).

CHECK

Using Marchant Method MM-189, Example IIa, we have:

	f_0 0.986 798 8
0.330 04 X (-.006 535 1)	-.002 156 8
	0.984 642 0
-0.055 X (0.000 086 9)	-.000 004 8
	0.984 637 2 = f_x

NOTE

The fact that the tabular values are rounded precludes taking more figures in "n" than exist in d'. However, it is advisable to check to see how large the 3rd difference can be before it affects the interpolate. Inasmuch as the method applies to the Bessel formula, the following relation holds, assuming that the function tabulated is the usual sort with progressively smaller differences of higher order.

$$d''' \text{ should not exceed } \frac{60\, d'_{\frac{1}{2}}}{10^a}$$

in which "a" is the number of figures taken in the change of argument "n".

In this case "n" is taken to 5 figures, hence d''' will affect the value only if it is greater than

$$\frac{60 \times 0.006\ 535\ 1}{10^5} = .000\ 003\ 9$$

It is actually 1/13 of this, so it does not affect the amount. If we could depend upon the tabulated values as being exact instead of rounded, the change of argument could be taken to 6 figures; i.e., 0.330 036.

1 026 6436 7

MARCHANT METHODS

THE BIRGE-VIETA METHOD
of
FINDING REAL ROOTS OF RATIONAL INTEGRAL FUNCTION

PREFACE: *Few realize the extent that classical mathematical methods have evolved under the control of the "parameter" (to use a mathematician's word) that pencil-and-paper shall be used in the calculations required by such methods. If the modern calculating machine had been available to the mathematicians of the Renaissance, it is possible that even such a familiar tool as the Briggs Logarithm might not have been developed. Certainly the art would have progressed along far different lines if from the start there had been available a machine that could multiply or divide as rapidly as one could enter amounts in a keyboard.*

The disclosure herein is an interesting example of how an early method, which was discarded because it involved so much numerical computation that was "unfit for a Christian," to quote from a writer of that day, has now been found to possess decided advantages when compared with methods that displaced it. This is because present-day calculating machines remove the drudgery element which caused the method to be relegated to the shelf over 200 years ago.

The method to which we refer was originally proposed by Francis Vieta (1540-1603). Raymond T. Birge, Ph.D., Professor of Physics and Chairman of the Department, University of California, is responsible for re-establishing it as a modern computing tool. Dr. Birge has noted that it possessed many advantages over the methods that have been developed to take its place (merely because of the excessive amount of pencil-and-paper work that it entailed).

In applying the Vieta method to the modern calculating machine, Dr. Birge has reduced it to simple systematic procedure that permits speedy determination of the root under conditions of controlled accuracy.

USES OF THE BIRGE-VIETA METHOD: The method is ideal for finding a real root of the usual algebraic equation when rough approximation of the root is known, particularly if the equation is of higher degree than the second. It is also excellent for solving transcendental equations (those that involve logarithmic or trigonometric functions in combination with analytic functions), particularly when the equations are in such form that substitutions of odd amounts in the equations or in their first derivatives are difficult. Inasmuch as the usual problem of inverse curvilinear interpolation is one of finding the root when the value of the function is a given amount, it will be seen that the Birge-Vieta method is adapted to such work, assuming of course that the tabular values are first expressed as an Interpolation Polynomial of degree "n" that fits n + 1 equidistant values of such tabulated function (See Marchant Method MM-226).

In the case of solving equations involving transcendental functions, tabular values are, likewise, obtained. An Interpolation Polynomial is then fitted to the values and then solved for the desired root. If, however, the equation has a simple first derivative and substitution of amounts in the original equation or its first derivative is not too difficult, the Newton-Raphson Method of obtaining the root is to be preferred.

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OUTLINE: It is assumed that the reader is familiar with the usual Horner Synthetic Division process which is described in most College Algebra texts. However, a Note is appended which describes this procedure in a way that will enable it to be understood by a computer who is not familiar with it. (See top of Page 4).

An algebraic statement of the sample computation is given. This is followed by detailed instructions for performing the work on a Marchant Silent-Speed calculator. An Appendix then states the particular advantages of the Birge-Vieta Method, as compared with methods that are ordinarily used for such work.

The symbolism of the Horner Method is employed insofar as possible.

EXAMPLE: Find correctly to nine figures the real root nearest to $x = 1.0$ of the following equation:

$y = g(x) = x^5 - x - 0.2$ (true value is $1.044\ 761\ 700_{07}$)

Assume $x = +1 = p_1$ as first approximation of the root.

I Transfer from $g(x)$ to $g'(x - p_1) = g'(x - 1) = g'(u)$
Transfer factor, $p_1 = +1$. Apply Horner Shift for A_0 and A_1 (See Note A, Page 4).

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	1	0	0	0	−1	−0.2
		1	1	1	1	0
	1	1	1	1	0	$-0.2 = A_0$
		1	2	3	4	
	1	2	3	4	$4 = A_1$	

Therefore $u = -\frac{A_0}{A_1} = -\frac{-0.2}{4} = +0.05 = x - p_1$

or $x = p_1 - \frac{A_0}{A_1} = 1.0 + 0.05 = 1.05 = p_2$, as second approximation.

It will be noted that the above represents the first steps of an ordinary Horner synthetic division. Only A_0 and A_1 need be found.

II Transfer from $g(x) = g''(x - p_2) = g''(x - 1.05) = g''(v)$.

It is a characteristic of this method that the calculations need be carried <u>only</u> to the reliability that the ratio of the next coefficients (in this case, B_0 and B_1) is likely to have. A practical rule is to carry <u>twice</u> as many <u>decimal</u> places in all sums and products used in obtaining B_0 as there are decimal places in the transfer factor. Hence, since 1.05 is the transfer factor, carry B_0 calculations to <u>four</u> decimal places. We find, in this problem, three significant figures for B_0, and hence carry all calculations for B_1 to at least <u>three</u> significant figures (it is really simpler to carry four and round off to three).

We now return to the <u>original</u> coefficients, an essential of the method, and one of its best features from the viewpoint of accuracy control.

$p_2 = 1.05$ transfer factor.

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	1	0	0	0	− 1	− 0.2
		1.05	1.1025	1.1576_2	$+ 1.2155_{01}$	$+ 0.2262_{76}$
	1	1.05	1.1025	1.1576_2	$+ 0.2155_{01}$	$+ 0.0263 = B_0$
		1.05	2.2050	3.472	4.860	
	1	2.10	3.307	4.629	$5.075 = B_1$	

By inspection $v = -B_0/B_1$ will have two ciphers. Therefore, by rule given, the ratio should be correct to two significant figures.

Therefore $$v = -\frac{B_0}{B_1} = -\frac{+0.0263}{5.07} = -0.005187$$

rounded to -0.0052

It will be noted that four decimal places carried in the B_0 calculations were sufficient to give B_0 to three significant figures, as is desired in order to be sure that B_0/B_1 is correctly calculated to two significant figures (i.e., in addition to the two ciphers with which it starts).

Therefore $$x = p_2 - \frac{B_0}{B_1} = 1.05 - 0.0052 = 1.0448 = p_3,$$ as second approximation.

III Transfer from $g(x)$ to $g'''(x - p_3) = g'''(x - 1.0448) = g'''(w)$
$p_3 = 1.0448$ transfer factor

As before, since there are four decimal places in $v = -B_0/B_1$ or in p_3, the next transfer factor, we carry eight decimal places in getting C_0, i.e., close to full capacity of a ten-key calculator, so for simplicity the full ten-key capacity is utilized. Then from the C_0 result, carry only six significant figures in computing C_1.

x^5	x^4	x^3	x^2	x^1	x^0
1	0	0	0	-1	-0.2
	1.0448	1.091 607 04	$1.140\ 511\ 035_4$	$+1.191\ 605\ 929_4$	$+0.200\ 189\ 874_6$
1	1.0448	1.091 607 04	$1.140\ 511\ 035_4$	$+0.191\ 605\ 929_4$	$+0.000\ 189\ 875 = C_0$
	1.0448	$2.183\ 21_4$	$3.421\ 53_2$	$+4.766\ 41_9$	
1	2.0896	3.274 82	4.562 04	4.958 02 $= C_1$	

There will be four ciphers in C_0/C_1, therefore carry five or six significant figures.

Therefore $$w = -\frac{C_0}{C_1} = -\frac{+0.000\ 189\ 875}{4.958\ 02} = -0.000\ 038\ 296\ 5_4$$

This ratio should be satisfactory to four significant figures. However, we retain five as this is to be the final approximation.

Therefore $$x = p_3 - \frac{C_0}{C_1} = 1.0448 - 0.000\ 038\ 296_{54} = 1.044\ 761\ 703_{46} = p_4$$

This root should be accurate to nine digits. It is seen that the error is 0.3 in the 9th digit.

A continuation of this process with transfer factor 1.044 761 703 gives $D_0 = +0.000\ 000\ 014_3$ and $D_1 = C_1$ (closely enough) $= 4.958$

Therefore $$-D_0/D_1 = -0.000\ 000\ 002_9 \text{ or } p_5 = p_4 - D_0/D_1 = 1.044\ 761\ 700_1$$

which is correct to ten figures.

An alternate continuation process is to use $p_4 = 1.044\ 761\ 7$ as transfer factor and by double multiplication (see Marchant Method MM-85) carry all products to full 20-

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digit capacity of the calculator, thus producing p_5 correct to 18 or 19 digits (1.044 761 700 075 552 795).

Note that the actual error in p_1 is -0.045, in p_2 is +0.0052, in p_3 is +0.000 038, and in p_4 is +0.000 000 003. Thus, each approximation is correct to about double the number of digits of its predecessor. This is a characteristic feature of the present method. For this reason, p_5 <u>should</u> be correct to about 18 digits.

NOTE A - THE HORNER SHIFT

For those not familiar with the Horner Shift, the procedure is easily understood by reference to the calculation for B_0/B_1 on Page 2, with factors manipulated as below:

						Transfer factor p
	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	a	b	c	d	e	f
of x^n		pa	pm	pn	po	pq
	a	m	n	o	q	$r = B_0$

in which $m = b + pa$, $n = c + pm$, $o = d + pn$, etc. and similarly for the next row that produces B_1.

MATHEMATICAL BASIS OF METHOD

The Birge-Vieta process obtains the value of the function and of its first derivative when the approximate roots (the transfer factors) are substituted for "x." That part of the process which obtains A_0, B_0, C_0, etc., obtains successively more accurate values of the function, and A_0, B_0, and C_0, etc. are these successive values. The step that obtains A_1, B_1, C_1, etc., similarly obtains successively more accurate values of the first derivative when the transfer factors are substituted for "x." This is done, however, not by duplicating the first step with respect to the equation of the first derivative of the function but by taking advantage of partial products and sums developed during the first step. This makes it unnecessary to set up the equation of the derived function.

The successive transfer factors may have the same or different signs. Under some conditions they may alternate in sign.

COMBINING SUBSTITUTION METHODS WITH THE BIRGE-VIETA PROCESS

Inasmuch as A_0, B_0, C_0, etc are the values of the function when the transfer factor is substituted for "x," and A_1, B_1, C_1, etc., are the first derivatives of the function with respect to "x" when the transfer factor is likewise substituted for "x," there will be cases in which the first two steps of the computation may be more easily done by taking advantage of these facts, using a Table of Powers for direct computation of these amounts. This plan reserves the Birge-Vieta process for cases in which direct substitution is not easy and where the first derivative also is not easy to compute, which by the premise at bottom of Page 1 is its indicated use, anyway. These conditions usually are met when the transfer factor exceeds three figures if a Table of Powers of three-figure amounts is available. It is met with two-digit transfer factors if a Table of Powers is not available, (assuming, of course, that the usual small powers of integers 1 to 9 are known).

A readily available table of "First Ten Powers of the Integers from 1 to 1000" is that of Works Project for Computation of Mathematical Tables, Table MT-1, Information Section, National Bureau of Standards, Washington, D. C.; price 50 cents.

It happens that the example used to illustrate this method is in such form that with the aid of a Table of Powers of three-figure amounts the results of the second section may be obtained somewhat faster by substitution. (The work of the first section is obviously merely a matter of inspection.)

As an example of this straight substitution, let us apply it to this second section. We first note that the powers of 1.05, to four decimals, are $x^5 = 1.2763$ and $x^4 = 1.2155$, (these are the only powers needed for substituting in the equation or in its first derivative).

From this, we have $1.05^5 - 1.05 - 0.2 = 0.0263 = B_0$

and its first derivative

$$5 \times 1.05^4 - 1 = 5.0775 = B_1$$

APPLICATION OF THE BIRGE-VIETA METHOD TO THE MARCHANT CALCULATOR

The skilled computer who prefers to add or subtract mentally, or who wishes to use auxiliary means for such addition or subtraction doubtless would prefer to set up the transfer factor as a constant in the Keyboard Dial and multiply by the various factors as needed. The amounts are then entered on a work sheet exactly as shown in the above analysis.

Others will wish to perform all additions and subtractions on the Marchant. The detailed Marchant operations for this procedure, when applied to the calculation of p_4, are as follows:

OPERATIONS: Decimals: Upper Dial 9, Middle Dial 18, Keyboard Dial 9. Use any 10 column Model M Marchant with Upper Green Shift Key down.

Inasmuch as the coefficient of x^5 is 1.0, the calculator computation is started for development of the x^3 column; thus,

(1) Set up 1.0448 and multiply by transfer factor (1.0448).

Copy 1.091 607 04 from Middle Dial to x^3 column.

(2) As there is no amount to add to this, the normal adding step is skipped. Shift to 10th position, clear Upper and Keyboard Dials, and copy Middle Dial amount (1.091 607 04) into Keyboard Dial, clear Middle Dial and multiply by transfer factor (1.0448).

Copy $1.140\ 511\ 035_4$ from Middle Dial to x^2 column.

(3) Repeat Step (2) with Keyboard Dial setup of 1.140 511 035.

(4) Clear Keyboard Dial, shift to 10th position, set up 1.0, and subtract.

Copy $0.191\ 605\ 929_4$ from Middle Dial to x^1 column.

(5) Repeat Step (2) with Keyboard Dial setup of 0.191 605 929.

(6) Clear Keyboard Dial, shift to 10th position, set up 0.2, and subtract.

Copy C_0 (0.000 189 875) to x^0 column.

(7) Clear all dials, set up 1.0448, and multiply by 2.0.

Copy 2.0896 from Middle Dial to x^4 column.

(8) Shift to 10th position, clear Keyboard Dial and transfer Middle Dial amount (2.0896) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(over)

(9) Shift to 10th position, clear Keyboard Dial, set up 1.091 607 04, and add.

Copy 3.274 82 from Middle Dial to x^3 column.

(10) Transfer Middle Dial amount (3.274 82) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(11) Shift to 10th position, clear Keyboard Dial, set up 1.140 511 035, and add.

(12) Transfer Middle Dial amount (4.562 04) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(13) Shift to 10th position, clear Keyboard Dial, set up 0.191 605 929, and add.

Copy 4.958 03 from Middle Dial as C_1.

(14) Clear dials, set up C_0 (0.000 189 875) and add.

(15) Set up C_1 (4.958 03) and divide.

$- w = 0.000\ 038\ 296$ appears in Upper Dial.

(16) Clear Middle and Keyboard Dials, shift to 10th position, set up 1.0448, and add.

(17) Set up 1.0 and reverse multiply by Upper Dial amount that is at right of decimal (.000 038 296), reducing it to ciphers.

Root (1.044 761 704) appears in Middle Dial.

That the error is "4" in the 10th significant figure, whereas the analysis on Page 3 shows it to be "3," comes about because the Marchant does not drop off right-hand figures in producing $4.766\ 41_9$ of Step (12). Slight variations of this sort from the analysis are to be expected. The root, however, is still accurate to 9 figures, which is all that this stage of the computation is expected to obtain.

The continuation of the process with transfer factor 1.044 761 704, if it is desired to go so far, may be done in the same manner as above.

Reference is made in the analysis to "double multiplication" with carrying all products to 20 digits. This is assisted by the means mentioned in Marchant Method MM-85, "Multiplication of Large Factors."

APPENDIX -- ADVANTAGES OF THE BIRGE-VIETA METHOD

Dr. Birge gives the following reasons why the Vieta process, when adapted to a calculator, is to be preferred, as compared with the more commonly used Ruffini-Horner Method. These advantages are in addition to the extra speed of the Vieta process because of there being fewer steps.

(1) One always deals with the <u>same</u> original coefficients (which often contain relatively few significant figures), instead of with constantly new sets of coefficients, which inevitably get more complex, as in the R-H method.

(2) Any <u>error</u> in the calculation affects <u>only</u> the particular transfer being made, and can never affect the final result. The same thing is true for the Newton iteration method, and constitutes the greatest advantage of that method. Thus, due to an error, a certain approximation may be poorer than the preceding approximation, but this fact immediately shows up in the <u>next</u> approximation. In other words, $p_1\ p_2\ p_3$ should constitute a series of numbers that rapidly settles down to a <u>constant</u> value, just as $x_1\ x_2\ x_3$ etc. in Newton's iteration method (for square roots, etc.) rapidly become constant.

But in the R-H method, <u>any</u> error makes the new function incorrect, and since we <u>then</u> proceed to get the root of the <u>new</u> function, the final result is necessarily incorrect. In other words, any such error carries through to the end. This advantage of the Vieta method over the R-H method can scarcely be overemphasized, and should be alone sufficient to make the R-H method completely obsolete.

(3) In the Vieta method the transfer factors p_1 p_2 etc., are all approximately the same size, and since the original coefficients are always used (advantage (1), <u>all</u> corresponding products and sums appearing in successive Horner shifts are approximately the same. Hence we do not need to figure the position of the decimal point, after the first Horner shift has been made. This fact is of great advantage in avoiding errors, and it results in much time saved.

(4) As already stated, one needs to calculate <u>only</u> the first two coefficients of each new function, whereas all coefficients must be calculated in the R-H method.

(5) In calculating these first two coefficients, we do not need, at first, to get the various sums and products to the <u>final</u> desired accuracy (as <u>is</u> necessary in the R-H method).

MARCHANT METHODS

THE BIRGE-VIETA METHOD
of
FINDING REAL ROOTS OF RATIONAL INTEGRAL FUNCTION

PREFACE: *Few realize the extent that classical mathematical methods have evolved under the control of the "parameter" (to use a mathematician's word) that pencil-and-paper shall be used in the calculations required by such methods. If the modern calculating machine had been available to the mathematicians of the Renaissance, it is possible that even such a familiar tool as the Briggs Logarithm might not have been developed. Certainly the art would have progressed along far different lines if from the start there had been available a machine that could multiply or divide as rapidly as one could enter amounts in a keyboard.*

The disclosure herein is an interesting example of how an early method, which was discarded because it involved so much numerical computation that was "unfit for a Christian," to quote from a writer of that day, has now been found to possess decided advantages when compared with methods that displaced it. This is because present-day calculating machines remove the drudgery element which caused the method to be relegated to the shelf over 200 years ago.

The method to which we refer was originally proposed by Francis Vieta (1540-1603). Raymond T. Birge, Ph.D., Professor of Physics and Chairman of the Department, University of California, is responsible for re-establishing it as a modern computing tool. Dr. Birge has noted that it possessed many advantages over the methods that have been developed to take its place (merely because of the excessive amount of pencil-and-paper work that it entailed).

In applying the Vieta method to the modern calculating machine, Dr. Birge has reduced it to simple systematic procedure that permits speedy determination of the root under conditions of controlled accuracy.

USES OF THE BIRGE-VIETA METHOD: The method is ideal for finding a real root of the usual algebraic equation when rough approximation of the root is known, particularly if the equation is of higher degree than the second. It is also excellent for solving transcendental equations (those that involve logarithmic or trigonometric functions in combination with analytic functions), particularly when the equations are in such form that substitutions of odd amounts in the equations or in their first derivatives are difficult. Inasmuch as the usual problem of inverse curvilinear interpolation is one of finding the root when the value of the function is a given amount, it will be seen that the Birge-Vieta method is adapted to such work, assuming of course that the tabular values are first expressed as an Interpolation Polynomial of degree "n" that fits n + 1 equidistant values of such tabulated function (See Marchant Method MM-226).

In the case of solving equations involving transcendental functions, tabular values are, likewise, obtained. An Interpolation Polynomial is then fitted to the values and then solved for the desired root. If, however, the equation has a simple first derivative and substitution of amounts in the original equation or its first derivative is not too difficult, the Newton-Raphson Method of obtaining the root is to be preferred.

(over)

OUTLINE: It is assumed that the reader is familiar with the usual Horner Synthetic Division process which is described in most College Algebra texts. However, a Note is appended which describes this procedure in a way that will enable it to be understood by a computer who is not familiar with it. (See top of Page 4).

An algebraic statement of the sample computation is given. This is followed by detailed instructions for performing the work on a Marchant Silent-Speed calculator. An Appendix then states the particular advantages of the Birge-Vieta Method, as compared with methods that are ordinarily used for such work.

The symbolism of the Horner Method is employed insofar as possible.

EXAMPLE: Find correctly to nine figures the real root nearest to $x = 1.0$ of the following equation:

$$y = g\ (x) = x^5 - x - 0.2 \text{ (true value is } 1.044\ 761\ 700_{07})$$

Assume $x = +1 = p_1$ as first approximation of the root.

I Transfer from $g\ (x)$ to $g'\ (x - p_1) = g'\ (x - 1) = g'\ (u)$
Transfer factor, $p_1 = +1$. Apply Horner Shift for A_0 and A_1 (See Note A, Page 4).

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	1	0	0	0	-1	-0.2
		1	1	1	1	0
	1	1	1	1	0	$-0.2 = A_0$
		1	2	3	4	
	1	2	3	4	$4 = A_1$	

Therefore $\quad u = -\frac{A_0}{A_1} = -\frac{-0.2}{4} = +0.05 = x - p_1$

or $\quad x = p_1 - \frac{A_0}{A_1} = 1.0 + 0.05 = 1.05 = p_2$, as second approximation.

It will be noted that the above represents the first steps of an ordinary Horner synthetic division. Only A_0 and A_1 need be found.

II Transfer from $g(x) = g''(x - p_2) = g''(x - 1.05) = g''(v)$.

It is a characteristic of this method that the calculations need be carried <u>only</u> to the reliability that the ratio of the next coefficients (in this case, B_0 and B_1) is likely to have. A practical rule is to carry <u>twice</u> as many <u>decimal</u> places in all sums and products used in obtaining B_0 as there are decimal places in the transfer factor. Hence, since 1.05 is the transfer factor, carry B_0 calculations to <u>four</u> decimal places. We find, in this problem, three significant figures for B_0, and hence carry all calculations for B_1 to at least <u>three</u> significant figures (it is really simpler to carry four and round off to three).

We now return to the <u>original</u> coefficients, an essential of the method, and one of its best features from the viewpoint of accuracy control.

$p_2 = 1.05$ transfer factor.

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	1	0	0	0	- 1	- 0.2
		1.05	1.1025	1.1576_2	$+ 1.2155_{01}$	$+ 0.2262_{76}$
	1	1.05	1.1025	1.1576_2	$+ 0.2155_{01}$	$+ 0.0263 = B_0$
		1.05	2.2050	3.472	4.860	
	1	2.10	3.307	4.629	$5.075 = B_1$	

By inspection $v = - B_0/B_1$ will have two ciphers. Therefore, by rule given, the ratio should be correct to two significant figures.

$$\text{Therefore} \qquad v = -\frac{B_0}{B_1} = -\frac{+0.0263}{5.07} = -0.005187$$

rounded to -0.0052

It will be noted that four decimal places carried in the B_0 calculations were sufficient to give B_0 to three significant figures, as is desired in order to be sure that B_0/B_1 is correctly calculated to two significant figures (i.e., in addition to the two ciphers with which it starts).

$$\text{Therefore} \qquad x = p_2 - \frac{B_0}{B_1} = 1.05 - 0.0052 = 1.0448 = p_3, \text{ as second approximation.}$$

III Transfer from $g(x)$ to $g'''(x - p_3) = g'''(x - 1.0448) = g'''(w)$
$p_3 = 1.0448$ transfer factor

As before, since there are four decimal places in $v = - B_0/B_1$ or in p_3, the next transfer factor, we carry eight decimal places in getting C_0, i.e., close to full capacity of a ten-key calculator, so for simplicity the full ten-key capacity is utilized. Then from the C_0 result, carry only six significant figures in computing C_1.

x^5	x^4	x^3	x^2	x^1	x^0
1	0	0	0	-1	-0.2
	1.0448	1.091 607 04	$1.140\ 511\ 035_4$	$+1.191\ 605\ 929_4$	$+0.200\ 189\ 874_6$
1	1.0448	1.091 607 04	$1.140\ 511\ 035_4$	$+0.191\ 605\ 929_4$	$+0.000\ 189\ 875 = C_0$
	1.0448	$2.183\ 21_4$	$3.421\ 53_2$	$+4.766\ 41_9$	
1	2.0896	3.274 82	4.562 04	$4.958\ 02 = C_1$	

There will be four ciphers in C_0/C_1, therefore carry five or six significant figures.

$$\text{Therefore} \qquad w = -\frac{C_0}{C_1} = -\frac{+0.000\ 189\ 875}{4.958\ 02} = -0.000\ 038\ 296\ 5_4$$

This ratio should be satisfactory to four significant figures. However, we retain five as this is to be the final approximation.

$$\text{Therefore} \qquad x = p_3 - \frac{C_0}{C_1} = 1.0448 - 0.000\ 038\ 296_{54} = 1.044\ 761\ 703_{46} = p_4$$

This root should be accurate to nine digits. It is seen that the error is 0.3 in the 9th digit.

A continuation of this process with transfer factor 1.044 761 703 gives $D_0 = +0.000\ 000\ 014_3$ and $D_1 = C_1$ (closely enough) $= 4.958$

$$\text{Therefore} \qquad -D_0/D_1 = -0.000\ 000\ 002_9 \text{ or } p_5 = p_4 - D_0/D_1 = 1.044\ 761\ 700_1$$

which is correct to ten figures.

An alternate continuation process is to use $p_4 = 1.044\ 761\ 7$ as transfer factor and by double multiplication (see Marchant Method MM-85) carry all products to full 20-

(over)

digit capacity of the calculator, thus producing p_5 correct to 18 or 19 digits (1.044 761 700 075 552 795).

Note that the actual error in p_1 is -0.045, in p_2 is +0.0052, in p_3 is +0.000 038, and in p_4 is +0.000 000 003. Thus, each approximation is correct to about double the number of digits of its predecessor. This is a characteristic feature of the present method. For this reason, p_5 <u>should</u> be correct to about 18 digits.

NOTE A - THE HORNER SHIFT

For those not familiar with the Horner Shift, the procedure is easily understood by reference to the calculation for B_0/B_1 on Page 2, with factors manipulated as below:

Transfer factor p

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	a	b	c	d	e	f
of x^n		pa	pm	pn	po	pq
	a	m	n	o	q	$r = B_0$

in which $m = b + pa$, $n = c + pm$, $o = d + pn$, etc. and similarly for the next row that produces B_1.

MATHEMATICAL BASIS OF METHOD

The Birge-Vieta process obtains the value of the function and of its first derivative when the approximate roots (the transfer factors) are substituted for "x." That part of the process which obtains A_0, B_0, C_0, etc., obtains successively more accurate values of the function, and A_0, B_0, and C_0, etc. are these successive values. The step that obtains A_1, B_1, C_1, etc., similarly obtains successively more accurate values of the first derivative when the transfer factors are substituted for "x." This is done, however, not by duplicating the first step with respect to the equation of the first derivative of the function but by taking advantage of partial products and sums developed during the first step. This makes it unnecessary to set up the equation of the derived function.

The successive transfer factors may have the same or different signs. Under some conditions they may alternate in sign.

COMBINING SUBSTITUTION METHODS WITH THE BIRGE-VIETA PROCESS

Inasmuch as A_0, B_0, C_0, etc are the values of the function when the transfer factor is substituted for "x," and A_1, B_1, C_1, etc., are the first derivatives of the function with respect to "x" when the transfer factor is likewise substituted for "x," there will be cases in which the first two steps of the computation may be more easily done by taking advantage of these facts, using a Table of Powers for direct computation of these amounts. This plan reserves the Birge-Vieta process for cases in which direct substitution is not easy and where the first derivative also is not easy to compute, which by the premise at bottom of Page 1 is its indicated use, anyway. These conditions usually are met when the transfer factor exceeds three figures if a Table of Powers of three-figure amounts is available. It is met with two-digit transfer factors if a Table of Powers is not available, (assuming, of course, that the usual small powers of integers 1 to 9 are known).

A readily available table of "First Ten Powers of the Integers from 1 to 1000" is that of Works Project for Computation of Mathematical Tables, Table MT-1, Information Section, National Bureau of Standards, Washington, D. C.; price 50 cents.

It happens that the example used to illustrate this method is in such form that with the aid of a Table of Powers of three-figure amounts the results of the second section may be obtained somewhat faster by substitution. (The work of the first section is obviously merely a matter of inspection.)

As an example of this straight substitution, let us apply it to this second section. We first note that the powers of 1.05, to four decimals, are $x^5 = 1.2763$ and $x^4 = 1.2155$, (these are the only powers needed for substituting in the equation or in its first derivative).

From this, we have $1.05^5 - 1.05 - 0.2 = 0.0263 = B_0$

and its first derivative

$$5 \times 1.05^4 - 1 = 5.0775 = B_1$$

APPLICATION OF THE BIRGE-VIETA METHOD TO THE MARCHANT CALCULATOR

The skilled computer who prefers to add or subtract mentally, or who wishes to use auxiliary means for such addition or subtraction doubtless would prefer to set up the transfer factor as a constant in the Keyboard Dial and multiply by the various factors as needed. The amounts are then entered on a work sheet exactly as shown in the above analysis.

Others will wish to perform all additions and subtractions on the Marchant. The detailed Marchant operations for this procedure, when applied to the calculation of p_4, are as follows:

OPERATIONS: Decimals: Upper Dial 9, Middle Dial 18, Keyboard Dial 9. Use any 10 column Model M Marchant with Upper Green Shift Key down.

Inasmuch as the coefficient of x^5 is 1.0, the calculator computation is started for development of the x^3 column; thus,

(1) Set up 1.0448 and multiply by transfer factor (1.0448).

Copy 1.091 607 04 from Middle Dial to x^3 column.

(2) As there is no amount to add to this, the normal adding step is skipped. Shift to 10th position, clear Upper and Keyboard Dials, and copy Middle Dial amount (1.091 607 04) into Keyboard Dial, clear Middle Dial and multiply by transfer factor (1.0448).

Copy $1.140\ 511\ 035_4$ from Middle Dial to x^2 column.

(3) Repeat Step (2) with Keyboard Dial setup of 1.140 511 035.

(4) Clear Keyboard Dial, shift to 10th position, set up 1.0, and subtract.

Copy $0.191\ 605\ 929_4$ from Middle Dial to x^1 column.

(5) Repeat Step (2) with Keyboard Dial setup of 0.191 605 929.

(6) Clear Keyboard Dial, shift to 10th position, set up 0.2, and subtract.

Copy C_0 (0.000 189 875) to x^0 column.

(7) Clear all dials, set up 1.0448, and multiply by 2.0.

Copy 2.0896 from Middle Dial to x^4 column.

(8) Shift to 10th position, clear Keyboard Dial and transfer Middle Dial amount (2.0896) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(over)

(9) Shift to 10th position, clear Keyboard Dial, set up 1.091 607 04, and add.

Copy 3.274 82 from Middle Dial to x^3 column.

(10) Transfer Middle Dial amount (3.274 82) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(11) Shift to 10th position, clear Keyboard Dial, set up 1.140 511 035, and add.

(12) Transfer Middle Dial amount (4.562 04) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(13) Shift to 10th position, clear Keyboard Dial, set up 0.191 605 929, and add.

Copy 4.958 03 from Middle Dial as C_1.

(14) Clear dials, set up C_0 (0.000 189 875) and add.

(15) Set up C_1 (4.958 03) and divide.

$- w = 0.000\ 038\ 296$ appears in Upper Dial.

(16) Clear Middle and Keyboard Dials, shift to 10th position, set up 1.0448, and add.

(17) Set up 1.0 and reverse multiply by Upper Dial amount that is at right of decimal (.000 038 296), reducing it to ciphers.

Root (1.044 761 704) appears in Middle Dial.

That the error is "4" in the 10th significant figure, whereas the analysis on Page 3 shows it to be "3," comes about because the Marchant does not drop off right-hand figures in producing $4.766\ 41_9$ of Step (12). Slight variations of this sort from the analysis are to be expected. The root, however, is still accurate to 9 figures, which is all that this stage of the computation is expected to obtain.

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MARCHANT — SILENT SPEED — METHODS

MM-228
MATHEMATICS
Sept., 1942

TABLE OF 5-POINT LAGRANGEAN INTERPOLATION COEFFICIENTS
(From 0 to 2, Argument 0.001, 7-Place)

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PREFACE

The tables reproduced herein were computed at the request of the Chief of Ordnance, U. S. Army (O.O. 063.2/33, July 24, 1940, from the Chief of Ordnance to Dr. Arnold N. Lowan, Project Supervisor for Works Projects Administration for New York City) by the project for "Computation of Mathematical Tables," O.P. 65-2-97-33, conducted by the Works Projects Administration for New York City under the sponsorship of the National Bureau of Standards, Dr. Lyman J. Briggs, Director.

These 5-Point Values were originally issued in June, 1941 for private distribution in the War Department. They have just been released for public reference, but inasmuch as the volume of tables which is to include these 5-Point Values will not be ready for some time, permission has been obtained to reproduce them in this Advance Form.

From the Preface of the original privately circulated edition of these tables, signed by C. M. Wesson, Major General, Chief of Ordnance, we quote:

"Acknowledgment is made to Arnold N. Lowan, Ph. D., Murray Pfeferman, Milton Abramowitz, M.S.; Gertrude Blanch, Ph.D; Jack Laderman, M.A.; Jacob L. Miller, B.A.; Matilda M. Persily, B.A.; and Hyman Serbin, Ph.D., of the administrative and technical staff of the project, for their work in computing and editing this table. Further acknowledgment is made to Professor J.A. Shohat, Mathematics Department, University of Pennsylvania, and to Dr. L. W. Dederick of the Aberdeen Proving Ground, Maryland, for their many valuable suggestions."

Particular attention is directed to the "Introduction" written by Dr. Arnold N. Lowan, which appears on the following pages.

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BIBLIOGRAPHY OF TABLES

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E. V. Huntington. Tables of Lagrangean Coefficients for Interpolating without Differences. Am. Acad. Arts & Sci., vol. 63 (1929) pp. 421-437. Cubic (four-point) interpolants. Tables 3 and 3a; [0(.01)1] for interpolating in main part of a table, and at the beginning or end, respectively. Quintic (six-point). Tables 5, 5a, 5b; [0(.01)1;8D] for interpolating in main part of a table in first interval, and in second interval of a table, respectively. Tables 30a, 30, 50a, 50b, 50 (cumulative): - short tables for continuous interpolation at intervals of 1/5, 2/5, 3/4, 4/5, 1.

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Karl Pearson (editor). Tracts for Computers. No. II, Part I pp. 31-46. Cambridge Univ. Press, London (1920). Short tables of coefficients in the 4, 5, 6, 7, 8, 9, 10, 11-point Lagrangean formulae, with considerable expository material.

NOTE: The figures in the square brackets indicate the range, interval, and number of decimal places in the tabulated entries. For example: In Kelley's table, the cubic interpolants are given for n ranging from 0 to 1, at intervals of 0.001 to 10 decimal places.

INTRODUCTION

The general Lagrangean interpolation formula of degree (n-1) is expressible as follows:

$$(1)\quad f(x) = \frac{(x-a_2)(x-a_3)\dots(x-a_n)}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)} f(a_1) + \frac{(x-a_1)(x-a_3)\dots(x-a_n)}{(a_2-a_1)(a_2-a_3)\dots(a_2-a_n)} f(a_2)$$

$$+ \dots + \frac{(x-a_1)(x-a_2)\dots(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\dots(a_n-a_{n-1})} f(a_n) + R_n ,$$

$$(2)\quad R_n = \frac{(x-a_1)(x-a_2)\dots(x-a_n)}{n!} f^{(n)}(y) .$$

In the above expressions, $a_1, a_2, \dots a_n$ are n arguments or points for which the values of f(x) are known, and $f^{(n)}(y)$ is the n^{th} derivative of f(x) at some point y in the interval enclosing the points $a_1, a_2, \dots a_n$ and x.

When f(x) is a polynomial of degree no higher than (n-1), R_n is identically zero. In the case where f(x) is a continuous function with n continuous derivatives, an approximate value of f(x) may be calculated from (1) by neglecting R_n. In that case, an upper bound of the error in the approximation is obtainable by replacing $f^{(n)}(y)$, in (2), by the upper bound of the numerical value of the n^{th} derivative in the range under consideration. Formula (1) is sometimes called an "n-point" interpolation formula, because the value of f(x) is made to depend on n given values of the function.

The Lagrangean formula may also be used for interpolation in a table where the points $a_1, a_2, \dots a_n$ are uniformly spaced -- at intervals h, say. In that case formula (1) becomes considerably simplified. For the case when n = 5, let the five consecutive arguments be designated by a_{-2}, a_{-1}, a_0, a_1, a_2, and let

$$(3)\quad x = a_0 + ph .$$

Replacing the above value of x in formulae (1) and (2), the following formulae are obtained when n = 5:

$$(4)\quad f(x) = \frac{p(p^2-1)(p-2)}{24} f(a_{-2}) - \frac{p(p-1)(p^2-4)}{6} f(a_{-1}) + \frac{(p^2-1)(p^2-4)}{4} f(a_0)$$

$$- \frac{p(p+1)(p^2-4)}{6} f(a_1) + \frac{p(p^2-1)(p+2)}{24} f(a_2) + R_5 ,$$

$$(4a)\quad R_5 = \frac{p(p^2-1)(p^2-4)\, h^5 f^{(5)}(y)}{120}$$

In practice, the value of $f^{(5)}(y)$ is sometimes difficult to determine. However, when the interval h is relatively small, and the values of the function are known for several more adjacent (equally spaced) arguments, $h^5 f^{(5)}(y)$ may be estimated from the fifth differences of the entries in the region under consideration. Whenever R_5 is relatively small, these fifth differences will generally not vary appreciably among themselves. In that case, the fifth difference corresponding to the argument a_{-2}, say, may replace $h^5 f^5(y)$, if $f(a_3)$ is known. In other words

$$(4b)^*\quad R_5 \sim \frac{p(p^2-1)(p^2-4)}{120} [-f(a_{-2})+5f(a_{-1})-10f(a_0)+10f(a_1)-5f(a_2)+f(a_3)]$$

Formula (4), with R_5 omitted, is designated as the <u>five-point Lagrangean</u> interpolation formula, and the coefficients of $f(a_k)$, k = -2, -1, 0, 1, 2, are tabulated in this volume to seven decimals. The values of p range from 0 to 2 at intervals of 0.001. Specifically

$$A_{-2} = p(p^2-1)(p-2)/24 \ ; \ A_{-1} = -p(p-1)(p^2-4)/6 \ ;$$

$$A_0 = (p^2-1)(p^2-4)/4 \ ; \ A_1 = -p(p+1)(p^2-4)/6 \ ;$$

$$A_2 = p(p^2-1)(p+2)/24$$

It should be noted that for every value of p, $(A_{-2}+A_{-1}+A_0+A_1+A_2)$ = 1 theoretically. Since the entries in this table are rounded numbers, the sum of the five coefficients may occasionally differ from unity by as much as two units in the last decimal place.

INTERPOLATION BY MEANS OF LAGRANGEAN COEFFICIENTS

The following examples will illustrate the application of formula (4). Let it be required to find x^8 corresponding to x = 4.3239 from a table listing x^8 at intervals of 0.1.

*The symbol $\sim$ will be used to denote approximate equality.

LAGRANGEAN INTERPOLATION FORMULA

Solution. The work may be conveniently arranged as follows:

$$p = .239$$

a_k	$a_{-2} = 4.1$	$a_{-1} = 4.2$	$a_0 = 4.3$	$a_1 = 4.4$	$a_2 = 4.5$
$f(a_k)$	79849.3	96826.5	116882.0	140482.2	168151.3
$A_k(p)$	.0165349	-.1195211	.9294145	.1945949	-.0210231

(5) $f(x) = (79849.3)(.0165349) - (96826.5)(.1195211) + (116882.0)(.9294145)$
$+ (140482.2)(.1945949) - (168151.3)(.0210231) = 122181.4$.

An upper bound of R_5 may be obtained either from (4a) or (4b). Using (4a), we have

$$h^5 f^{(5)}(y) < h^5 f^{(5)}(4.5) < 6.2;\ \frac{p(p^2-1)(p^2-4)}{120} < 0.0075$$

hence $0 < R_5 < (6.2)(0.0075) < 0.05$

When R_5 is estimated from (4b), $h^5 f^{(5)}(y)$ may be replaced by the fifth difference of the entries in the region under consideration; the value of f(x) for another point in the region - say $f(a_3)$ - is therefore needed. From a table of x^8, it is found that f(4.6) = 200476.1, and it may be verified that the fifth differences in this region are less than 6.4; hence the upper bound of R_5 previously obtained is verified from (4b). By direct computation, $(4.3239)^8 = 122181.41$. The computed value is therefore correct in this instance to seven significant figures. Rounding errors inherent in the terms of (5) may sometimes occasion an error of somewhat more than a unit in the seventh significant figure of the computed value of f(x), even when R_5 is negligible; but this rounding error will generally be less than two units in the seventh significant figure of f(x).

Sometimes it is required to interpolate between the last two entries of a table, or between the first two entries. In that case, interpolation must be made to depend on entries all of which are on one side of a_o and a_1. The problem of interpolating near the end of a table may be easily solved. For example, in the previous problem, let it be assumed that the table ends with x = 4.4. The points on which interpolation is made to depend are now as follows: $a_{-2} = 4.0$; $a_{-1} = 4.1$; $a_o = 4.2$; $a_1 = 4.3$; $a_2 = 4.4$. As before $x = a_o + ph = 4.2 + 0.1239$ hence ph = 0.1239 and p = 1.239. Since the coefficients corresponding to the above value of p are tabulated, the problem may be solved by the same method as before.

LAGRANGEAN INTERPOLATION FORMULA

The problem of interpolating near the beginning of a table remains to be considered. In the former example, let it be assumed that f(x) is known for the arguments 4.3, 4.4, 4.5, 4.6 and 4.7. If the points are taken in the above order, $a_o = 4.5$ and $x = a_o - .1761$; hence p = -1.761. It may be readily verified that $A_k(-p) = A_{-k}(p)$; that is, the coefficients A_{-2} and A_2 as well as A_{-1} and A_1 must be interchanged. One way of accomplishing this is to reverse the order of the points as follows:

$a_{-2} = 4.7$, $a_{-1} = 4.6$, $a_o = 4.5$, $a_1 = 4.4$, $a_2 = 4.3$, $x = a_o - .1761$, ph = -.1761 and since h is negative, p = 1.761. Hence the coefficients A_k may be found from this table. The actual work follows:

$$p = 1.761$$

a_k	$a_{-2} = 4.7$	$a_{-1} = 4.6$	$a_o = 4.5$	$a_1 = 4.4$	$a_2 = 4.3$
$f(a_k)$	238112.9	200476.1	168151.3	140482.2	116882.0
A_k	-.0368466	.2007678	-.4721634	.7284097	.5798324

$f(x) = -(238112.9)(.0368466) + (200476.1)(.2007678) - (168151.3)(.4721634)$
$+ (140482.2)(.7284097) + (116882.0)(.5798324)$
$= 122181.2$.

The last place is incorrect by two units. This error is partly due to the higher value of R_5, arising from the larger value of p.

If p in formula (3) is not a tabulated argument of this table, interpolation for f(x) may be performed by one of the following two methods:

a. Interpolation for coefficients. Let F_{-1}, F_o, F_1 be the values of any one of the five interpolants, corresponding to the arguments $p_o - h$, p_o, and $p_o + h$. Let, further, F_p designate the value of the interpolant for the argument $p = p_o + th$. The value of F_p may be determined as follows:

(6) $$F_p = F_o + t(F_1 - F_o) + \tfrac{1}{2}t(t-1)\,[F_{-1} - 2F_o + F_1] .$$

If the term involving $\frac{1}{2}t(t-1)$ is neglected, the value of F_p will generally be correct to within two units in the sixth decimal place.

Example: Let it be required to find the coefficients A_k corresponding to p = .2394762.

Solution: $p_o = .239$, $p_o + h = .240$, $p_o - h = .238$; t = .4762; $\frac{1}{2}t(t-1) =$

-.137. With the aid of the tables of A_k, the calculations may be conveniently arranged as follows:

$$p = .2394762$$

		A_{-2}	A_{-1}	A_0	A_1	A_2
p = .238	F_{-1} =	.0164834	-.1191919	.9299971	.1936477	-.0209364
p = .239	F_0 =	.0165349	-.1195211	.9294145	.1945949	-.0210231
p = .240	F_1 =	.0165862	-.1198490	.9288294	.1955430	-.0211098
$C = F_0 - F_{-1}$	=	.0000515	-.0003292	-.0005826	.0009472	-.0000867
$D = F_1 - F_0$	=	.0000513	-.0003279	-.0005851	.0009481	-.0000867
D-C	=		.0000013	-.0000025	.0000009	
t D	=	.00002442	-.00015614	-.00027862	.00045149	-.00004129
$\frac{1}{2}t(t-1)(D-C) = E$	=		-.00000018	+.00000034	-.00000012	
$F + tD + E = A_k(p)$	=	.0165593	-.1196774	.9291362	.1950463	-.0210644

As a check on the accuracy of the computations, it may be verified that the sum of the coefficients $A_k(p)$ is equal to unity. The value of the function may then be obtained by the method already illustrated.

b. Alternative method. In the given example, the values of f(x) may be found for p = 0.239, 0.240, and 0.241 by the usual method. The results are as follows:

x	f(x)	diff.
4.3239	122181.4	
		22.5
4.3240	122203.9	
		22.7
4.3241	122226.6	

Since the first differences are almost constant, linear interpolation for f(x) in the above values is adequate. The calculation follows:

$$f(4.32394762) = 122181.4 + .4762(22.5) = 122192.1\,.$$

If the first differences are appreciably different, a closer approximation to f(x) may be obtained with the aid of formula (6), where now the F's refer to the function f(x) under consideration.

ADVANTAGES OF THE LAGRANGEAN FORMULA

When adequate tables of Lagrangean interpolants and a calculating machine are at hand, interpolation by means of the Lagrangean formula is simple and expedient. The partial products $A_k f(a_k)$ can be accumulated in the machine and the final result obtained with a minimum of subsidiary hand computations. This method is particularly advantageous when interpolation is performed in a table which does not list differences. Several good tables of four-point interpolants are already available* and it is hoped the present tables will give the Lagrangean formulae the greater prominence they deserve.

ARNOLD N. LOWAN

*See Bibliography

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.000	+0.0000 000	-0.0000 000	+1.0000 000	+0.0000 000	-0.0000 000
01	.0000 833	.0006 660	+0.9999 988	.0006 673	.0000 834
02	.0001 665	.0013 307	.9999 950	.0013 360	.0001 668
03	.0002 496	.0019 940	.9999 888	.0020 060	.0002 504
04	.0003 327	.0026 560	.9999 800	.0026 773	.0003 340
0.005	+0.0004 156	-0.0033 166	+0.9999 688	+0.0033 500	-0.0004 177
06	.0004 985	.0039 760	.9999 550	.0040 240	.0005 015
07	.0005 813	.0046 339	.9999 388	.0046 993	.0005 853
08	.0006 640	.0052 906	.9999 200	.0053 759	.0006 693
09	.0007 466	.0059 459	.9998 988	.0060 539	.0007 533
0.010	+0.0008 291	-0.0065 998	+0.9998 750	+0.0067 332	-0.0008 374
11	.0009 115	.0072 524	.9998 488	.0074 138	.0009 216
12	.0009 939	.0079 037	.9998 200	.0080 957	.0010 059
13	.0010 761	.0085 536	.9997 888	.0087 790	.0010 902
14	.0011 583	.0092 022	.9997 550	.0094 635	.0011 746
0.015	+0.0012 403	-0.0098 494	+0.9997 188	+0.0101 494	-0.0012 591
16	.0013 223	.0104 953	.9996 800	.0108 366	.0013 437
17	.0014 042	.0111 399	.9996 388	.0115 252	.0014 283
18	.0014 860	.0117 830	.9995 950	.0122 150	.0015 130
19	.0015 677	.0124 249	.9995 488	.0129 062	.0015 978
0.020	+0.0016 493	-0.0130 654	+0.9995 000	+0.0135 986	-0.0016 827
21	.0017 309	.0137 045	.9994 488	.0142 924	.0017 676
22	.0018 123	.0143 423	.9993 951	.0149 875	.0018 526
23	.0018 936	.0149 787	.9993 388	.0156 839	.0019 377
24	.0019 749	.0156 138	.9992 801	.0163 816	.0020 228
0.025	+0.0020 560	-0.0162 475	+0.9992 188	+0.0170 807	-0.0021 081
26	.0021 371	.0168 798	.9991 551	.0177 810	.0021 933
27	.0022 180	.0175 108	.9990 889	.0184 826	.0022 787
28	.0022 989	.0181 404	.9990 202	.0191 856	.0023 641
29	.0023 796	.0187 687	.9989 489	.0198 898	.0024 496
0.030	+0.0024 603	-0.0193 956	+0.9988 752	+0.0205 954	-0.0025 352
31	.0025 408	.0200 212	.9987 990	.0213 022	.0026 209
32	.0026 213	.0206 454	.9987 203	.0220 104	.0027 066
33	.0027 017	.0212 682	.9986 390	.0227 198	.0027 923
34	.0027 819	.0218 897	.9985 553	.0234 306	.0028 782
0.035	+0.0028 621	-0.0225 098	+0.9984 691	+0.0241 426	-0.0029 641
36	.0029 422	.0231 285	.9983 804	.0248 559	.0030 500
37	.0030 221	.0237 459	.9982 892	.0255 706	.0031 361
38	.0031 020	.0243 619	.9981 955	.0262 865	.0032 222
39	.0031 818	.0249 765	.9980 993	.0270 037	.0033 083
0.040	+0.0032 614	-0.0255 898	+0.9980 006	+0.0277 222	-0.0033 946
41	.0033 410	.0262 017	.9978 995	.0284 420	.0034 808
42	.0034 205	.0268 122	.9977 958	.0291 631	.0035 672
43	.0034 998	.0274 213	.9976 896	.0298 855	.0036 536
44	.0035 791	.0280 291	.9975 809	.0306 092	.0037 401
0.045	+0.0036 582	-0.0286 355	+0.9974 698	+0.0313 341	-0.0038 266
46	.0037 372	.0292 405	.9973 561	.0320 604	.0039 132
47	.0038 162	.0298 442	.9972 400	.0327 879	.0039 999
48	.0038 950	.0304 465	.9971 213	.0335 167	.0040 866
49	.0039 737	.0310 474	.9970 002	.0342 468	.0041 733
0.050	+0.0040 523	-0.0316 469	+0.9968 766	+0.0349 781	-0.0042 602

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.050	+0.0040 523	-0.0316 469	+0.9968 766	+0.0349 781	-0.0042 602
51	.0041 309	.0322 450	.9967 504	.0357 108	.0043 470
52	.0042 093	.0328 418	.9966 218	.0364 447	.0044 340
53	.0042 875	.0334 372	.9964 907	.0371 799	.0045 210
54	.0043 657	.0340 312	.9963 571	.0379 163	.0046 080
0.055	+0.0044 438	-0.0346 238	+0.9962 210	+0.0386 541	-0.0046 951
56	.0045 218	.0352 150	.9960 825	.0393 931	.0047 823
57	.0045 996	.0358 049	.9959 414	.0401 334	.0048 695
58	.0046 774	.0363 934	.9957 978	.0408 749	.0049 568
59	.0047 550	.0369 805	.9956 518	.0416 178	.0050 441
0.060	+0.0048 325	-0.0375 662	+0.9955 032	+0.0423 618	-0.0051 315
61	.0049 100	.0381 505	.9953 522	.0431 072	.0052 189
62	.0049 873	.0387 334	.9951 987	.0438 538	.0053 064
63	.0050 644	.0393 150	.9950 427	.0446 017	.0053 939
64	.0051 415	.0398 951	.9948 842	.0453 508	.0054 815
0.065	+0.0052 185	-0.0404 739	+0.9947 232	+0.0461 013	-0.0055 691
66	.0052 953	.0410 512	.9945 597	.0468 529	.0056 568
67	.0053 721	.0416 272	.9943 938	.0476 058	.0057 445
68	.0054 487	.0422 018	.9942 253	.0483 600	.0058 322
69	.0055 252	.0427 750	.9940 544	.0491 155	.0059 201
0.070	+0.0056 016	-0.0433 468	+0.9938 810	+0.0498 722	-0.0060 079
71	.0056 779	.0439 173	.9937 051	.0506 301	.0060 958
72	.0057 540	.0444 863	.9935 267	.0513 893	.0061 838
73	.0058 301	.0450 539	.9933 458	.0521 498	.0062 718
74	.0059 060	.0456 201	.9931 625	.0529 115	.0063 598
0.075	+0.0059 818	-0.0461 850	+0.9929 767	+0.0536 744	-0.0064 479
76	.0060 575	.0467 484	.9927 883	.0544 386	.0065 360
77	.0061 330	.0473 104	.9925 975	.0552 041	.0066 242
78	.0062 085	.0478 711	.9924 043	.0559 707	.0067 124
79	.0062 838	.0484 303	.9922 085	.0567 387	.0068 007
0.080	+0.0063 590	-0.0489 882	+0.9920 102	+0.0575 078	-0.0068 890
81	.0064 341	.0495 446	.9918 095	.0582 783	.0069 773
82	.0065 091	.0500 996	.9916 063	.0590 499	.0070 657
83	.0065 840	.0506 533	.9914 006	.0598 228	.0071 541
84	.0066 587	.0512 055	.9911 924	.0605 969	.0072 425
0.085	+0.0067 333	-0.0517 563	+0.9909 818	+0.0613 723	-0.0073 310
86	.0068 078	.0523 058	.9907 687	.0621 489	.0074 195
87	.0068 821	.0528 538	.9905 531	.0629 267	.0075 081
88	.0069 564	.0534 004	.9903 350	.0637 058	.0075 967
89	.0070 305	.0539 456	.9901 144	.0644 860	.0076 853
0.090	+0.0071 045	-0.0544 894	+0.9898 914	+0.0652 676	-0.0077 740
91	.0071 784	.0550 318	.9896 659	.0660 503	.0078 627
92	.0072 521	.0555 728	.9894 379	.0668 343	.0079 515
93	.0073 257	.0561 124	.9892 075	.0676 195	.0080 402
94	.0073 992	.0566 506	.9889 745	.0684 059	.0081 290
0.095	+0.0074 726	-0.0571 873	+0.9887 391	+0.0691 935	-0.0082 179
96	.0075 458	.0577 227	.9885 012	.0699 824	.0083 067
97	.0076 189	.0582 566	.9882 609	.0707 725	.0083 956
98	.0076 919	.0587 892	.9880 181	.0715 638	.0084 846
99	.0077 648	.0593 203	.9877 728	.0723 563	.0085 735
0.100	+0.0078 375	-0.0598 500	+0.9875 250	+0.0731 500	-0.0086 625

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.100	+0.0078 375	-0.0598 500	+0.9875 250	+0.0731 500	-0.0086 625
01	.0079 101	.0603 783	.9872 748	.0739 449	.0087 515
02	.0079 826	.0609 052	.9870 221	.0747 411	.0088 406
03	.0080 549	.0614 306	.9867 669	.0755 385	.0089 296
04	.0081 271	.0619 547	.9865 092	.0763 370	.0090 187
0.105	+0.0081 992	-0.0624 773	+0.9862 491	+0.0771 368	-0.0091 078
06	.0082 712	.0629 985	.9859 866	.0779 378	.0091 970
07	.0083 430	.0635 183	.9857 215	.0787 400	.0092 862
08	.0084 147	.0640 367	.9854 540	.0795 434	.0093 754
09	.0084 863	.0645 537	.9851 840	.0803 480	.0094 646
0.110	+0.0085 577	-0.0650 692	+0.9849 116	+0.0811 538	-0.0095 538
11	.0086 290	.0655 834	.9846 367	.0819 608	.0096 431
12	.0087 001	.0660 961	.9843 593	.0827 690	.0097 324
13	.0087 712	.0666 074	.9840 795	.0835 783	.0098 217
14	.0088 421	.0671 172	.9837 972	.0843 889	.0099 110
0.115	+0.0089 128	-0.0676 257	+0.9835 125	+0.0852 007	-0.0100 003
16	.0089 835	.0681 327	.9832 253	.0860 137	.0100 897
17	.0090 540	.0686 383	.9829 356	.0868 278	.0101 791
18	.0091 243	.0691 425	.9826 435	.0876 432	.0102 685
19	.0091 946	.0696 452	.9823 489	.0884 597	.0103 579
0.120	+0.0092 646	-0.0701 466	+0.9820 518	+0.0892 774	-0.0104 474
21	.0093 346	.0706 465	.9817 523	.0900 963	.0105 368
22	.0094 044	.0711 449	.9814 504	.0909 164	.0106 263
23	.0094 741	.0716 420	.9811 460	.0917 377	.0107 158
24	.0095 436	.0721 376	.9808 391	.0925 602	.0108 053
0.125	+0.0096 130	-0.0726 318	+0.9805 298	+0.0933 838	-0.0108 948
26	.0096 823	.0731 246	.9802 180	.0942 086	.0109 843
27	.0097 514	.0736 160	.9799 038	.0950 346	.0110 738
28	.0098 204	.0741 059	.9795 871	.0958 617	.0111 634
29	.0098 893	.0745 944	.9792 680	.0966 901	.0112 529
0.130	+0.0099 580	-0.0750 814	+0.9789 464	+0.0975 196	-0.0113 425
31	.0100 266	.0755 671	.9786 224	.0983 502	.0114 321
32	.0100 950	.0760 513	.9782 959	.0991 821	.0115 217
33	.0101 633	.0765 340	.9779 670	.1000 151	.0116 113
34	.0102 314	.0770 154	.9776 356	.1008 492	.0117 009
0.135	+0.0102 994	-0.0774 953	+0.9773 018	+0.1016 846	-0.0117 905
36	.0103 673	.0779 738	.9769 655	.1025 211	.0118 801
37	.0104 350	.0784 508	.9766 268	.1033 587	.0119 698
38	.0105 026	.0789 264	.9762 857	.1041 975	.0120 594
39	.0105 700	.0794 006	.9759 421	.1050 375	.0121 490
0.140	+0.0106 373	-0.0798 734	+0.9755 960	+0.1058 786	-0.0122 387
41	.0107 045	.0803 447	.9752 476	.1067 209	.0123 283
42	.0107 715	.0808 145	.9748 966	.1075 644	.0124 180
43	.0108 384	.0812 830	.9745 433	.1084 089	.0125 076
44	.0109 051	.0817 500	.9741 875	.1092 547	.0125 973
0.145	+0.0109 717	-0.0822 156	+0.9738 293	+0.1101 016	-0.0126 869
46	.0110 381	.0826 797	.9734 686	.1109 496	.0127 766
47	.0111 044	.0831 424	.9731 055	.1117 988	.0128 662
48	.0111 705	.0836 037	.9727 399	.1126 491	.0129 559
49	.0112 365	.0840 635	.9723 720	.1135 005	.0130 455
0.150	+0.0113 023	-0.0845 219	+0.9720 016	+0.1143 531	-0.0131 352

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.150	+0.0113 023	-0.0845 219	+0.9720 016	+0.1143 531	-0.0131 352
51	.0113 680	.0849 788	.9716 287	.1152 069	.0132 248
52	.0114 336	.0854 343	.9712 534	.1160 617	.0133 144
53	.0114 990	.0858 884	.9708 757	.1169 177	.0134 041
54	.0115 642	.0863 410	.9704 956	.1177 749	.0134 937
0.155	+0.0116 294	-0.0867 922	+0.9701 131	+0.1186 332	-0.0135 833
56	.0116 943	.0872 420	.9697 281	.1194 926	.0136 730
57	.0117 591	.0876 903	.9693 406	.1203 531	.0137 626
58	.0118 238	.0881 371	.9689 508	.1212 147	.0138 522
59	.0118 883	.0885 826	.9685 585	.1220 775	.0139 418
0.160	+0.0119 526	-0.0890 266	+0.9681 638	+0.1229 414	-0.0140 314
61	.0120 168	.0894 691	.9677 667	.1238 065	.0141 209
62	.0120 809	.0899 102	.9673 672	.1246 726	.0142 105
63	.0121 448	.0903 499	.9669 652	.1255 399	.0143 001
64	.0122 086	.0907 881	.9665 608	.1264 083	.0143 896
0.165	+0.0122 722	-0.0912 248	+0.9661 541	+0.1272 778	-0.0144 791
66	.0123 356	.0916 602	.9657 448	.1281 484	.0145 687
67	.0123 989	.0920 941	.9653 332	.1290 201	.0146 582
68	.0124 621	.0925 265	.9649 191	.1298 930	.0147 477
69	.0125 250	.0929 575	.9645 027	.1307 669	.0148 372
0.170	+0.0125 879	-0.0933 870	+0.9640 838	+0.1316 420	-0.0149 266
71	.0126 506	.0938 151	.9636 625	.1325 181	.0150 161
72	.0127 131	.0942 418	.9632 388	.1333 954	.0151 055
73	.0127 755	.0946 670	.9628 127	.1342 738	.0151 949
74	.0128 377	.0950 908	.9623 842	.1351 532	.0152 843
0.175	+0.0128 998	-0.0955 131	+0.9619 532	+0.1360 338	-0.0153 737
76	.0129 617	.0959 340	.9615 199	.1369 155	.0154 630
77	.0130 234	.0963 534	.9610 841	.1377 982	.0155 524
78	.0130 850	.0967 714	.9606 460	.1386 821	.0156 417
79	.0131 465	.0971 879	.9602 054	.1395 670	.0157 310
0.180	+0.0132 077	-0.0976 030	+0.9597 624	+0.1404 530	-0.0158 203
81	.0132 689	.0980 166	.9593 171	.1413 402	.0159 095
82	.0133 298	.0984 288	.9588 693	.1422 284	.0159 987
83	.0133 906	.0988 395	.9584 191	.1431 177	.0160 879
84	.0134 513	.0992 488	.9579 666	.1440 080	.0161 771
0.185	+0.0135 118	-0.0996 566	+0.9575 116	+0.1448 995	-0.0162 663
86	.0135 721	.1000 630	.9570 542	.1457 920	.0163 554
87	.0136 323	.1004 679	.9565 945	.1466 857	.0164 445
88	.0136 923	.1008 714	.9561 323	.1475 804	.0165 336
89	.0137 522	.1012 735	.9556 677	.1484 761	.0166 226
0.190	+0.0138 119	-0.1016 740	+0.9552 008	+0.1493 730	-0.0167 116
91	.0138 714	.1020 732	.9547 315	.1502 709	.0168 006
92	.0139 308	.1024 708	.9542 597	.1511 699	.0168 896
93	.0139 900	.1028 671	.9537 856	.1520 699	.0169 785
94	.0140 491	.1032 618	.9533 091	.1529 710	.0170 674
0.195	+0.0141 080	-0.1036 552	+0.9528 302	+0.1538 732	-0.0171 562
96	.0141 667	.1040 470	.9523 489	.1547 764	.0172 450
97	.0142 253	.1044 375	.9518 653	.1556 807	.0173 338
98	.0142 837	.1048 264	.9513 792	.1565 861	.0174 226
99	.0143 419	.1052 139	.9508 908	.1574 925	.0175 113
0.200	+0.0144 000	-0.1056 000	+0.9504 000	+0.1584 000	-0.0176 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.200	+0.0144 000	-0.1056 000	+0.9504 000	+0.1584 000	-0.0176 000
01	.0144 579	.1059 846	.9499 068	.1593 085	.0176 886
02	.0145 157	.1063 678	.9494 112	.1602 181	.0177 773
03	.0145 733	.1067 495	.9489 133	.1611 287	.0178 658
04	.0146 307	.1071 297	.9484 130	.1620 404	.0179 544
0.205	+0.0146 880	-0.1075 085	+0.9479 103	+0.1629 531	-0.0180 429
06	.0147 450	.1078 858	.9474 052	.1638 669	.0181 313
07	.0148 020	.1082 617	.9468 978	.1647 817	.0182 197
08	.0148 587	.1086 361	.9463 879	.1656 976	.0183 081
09	.0149 153	.1090 091	.9458 758	.1666 144	.0183 964
0.210	+0.0149 718	-0.1093 806	+0.9453 612	+0.1675 324	-0.0184 847
11	.0150 281	.1097 507	.9448 443	.1684 513	.0185 730
12	.0150 842	.1101 193	.9443 250	.1693 713	.0186 612
13	.0151 401	.1104 865	.9438 033	.1702 923	.0187 493
14	.0151 959	.1108 522	.9432 793	.1712 144	.0188 374
0.215	+0.0152 515	-0.1112 164	+0.9427 529	+0.1721 375	-0.0189 255
16	.0153 069	.1115 792	.9422 242	.1730 616	.0190 135
17	.0153 622	.1119 405	.9416 931	.1739 867	.0191 015
18	.0154 173	.1123 004	.9411 596	.1749 129	.0191 894
19	.0154 722	.1126 588	.9406 238	.1758 400	.0192 772
0.220	+0.0155 269	-0.1130 158	+0.9400 856	+0.1767 682	-0.0193 651
21	.0155 815	.1133 713	.9395 451	.1776 974	.0194 528
22	.0156 360	.1137 253	.9390 022	.1786 277	.0195 405
23	.0156 902	.1140 779	.9384 570	.1795 589	.0196 282
24	.0157 443	.1144 290	.9379 094	.1804 912	.0197 158
0.225	+0.0157 982	-0.1147 787	+0.9373 595	+0.1814 244	-0.0198 034
26	.0158 519	.1151 269	.9368 072	.1823 587	.0198 909
27	.0159 055	.1154 737	.9362 526	.1832 939	.0199 783
28	.0159 589	.1158 190	.9356 956	.1842 302	.0200 657
29	.0160 121	.1161 628	.9351 363	.1851 675	.0201 530
0.230	+0.0160 652	-0.1165 052	+0.9345 746	+0.1861 058	-0.0202 403
31	.0161 181	.1168 462	.9340 106	.1870 450	.0203 275
32	.0161 708	.1171 856	.9334 443	.1879 853	.0204 147
33	.0162 233	.1175 237	.9328 756	.1889 266	.0205 018
34	.0162 757	.1178 602	.9323 046	.1898 688	.0205 888
0.235	+0.0163 279	-0.1181 953	+0.9317 312	+0.1908 121	-0.0206 758
36	.0163 799	.1185 290	.9311 555	.1917 563	.0207 627
37	.0164 317	.1188 612	.9305 775	.1927 015	.0208 496
38	.0164 834	.1191 919	.9299 971	.1936 477	.0209 364
39	.0165 349	.1195 211	.9294 145	.1945 949	.0210 231
0.240	+0.0165 862	-0.1198 490	+0.9288 294	+0.1955 430	-0.0211 098
41	.0166 374	.1201 753	.9282 421	.1964 922	.0211 964
42	.0166 884	.1205 002	.9276 524	.1974 423	.0212 829
43	.0167 392	.1208 236	.9270 604	.1983 934	.0213 694
44	.0167 898	.1211 456	.9264 661	.1993 454	.0214 557
0.245	+0.0168 402	-0.1214 661	+0.9258 695	+0.2002 985	-0.0215 421
46	.0168 905	.1217 852	.9252 705	.2012 525	.0216 283
47	.0169 406	.1221 028	.9246 693	.2022 074	.0217 145
48	.0169 905	.1224 190	.9240 657	.2031 634	.0218 006
49	.0170 403	.1227 336	.9234 598	.2041 203	.0218 867
0.250	+0.0170 898	-0.1230 469	+0.9228 516	+0.2050 781	-0.0219 727

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.250	+0.0170 898	-0.1230 469	+0.9228 516	+0.2050 781	-0.0219 727
51	.0171 392	.1233 586	.9222 410	.2060 369	.0220 586
52	.0171 884	.1236 690	.9216 282	.2069 967	.0221 444
53	.0172 375	.1239 778	.9210 130	.2079 574	.0222 301
54	.0172 863	.1242 852	.9203 956	.2089 191	.0223 158
0.255	+0.0173 350	-0.1245 911	+0.9197 758	+0.2098 817	-0.0224 014
56	.0173 835	.1248 956	.9191 537	.2108 453	.0224 869
57	.0174 318	.1251 986	.9185 294	.2118 098	.0225 724
58	.0174 800	.1255 002	.9179 027	.2127 753	.0226 578
59	.0175 280	.1258 003	.9172 737	.2137 417	.0227 430
0.260	+0.0175 757	-0.1260 990	+0.9166 424	+0.2147 090	-0.0228 283
61	.0176 233	.1263 961	.9160 089	.2156 773	.0229 134
62	.0176 708	.1266 919	.9153 730	.2166 465	.0229 984
63	.0177 180	.1269 862	.9147 348	.2176 167	.0230 834
64	.0177 651	.1272 790	.9140 944	.2185 878	.0231 683
0.265	+0.0178 120	-0.1275 703	+0.9134 516	+0.2195 598	-0.0232 531
66	.0178 587	.1278 602	.9128 066	.2205 327	.0233 378
67	.0179 052	.1281 487	.9121 593	.2215 066	.0234 224
68	.0179 515	.1284 356	.9115 097	.2224 814	.0235 070
69	.0179 977	.1287 212	.9108 578	.2234 571	.0235 914
0.270	+0.0180 437	-0.1290 052	+0.9102 036	+0.2244 338	-0.0236 758
71	.0180 895	.1292 878	.9095 471	.2254 113	.0237 601
72	.0181 351	.1295 690	.9088 884	.2263 898	.0238 443
73	.0181 805	.1298 487	.9082 274	.2273 692	.0239 284
74	.0182 258	.1301 269	.9075 641	.2283 495	.0240 124
0.275	+0.0182 708	-0.1304 037	+0.9068 985	+0.2293 307	-0.0240 963
76	.0183 157	.1306 790	.9062 307	.2303 128	.0241 802
77	.0183 604	.1309 529	.9055 606	.2312 958	.0242 639
78	.0184 050	.1312 253	.9048 882	.2322 797	.0243 476
79	.0184 493	.1314 963	.9042 136	.2332 645	.0244 311
0.280	+0.0184 934	-0.1317 658	+0.9035 366	+0.2342 502	-0.0245 146
81	.0185 374	.1320 338	.9028 575	.2352 369	.0245 979
82	.0185 812	.1323 004	.9021 760	.2362 244	.0246 812
83	.0186 248	.1325 655	.9014 923	.2372 128	.0247 643
84	0186 682	.1328 292	.9008 063	.2382 021	.0248 474
0.285	+0.0187 114	-0.1330 914	+0.9001 181	+0.2391 922	-0.0249 304
86	.0187 545	.1333 522	.8994 276	.2401 833	.0250 133
87	.0187 973	.1336 115	.8987 349	.2411 752	.0250 960
88	.0188 400	.1338 693	.8980 399	.2421 681	.0251 787
89	.0188 825	.1341 257	.8973 427	.2431 618	.0252 613
0.290	+0.0189 248	-0.1343 806	+0.8966 432	+0.2441 564	-0.0253 437
91	.0189 669	.1346 341	.8959 415	.2451 518	.0254 261
92	.0190 088	.1348 861	.8952 375	.2461 482	.0255 083
93	.0190 506	.1351 367	.8945 313	.2471 454	.0255 905
94	.0190 921	.1353 858	.8938 228	.2481 434	.0256 725
0.295	+0.0191 335	-0.1356 335	+0.8931 121	+0.2491 424	-0.0257 545
96	.0191 747	.1358 797	.8923 991	.2501 422	.0258 363
97	.0192 157	.1361 245	.8916 840	.2511 428	.0259 180
98	.0192 565	.1363 678	.8909 665	.2521 444	.0259 996
99	.0192 971	.1366 096	.8902 469	.2531 468	.0260 811
0.300	+0.0193 375	-0.1368 500	+0.8895 250	+0.2541 500	-0.0261 625

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.300	+0.0193 375	-0.1368 500	+0.8895 250	+0.2541 500	-0.0261 625
01	.0193 777	.1370 889	.8888 009	.2551 541	.0262 438
02	.0194 178	.1373 264	.8880 745	.2561 590	.0263 249
03	.0194 577	.1375 625	.8873 460	.2571 648	.0264 060
04	.0194 973	.1377 970	.8866 152	.2581 715	.0264 869
0.305	+0.0195 368	-0.1380 302	+0.8858 822	+0.2591 790	-0.0265 678
06	.0195 761	.1382 618	.8851 469	.2601 873	.0266 485
07	.0196 152	.1384 921	.8844 095	.2611 964	.0267 291
08	.0196 541	.1387 208	.8836 698	.2622 065	.0268 095
09	.0196 928	.1389 482	.8829 279	.2632 173	.0268 899
0.310	+0.0197 314	-0.1391 740	+0.8821 838	+0.2642 290	-0.0269 701
11	.0197 697	.1393 985	.8814 375	.2652 415	.0270 502
12	.0198 079	.1396 214	.8806 890	.2662 548	.0271 302
13	.0198 458	.1398 429	.8799 382	.2672 690	.0272 101
14	.0198 836	.1400 630	.8791 853	.2682 839	.0272 899
0.315	+0.0199 212	-0.1402 816	+0.8784 302	+0.2692 998	-0.0273 695
16	.0199 586	.1404 988	.8776 728	.2703 164	.0274 490
17	.0199 958	.1407 145	.8769 133	.2713 338	.0275 284
18	.0200 328	.1409 288	.8761 515	.2723 521	.0276 076
19	.0200 696	.1411 416	.8753 876	.2733 712	.0276 868
0.320	+0.0201 062	-0.1413 530	+0.8746 214	+0.2743 910	-0.0277 658
21	.0201 427	.1415 629	.8738 531	.2754 117	.0278 446
22	.0201 789	.1417 714	.8730 826	.2764 332	.0279 234
23	.0202 150	.1419 784	.8723 099	.2774 555	.0280 020
24	.0202 508	.1421 840	.8715 350	.2784 786	.0280 805
0.325	+0.0202 865	-0.1423 881	+0.8707 579	+0.2795 025	-0.0281 588
26	.0203 219	.1425 908	.8699 786	.2805 272	.0282 371
27	.0203 572	.1427 920	.8691 972	.2815 527	.0283 152
28	.0203 923	.1429 918	.8684 136	.2825 790	.0283 931
29	.0204 272	.1431 901	.8676 278	.2836 061	.0284 709
0.330	+0.0204 619	-0.1433 870	+0.8668 398	+0.2846 340	-0.0285 486
31	.0204 964	.1435 825	.8660 497	.2856 626	.0286 262
32	.0205 307	.1437 765	.8652 573	.2866 921	.0287 036
33	.0205 648	.1439 691	.8644 628	.2877 223	.0287 809
34	.0205 987	.1441 602	.8636 662	.2887 533	.0288 580
0.335	+0.0206 324	-0.1443 498	+0.8628 674	+0.2897 850	-0.0289 350
36	.0206 660	.1445 381	.8620 664	.2908 176	.0290 118
37	.0206 993	.1447 249	.8612 632	.2918 509	.0290 886
38	.0207 324	.1449 102	.8604 579	.2928 850	.0291 651
39	.0207 654	.1450 941	.8596 505	.2939 198	.0292 416
0.340	+0.0207 981	-0.1452 766	+0.8588 408	+0.2949 554	-0.0293 179
41	.0208 307	.1454 576	.8580 291	.2959 918	.0293 940
42	.0208 631	.1456 371	.8572 151	.2970 290	.0294 700
43	.0208 952	.1458 153	.8563 991	.2980 669	.0295 459
44	.0209 272	.1459 920	.8555 809	.2991 055	.0296 216
0.345	+0.0209 589	-0.1461 672	+0.8547 605	+0.3001 449	-0.0296 971
46	.0209 905	.1463 410	.8539 380	.3011 851	.0297 725
47	.0210 219	.1465 134	.8531 133	.3022 260	.0298 478
48	.0210 531	.1466 843	.8522 865	.3032 676	.0299 229
49	.0210 841	.1468 538	.8514 576	.3043 100	.0299 979
0.350	+0.0211 148	-0.1470 219	+0.8506 266	+0.3053 531	-0.0300 727

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.350	+0.0211 148	-0.1470 219	+0.8506 266	+0.3053 531	-0.0300 727
51	.0211 454	.1471 885	.8497 934	.3063 970	.0301 473
52	.0211 758	.1473 537	.8489 581	.3074 416	.0302 218
53	.0212 060	.1475 174	.8481 206	.3084 869	.0302 962
54	.0212 360	.1476 797	.8472 810	.3095 330	.0303 703
0.355	+0.0212 658	-0.1478 406	+0.8464 393	+0.3105 798	-0.0304 444
56	.0212 954	.1480 000	.8455 955	.3116 273	.0305 182
57	.0213 248	.1481 580	.8447 496	.3126 756	.0305 920
58	.0213 540	.1483 145	.8439 015	.3137 245	.0306 655
59	.0213 830	.1484 697	.8430 513	.3147 742	.0307 389
0.360	+0.0214 118	-0.1486 234	+0.8421 990	+0.3158 246	-0.0308 122
61	.0214 405	.1487 756	.8413 446	.3168 758	.0308 852
62	.0214 689	.1489 264	.8404 881	.3179 276	.0309 582
63	.0214 971	.1490 758	.8396 295	.3189 801	.0310 309
64	.0215 251	.1492 238	.8387 688	.3200 334	.0311 035
0.365	+0.0215 529	-0.1493 703	+0.8379 060	+0.3210 873	-0.0311 759
66	.0215 805	.1495 154	.8370 411	.3221 420	.0312 482
67	.0216 079	.1496 590	.8361 740	.3231 973	.0313 203
68	.0216 351	.1498 013	.8353 049	.3242 534	.0313 922
69	.0216 622	.1499 421	.8344 337	.3253 101	.0314 639
0.370	+0.0216 890	-0.1500 814	+0.8335 604	+0.3263 676	-0.0315 355
71	.0217 156	.1502 194	.8326 850	.3274 257	.0316 069
72	.0217 420	.1503 559	.8318 075	.3284 845	.0316 782
73	.0217 682	.1504 910	.8309 280	.3295 440	.0317 492
74	.0217 943	.1506 246	.8300 463	.3306 042	.0318 201
0.375	+0.0218 201	-0.1507 568	+0.8291 626	+0.3316 650	-0.0318 909
76	.0218 457	.1508 876	.8282 768	.3327 266	.0319 614
77	.0218 711	.1510 170	.8273 889	.3337 888	.0320 318
78	.0218 963	.1511 449	.8264 990	.3348 517	.0321 020
79	.0219 213	.1512 715	.8256 069	.3359 152	.0321 720
0.380	+0.0219 461	-0.1513 966	+0.8247 128	+0.3369 794	-0.0322 419
81	.0219 708	.1515 202	.8238 167	.3380 443	.0323 115
82	.0219 952	.1516 425	.8229 185	.3391 099	.0323 810
83	.0220 194	.1517 633	.8220 182	.3401 761	.0324 503
84	.0220 434	.1518 827	.8211 158	.3412 429	.0325 194
0.385	+0.0220 672	-0.1520 007	+0.8202 114	+0.3423 105	-0.0325 884
86	.0220 908	.1521 172	.8193 050	.3433 786	.0326 571
87	.0221 142	.1522 324	.8183 964	.3444 474	.0327 257
88	.0221 374	.1523 461	.8174 859	.3455 169	.0327 941
89	.0221 604	.1524 584	.8165 733	.3465 870	.0328 623
0.390	+0.0221 832	-0.1525 692	+0.8156 586	+0.3476 578	-0.0329 303
91	.0222 058	.1526 787	.8147 419	.3487 292	.0329 981
92	.0222 282	.1527 867	.8138 232	.3498 012	.0330 658
93	.0222 504	.1528 933	.8129 024	.3508 738	.0331 332
94	.0222 723	.1529 985	.8119 796	.3519 471	.0332 005
0.395	+0.0222 941	-0.1531 023	+0.8110 547	+0.3530 211	-0.0332 676
96	.0223 157	.1532 047	.8101 278	.3540 956	.0333 344
97	.0223 371	.1533 056	.8091 989	.3551 708	.0334 011
98	.0223 583	.1534 052	.8082 680	.3562 466	.0334 676
99	.0223 792	.1535 033	.8073 350	.3573 230	.0335 339
0.400	+0.0224 000	-0.1536 000	+0.8064 000	+0.3584 000	-0.0336 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.400	+0.0224 000	-0.1536 000	+0.8064 000	+0.3584 000	-0.0336 000
01	.0224 206	.1536 953	.8054 630	.3594 776	.0336 659
02	.0224 409	.1537 892	.8045 240	.3605 559	.0337 316
03	.0224 611	.1538 816	.8035 829	.3616 347	.0337 971
04	.0224 810	.1539 727	.8026 399	.3627 142	.0338 624
0.405	+0.0225 008	-0.1540 623	+0.8016 948	+0.3637 943	-0.0339 275
06	.0225 203	.1541 506	.8007 477	.3648 749	.0339 924
07	.0225 397	.1542 374	.7997 986	.3659 562	.0340 571
08	.0225 588	.1543 228	.7988 476	.3670 381	.0341 216
09	.0225 778	.1544 068	.7978 945	.3681 205	.0341 859
0.410	+0.0225 965	-0.1544 894	+0.7969 394	+0.3692 036	-0.0342 500
11	.0226 150	.1545 706	.7959 823	.3702 872	.0343 139
12	.0226 333	.1546 504	.7950 233	.3713 714	.0343 776
13	.0226 514	.1547 288	.7940 622	.3724 562	.0344 411
14	.0226 694	.1548 058	.7930 991	.3735 416	.0345 043
0.415	+0.0226 871	-0.1548 813	+0.7921 341	+0.3746 275	-0.0345 674
16	.0227 046	.1549 555	.7911 671	.3757 141	.0346 302
17	.0227 219	.1550 283	.7901 981	.3768 012	.0346 928
18	.0227 390	.1550 996	.7892 271	.3778 888	.0347 553
19	.0227 559	.1551 696	.7882 542	.3789 770	.0348 175
0.420	+0.0227 725	-0.1552 382	+0.7872 792	+0.3800 658	-0.0348 795
21	.0227 890	.1553 053	.7863 023	.3811 552	.0349 412
22	.0228 053	.1553 711	.7853 235	.3822 451	.0350 028
23	.0228 214	.1554 354	.7843 426	.3833 356	.0350 641
24	.0228 372	.1554 984	.7833 599	.3844 266	.0351 253
0.425	+0.0228 529	-0.1555 600	+0.7823 751	+0.3855 182	-0.0351 862
26	.0228 683	.1556 201	.7813 884	.3866 103	.0352 469
27	.0228 836	.1556 789	.7803 997	.3877 029	.0353 073
28	.0228 986	.1557 363	.7794 091	.3887 961	.0353 676
29	.0229 135	.1557 923	.7784 165	.3898 899	.0354 276
0.430	+0.0229 281	-0.1558 468	+0.7774 220	+0.3909 842	-0.0354 874
31	.0229 425	.1559 000	.7764 255	.3920 790	.0355 470
32	.0229 567	.1559 518	.7754 271	.3931 743	.0356 063
33	.0229 707	.1560 022	.7744 268	.3942 702	.0356 655
34	.0229 845	.1560 512	.7734 245	.3953 666	.0357 244
0.435	+0.0229 981	-0.1560 989	+0.7724 203	+0.3964 635	-0.0357 830
36	.0230 115	.1561 451	.7714 141	.3975 609	.0358 415
37	.0230 247	.1561 900	.7704 060	.3986 589	.0358 997
38	.0230 377	.1562 334	.7693 960	.3997 574	.0359 577
39	.0230 505	.1562 755	.7683 841	.4008 563	.0360 154
0.440	+0.0230 630	-0.1563 162	+0.7673 702	+0.4019 558	-0.0360 730
41	.0230 754	.1563 555	.7663 545	.4030 558	.0361 302
42	.0230 876	.1563 934	.7653 368	.4041 563	.0361 873
43	.0230 995	.1564 299	.7643 172	.4052 573	.0362 441
44	.0231 112	.1564 650	.7632 957	.4063 588	.0363 007
0.445	+0.0231 228	-0.1564 988	+0.7622 722	+0.4074 608	-0.0363 570
46	.0231 341	.1565 312	.7612 469	.4085 633	.0364 131
47	.0231 452	.1565 622	.7602 197	.4096 663	.0364 690
48	.0231 561	.1565 918	.7591 905	.4107 698	.0365 246
49	.0231 668	.1566 200	.7581 595	.4118 737	.0365 800
0.450	+0.0231 773	-0.1566 469	+0.7571 266	+0.4129 781	-0.0366 352

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.450	+0.0231 773	-0.1566 469	+0.7571 266	+0.4129 781	-0.0366 352
51	.0231 876	.1566 724	.7560 917	.4140 830	.0366 901
52	.0231 977	.1566 965	.7550 550	.4151 884	.0367 447
53	.0232 076	.1567 192	.7540 164	.4162 943	.0367 991
54	.0232 173	.1567 405	.7529 760	.4174 006	.0368 533
0.455	+0.0232 267	-0.1567 605	+0.7519 336	+0.4185 074	-0.0369 072
56	.0232 360	.1567 791	.7508 893	.4196 146	.0369 609
57	.0232 450	.1567 963	.7498 432	.4207 224	.0370 143
58	.0232 539	.1568 122	.7487 952	.4218 305	.0370 675
59	.0232 625	.1568 267	.7477 454	.4229 392	.0371 204
0.460	+0.0232 709	-0.1568 398	+0.7466 936	+0.4240 482	-0.0371 731
61	.0232 792	.1568 515	.7456 400	.4251 578	.0372 255
62	.0232 872	.1568 619	.7445 846	.4262 678	.0372 776
63	.0232 950	.1568 709	.7435 273	.4273 782	.0373 296
64	.0233 026	.1568 785	.7424 681	.4284 890	.0373 812
0.465	+0.0233 100	-0.1568 848	+0.7414 071	+0.4296 004	-0.0374 326
66	.0233 171	.1568 897	.7403 442	.4307 121	.0374 837
67	.0233 241	.1568 932	.7392 795	.4318 243	.0375 346
68	.0233 309	.1568 954	.7382 129	.4329 369	.0375 853
69	.0233 374	.1568 962	.7371 445	.4340 499	.0376 356
0.470	+0.0233 438	-0.1568 956	+0.7360 742	+0.4351 634	-0.0376 857
71	.0233 499	.1568 937	.7350 021	.4362 772	.0377 356
72	.0233 559	.1568 904	.7339 282	.4373 915	.0377 851
73	.0233 616	.1568 858	.7328 524	.4385 063	.0378 344
74	.0233 671	.1568 798	.7317 748	.4396 214	.0378 835
0.475	+0.0233 724	-0.1568 725	+0.7306 954	+0.4407 369	-0.0379 323
76	.0233 775	.1568 638	.7296 142	.4418 529	.0379 808
77	.0233 824	.1568 537	.7285 311	.4429 692	.0380 290
78	.0233 871	.1568 423	.7274 462	.4440 860	.0380 770
79	.0233 916	.1568 295	.7263 595	.4452 031	.0381 247
0.480	+0.0233 958	-0.1568 154	+0.7252 710	+0.4463 206	-0.0381 722
81	.0233 999	.1567 999	.7241 807	.4474 386	.0382 193
82	.0234 038	.1567 830	.7230 886	.4485 569	.0382 662
83	.0234 074	.1567 649	.7219 947	.4496 756	.0383 128
84	.0234 108	.1567 453	.7208 990	.4507 947	.0383 592
0.485	+0.0234 141	-0.1567 244	+0.7198 015	+0.4519 142	-0.0384 052
86	.0234 171	.1567 022	.7187 021	.4530 340	.0384 510
87	.0234 199	.1566 786	.7176 010	.4541 543	.0384 966
88	.0234 225	.1566 537	.7164 981	.4552 749	.0385 418
89	.0234 249	.1566 274	.7153 935	.4563 958	.0385 867
0.490	+0.0234 271	-0.1565 998	+0.7142 870	+0.4575 172	-0.0386 314
91	.0234 291	.1565 709	.7131 788	.4586 389	.0386 758
92	.0234 308	.1565 406	.7120 687	.4597 609	.0387 199
93	.0234 324	.1565 089	.7109 570	.4608 833	.0387 637
94	.0234 338	.1564 760	.7098 434	.4620 061	.0388 073
0.495	+0.0234 349	-0.1564 416	+0.7087 281	+0.4631 292	-0.0388 505
96	.0234 358	.1564 060	.7076 110	.4642 527	.0388 935
97	.0234 366	.1563 690	.7064 921	.4653 765	.0389 362
98	.0234 371	.1563 307	.7053 715	.4665 007	.0389 786
99	.0234 374	.1562 910	.7042 491	.4676 252	.0390 207
0.500	+0.0234 375	-0.1562 500	+0.7031 250	+0.4687 500	-0.0390 625

$$\begin{array}{llllll} a_{11} & a_{12} & a_{13} & a_{14} \cdots & a_{1n} & \sum_{i=1}^{n} a_{1i} \\ a_{21} & & & & & \sum_{i=1}^{n} a_{2i} \\ a_{31} \\ a_{41} \\ \vdots \\ a_{n1} \end{array}$$

$$\begin{array}{llllll} b_{11} & b_{12} & b_{13} & b_{14} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & b_{24} & \cdots & b_{2n} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ \vdots \\ b_{n1} & b_{n2} \end{array}$$

$b_{11} = a_{11} \quad b_{21} = a_{21} \quad b_{31} = a_{31} \quad \cdots \quad b_{n1} = a_{n1}$

$b_{12} = \frac{a_{12}}{b_{11}} \quad b_{13} = \frac{a_{13}}{b_{11}} \quad b_{14} = \frac{a_{14}}{b_{11}} \quad \cdots \quad b_{1n} = \frac{a_{1n}}{b_{11}}$

$b_{22} = a_{22} - b_{21} b_{12} \quad b_{32} = a_{32} - b_{31} b_{12}. \quad b_{42} = a_{42} - b_{41} b_{12} \quad \cdots \quad b_{n2} = a_{n2} - b_{n1} b_{12}$

$b_{23} = (a_{23} - b_{21} b_{13}) \div b_{22} \quad b_{24} = (a_{24} - b_{21} b_{14}) \div b_{22} \quad \cdots \quad b_{2n} = (a_{2n} - b_{21} b_{1n}) \div b_{22}$

$b_{33} = a_{33} - b_{31} b_{13} - b_{32} b_{23}\ ; \quad b_{43} = a_{43} - b_{41} b_{13} - b_{42} b_{23} \quad \cdots\cdots \quad b_{n3} = a_{n3} - b_{n1} b_{13} - b_{n2} b_{23}$

$b_{34} = (a_{34} - b_{31} b_{14} - b_{32} b_{24}) \div b_{33} \quad \cdots \cdots \quad b_{3n} = (a_{3n} - b_{31} b_{1n} - b_{32} b_{2n}) \div b_{33}.$

Value of the Determinant $D = b_{11} b_{22} b_{33} \cdots b_{nn}$

Auxiliary Matrix

$$
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & \\
\multicolumn{5}{c}{\left[\begin{array}{ccccc}
b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\
b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\
b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\
b_{41} & b_{42} & b_{43} & b_{44} & b_{45}
\end{array}\right]} \\
x_1 & x_2 & x_3 & x_4 &
\end{array}
$$

$$x_4 = b_{45}$$

$$x_3 = b_{35} - b_{34} x_4$$

$$x_2 = b_{25} - b_{23} x_3 - b_{24} x_4$$

$$x_1 = b_{15} - b_{12} x_2 - b_{13} x_3 - b_{14} x_4$$

in general if $\left\{ \begin{array}{l} n \text{ rows} \\ n+1 \text{ columns} \end{array} \right.$

$$m \leq n \qquad x_m = b_{m(n+1)} - b_{m(m+1)} x_{m+1} - \dots - b_{m(m+n)} x_n$$

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.500	+0.0234 375	-0.1562 500	+0.7031 250	+0.4687 500	-0.0390 625
01	.0234 374	.1562 077	.7019 991	.4698 752	.0391 040
02	.0234 371	.1561 640	.7008 715	.4710 007	.0391 452
03	.0234 366	.1561 190	.6997 421	.4721 265	.0391 862
04	.0234 358	.1560 727	.6986 110	.4732 526	.0392 268
0.505	+0.0234 349	-0.1560 250	+0.6974 782	+0.4743 791	-0.0392 672
06	.0234 338	.1559 760	.6963 436	.4755 059	.0393 072
07	.0234 324	.1559 257	.6952 073	.4766 330	.0393 470
08	.0234 308	.1558 741	.6940 693	.4777 604	.0393 864
09	.0234 291	.1558 211	.6929 295	.4788 881	.0394 256
0.510	+0.0234 271	-0.1557 668	+0.6917 880	+0.4800 162	-0.0394 644
11	.0234 249	.1557 112	.6906 448	.4811 445	.0395 030
12	.0234 225	.1556 543	.6894 999	.4822 731	.0395 412
13	.0234 199	.1555 960	.6883 532	.4834 021	.0395 792
14	.0234 171	.1555 365	.6872 049	.4845 313	.0396 168
0.515	+0.0234 141	-0.1554 756	+0.6860 548	+0.4856 608	-0.0396 541
16	.0234 108	.1554 134	.6849 031	.4867 906	.0396 911
17	.0234 074	.1553 498	.6837 496	.4879 207	.0397 279
18	.0234 038	.1552 850	.6825 944	.4890 511	.0397 643
19	.0233 999	.1552 188	.6814 376	.4901 817	.0398 004
0.520	+0.0233 958	-0.1551 514	+0.6802 790	+0.4913 126	-0.0398 362
21	.0233 916	.1550 826	.6791 188	.4924 438	.0398 716
22	.0233 871	.1550 125	.6779 569	.4935 753	.0399 068
23	.0233 824	.1549 411	.6767 933	.4947 070	.0399 416
24	.0233 775	.1548 684	.6756 280	.4958 390	.0399 762
0.525	+0.0233 724	-0.1547 943	+0.6744 610	+0.4969 713	-0.0400 104
26	.0233 671	.1547 190	.6732 924	.4981 038	.0400 443
27	.0233 616	.1546 424	.6721 221	.4992 366	.0400 779
28	.0233 559	.1545 644	.6709 501	.5003 696	.0401 111
29	.0233 499	.1544 852	.6697 765	.5015 029	.0401 441
0.530	+0.0233 438	-0.1544 046	+0.6686 012	+0.5026 364	-0.0401 767
31	.0233 374	.1543 228	.6674 243	.5037 701	.0402 090
32	.0233 309	.1542 396	.6662 456	.5049 041	.0402 410
33	.0233 241	.1541 552	.6650 654	.5060 383	.0402 726
34	.0233 171	.1540 694	.6638 835	.5071 728	.0403 040
0.535	+0.0233 100	-0.1539 824	+0.6626 999	+0.5083 075	-0.0403 350
36	.0233 026	.1538 941	.6615 147	.5094 424	.0403 657
37	.0232 950	.1538 044	.6603 279	.5105 775	.0403 960
38	.0232 872	.1537 135	.6591 395	.5117 129	.0404 260
39	.0232 792	.1536 213	.6579 494	.5128 485	.0404 557
0.540	+0.0232 709	-0.1535 278	+0.6567 576	+0.5139 842	-0.0404 851
41	.0232 625	.1534 330	.6555 643	.5151 202	.0405 141
42	.0232 539	.1533 369	.6543 693	.5162 564	.0405 428
43	.0232 450	.1532 395	.6531 727	.5173 928	.0405 711
44	.0232 360	.1531 408	.6519 745	.5185 294	.0405 991
0.545	+0.0232 267	-0.1530 409	+0.6507 747	+0.5196 663	-0.0406 268
46	.0232 173	.1529 396	.6495 733	.5208 033	.0406 542
47	.0232 076	.1528 371	.6483 703	.5219 404	.0406 812
48	.0231 977	.1527 333	.6471 656	.5230 778	.0407 078
49	.0231 876	.1526 282	.6459 594	.5242 154	.0407 342
0.550	+0.0231 773	-0.1525 219	+0.6447 516	+0.5253 531	-0.0407 602

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.550	+0.0231 773	-0.1525 219	+0.6447 516	+0.5253 531	-0.0407 602
51	.0231 668	.1524 142	.6435 421	.5264 910	.0407 858
52	.0231 561	.1523 053	.6423 311	.5276 291	.0408 111
53	.0231 452	.1521 951	.6411 185	.5287 674	.0408 360
54	.0231 341	.1520 837	.6399 044	.5299 059	.0408 607
0.555	+0.0231 228	-0.1519 709	+0.6386 886	+0.5310 445	-0.0408 849
56	.0231 112	.1518 569	.6374 713	.5321 832	.0409 088
57	.0230 995	.1517 416	.6362 524	.5333 221	.0409 324
58	.0230 876	.1516 251	.6350 319	.5344 612	.0409 556
59	.0230 754	.1515 072	.6338 098	.5356 005	.0409 785
0.560	+0.0230 630	-0.1513 882	+0.6325 862	+0.5367 398	-0.0410 010
61	.0230 505	.1512 678	.6313 611	.5378 794	.0410 231
62	.0230 377	.1511 462	.6301 344	.5390 190	.0410 449
63	.0230 247	.1510 233	.6289 061	.5401 589	.0410 664
64	.0230 115	.1508 992	.6276 763	.5412 988	.0410 874
0.565	+0.0229 981	-0.1507 737	+0.6264 449	+0.5424 389	-0.0411 082
66	.0229 845	.1506 471	.6252 120	.5435 791	.0411 285
67	.0229 707	.1505 192	.6239 775	.5447 194	.0411 486
68	.0229 567	.1503 900	.6227 416	.5458 599	.0411 682
69	.0229 425	.1502 595	.6215 040	.5470 005	.0411 875
0.570	+0.0229 281	-0.1501 278	+0.6202 650	+0.5481 412	-0.0412 064
71	.0229 135	.1499 949	.6190 244	.5492 820	.0412 250
72	.0228 986	.1498 607	.6177 823	.5504 229	.0412 432
73	.0228 836	.1497 252	.6165 387	.5515 639	.0412 610
74	.0228 683	.1495 885	.6152 936	.5527 051	.0412 785
0.575	+0.0228 529	-0.1494 506	+0.6140 470	+0.5538 463	-0.0412 956
76	.0228 372	.1493 114	.6127 988	.5549 876	.0413 123
77	.0228 214	.1491 709	.6115 492	.5561 290	.0413 286
78	.0228 053	.1490 293	.6102 980	.5572 706	.0413 446
79	.0227 890	.1488 863	.6090 454	.5584 122	.0413 602
0.580	+0.0227 725	-0.1487 422	+0.6077 912	+0.5595 538	-0.0413 755
81	.0227 559	.1485 967	.6065 356	.5606 956	.0413 903
82	.0227 390	.1484 501	.6052 785	.5618 374	.0414 048
83	.0227 219	.1483 022	.6040 199	.5629 794	.0414 189
84	.0227 046	.1481 531	.6027 598	.5641 214	.0414 326
0.585	+0.0226 871	-0.1480 027	+0.6014 982	+0.5652 634	-0.0414 460
86	.0226 694	.1478 511	.6002 352	.5664 055	.0414 590
87	.0226 514	.1476 983	.5989 707	.5675 477	.0414 715
88	.0226 333	.1475 442	.5977 047	.5686 899	.0414 838
89	.0226 150	.1473 890	.5964 373	.5698 322	.0414 956
0.590	+0.0225 965	-0.1472 324	+0.5951 684	+0.5709 746	-0.0415 070
91	.0225 778	.1470 747	.5938 981	.5721 170	.0415 181
92	.0225 588	.1469 157	.5926 263	.5732 594	.0415 287
93	.0225 397	.1467 555	.5913 530	.5744 019	.0415 390
94	.0225 203	.1465 941	.5900 783	.5755 444	.0415 489
0.595	+0.0225 008	-0.1464 315	+0.5888 022	+0.5766 869	-0.0415 584
96	.0224 810	.1462 676	.5875 246	.5778 295	.0415 675
97	.0224 611	.1461 025	.5862 456	.5789 721	.0415 762
98	.0224 409	.1459 362	.5849 652	.5801 147	.0415 845
99	.0224 206	.1457 687	.5836 833	.5812 573	.0415 925
0.600	+0.0224 000	-0.1456 000	+0.5824 000	+0.5824 000	-0.0416 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.600	+0.0224 000	-0.1456 000	+0.5824 000	+0.5824 000	-0.0416 000
01	.0223 792	.1454 301	.5811 153	.5835 427	.0416 071
02	.0223 583	.1452 589	.5798 292	.5846 854	.0416 139
03	.0223 371	.1450 865	.5785 416	.5858 280	.0416 202
04	.0223 157	.1449 130	.5772 527	.5869 707	.0416 261
0.605	+0.0222 941	-0.1447 382	+0.5759 623	+0.5881 134	-0.0416 317
06	.0222 723	.1445 622	.5746 706	.5892 561	.0416 368
07	.0222 504	.1443 850	.5733 774	.5903 988	.0416 416
08	.0222 282	.1442 066	.5720 829	.5915 415	.0416 459
09	.0222 058	.1440 270	.5707 869	.5926 841	.0416 498
0.610	+0.0221 832	-0.1438 462	+0.5694 896	+0.5938 268	-0.0416 533
11	.0221 604	.1436 642	.5681 909	.5949 694	.0416 564
12	.0221 374	.1434 810	.5668 908	.5961 120	.0416 591
13	.0221 142	.1432 967	.5655 893	.5972 545	.0416 614
14	.0220 908	.1431 111	.5642 865	.5983 971	.0416 633
0.615	+0.0220 672	-0.1429 243	+0.5629 823	+0.5995 396	-0.0416 648
16	.0220 434	.1427 363	.5616 767	.6006 820	.0416 658
17	.0220 194	.1425 472	.5603 698	.6018 245	.0416 664
18	.0219 952	.1423 568	.5590 615	.6029 668	.0416 667
19	.0219 708	.1421 653	.5577 518	.6041 092	.0416 665
0.620	+0.0219 461	-0.1419 726	+0.5564 408	+0.6052 514	-0.0416 659
21	.0219 213	.1417 787	.5551 285	.6063 937	.0416 648
22	.0218 963	.1415 836	.5538 148	.6075 358	.0416 634
23	.0218 711	.1413 873	.5524 998	.6086 779	.0416 615
24	.0218 457	.1411 898	.5511 834	.6098 200	.0416 592
0.625	+0.0218 201	-0.1409 912	+0.5498 657	+0.6109 619	-0.0416 565
26	.0217 943	.1407 914	.5485 467	.6121 038	.0416 534
27	.0217 682	.1405 904	.5472 264	.6132 456	.0416 498
28	.0217 420	.1403 883	.5459 047	.6143 874	.0416 458
29	.0217 156	.1401 849	.5445 817	.6155 290	.0416 414
0.630	+0.0216 890	-0.1399 804	+0.5432 574	+0.6166 706	-0.0416 365
31	.0216 622	.1397 748	.5419 318	.6178 120	.0416 312
32	.0216 351	.1395 679	.5406 049	.6189 534	.0416 255
33	.0216 079	.1393 599	.5392 767	.6200 947	.0416 194
34	.0215 805	.1391 508	.5379 472	.6212 359	.0416 128
0.635	+0.0215 529	-0.1389 404	+0.5366 164	+0.6223 770	-0.0416 058
36	.0215 251	.1387 289	.5352 843	.6235 179	.0415 983
37	.0214 971	.1385 163	.5339 509	.6246 588	.0415 905
38	.0214 689	.1383 025	.5326 162	.6257 995	.0415 821
39	.0214 405	.1380 875	.5312 803	.6269 401	.0415 734
0.640	+0.0214 118	-0.1378 714	+0.5299 430	+0.6280 806	-0.0415 642
41	.0213 830	.1376 541	.5286 045	.6292 210	.0415 545
42	.0213 540	.1374 356	.5272 648	.6303 613	.0415 444
43	.0213 248	.1372 161	.5259 238	.6315 014	.0415 339
44	.0212 954	.1369 953	.5245 815	.6326 413	.0415 229
0.645	+0.0212 658	-0.1367 734	+0.5232 380	+0.6337 812	-0.0415 115
46	.0212 360	.1365 504	.5218 932	.6349 209	.0414 996
47	.0212 060	.1363 262	.5205 471	.6360 604	.0414 873
48	.0211 758	.1361 009	.5191 998	.6371 998	.0414 745
49	.0211 454	.1358 745	.5178 513	.6383 390	.0414 613
0.650	+0.0211 148	-0.1356 469	+0.5165 016	+0.6394 781	-0.0414 477

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.650	+0.0211 148	0.1356 469	+0.5165 016	+0.6394 781	-0.0414 477
51	.0210 841	.1354 181	.5151 506	.6406 170	.0414 335
52	.0210 531	.1351 883	.5137 984	.6417 558	.0414 190
53	.0210 219	.1349 573	.5124 449	.6428 944	.0414 039
54	.0209 905	.1347 251	.5110 902	.6440 328	.0413 884
0.655	+0.0209 589	-0.1344 918	+0.5097 344	+0.6451 710	-0.0413 725
56	.0209 272	.1342 574	.5083 773	.6463 091	.0413 561
57	.0208 952	.1340 219	.5070 190	.6474 470	.0413 392
58	.0208 631	.1337 853	.5056 595	.6485 846	.0413 219
59	.0208 307	.1335 475	.5042 987	.6497 221	.0413 041
0.660	+0.0207 981	-0.1333 086	+0.5029 368	+0.6508 594	-0.0412 859
61	.0207 654	.1330 685	.5015 737	.6519 965	.0412 671
62	.0207 324	.1328 274	.5002 095	.6531 334	.0412 480
63	.0206 993	.1325 851	.4988 440	.6542 701	.0412 283
64	.0206 660	.1323 417	.4974 773	.6554 066	.0412 082
0.665	+0.0206 324	-0.1320 972	+0.4961 095	+0.6565 429	-0.0411 876
66	.0205 987	.1318 516	.4947 405	.6576 790	.0411 666
67	.0205 648	.1316 049	.4933 703	.6588 148	.0411 450
68	.0205 307	.1313 570	.4919 990	.6599 504	.0411 230
69	.0204 964	.1311 081	.4906 265	.6610 858	.0411 006
0.670	+0.0204 619	-0.1308 580	+0.4892 528	+0.6622 210	-0.0410 776
71	.0204 272	.1306 069	.4878 780	.6633 559	.0410 542
72	.0203 923	.1303 546	.4865 020	.6644 906	.0410 303
73	.0203 572	.1301 012	.4851 249	.6656 250	.0410 059
74	.0203 219	.1298 468	.4837 467	.6667 592	.0409 811
0.675	+0.0202 865	-0.1295 912	+0.4823 673	+0.6678 932	-0.0409 557
76	.0202 508	.1293 345	.4809 868	.6690 269	.0409 299
77	.0202 150	.1290 768	.4796 051	.6701 603	.0409 036
78	.0201 789	.1288 179	.4782 223	.6712 935	.0408 768
79	.0201 427	.1285 580	.4768 385	.6724 264	.0408 495
0.680	+0.0201 062	-0.1282 970	+0.4754 534	+0.6735 590	-0.0408 218
81	.0200 696	.1280 348	.4740 673	.6746 914	.0407 935
82	.0200 328	.1277 716	.4726 801	.6758 235	.0407 648
83	.0199 958	.1275 073	.4712 917	.6769 553	.0407 355
84	.0199 586	.1272 420	.4699 023	.6780 869	.0407 058
0.685	+0.0199 212	-0.1269 755	+0.4685 118	+0.6792 181	-0.0406 756
86	.0198 836	.1267 080	.4671 201	.6803 491	.0406 449
87	.0198 458	.1264 393	.4657 274	.6814 798	.0406 137
88	.0198 079	.1261 696	.4643 336	.6826 101	.0405 820
89	.0197 697	.1258 989	.4629 388	.6837 402	.0405 498
0.690	+0.0197 314	-0.1256 270	+0.4615 428	+0.6848 700	-0.0405 171
91	.0196 928	.1253 541	.4601 458	.6859 994	.0404 839
92	.0196 541	.1250 801	.4587 477	.6871 286	.0404 502
93	.0196 152	.1248 051	.4573 485	.6882 574	.0404 160
94	.0195 761	.1245 290	.4559 483	.6893 859	.0403 813
0.695	+0.0195 368	-0.1242 518	+0.4545 470	+0.6905 141	-0.0403 461
96	.0194 973	.1239 736	.4531 447	.6916 419	.0403 104
97	.0194 577	.1236 943	.4517 413	.6927 695	.0402 742
98	.0194 178	.1234 139	.4503 369	.6938 966	.0402 375
99	.0193 777	.1231 325	.4489 315	.6950 235	.0402 002
0.700	+0.0193 375	-0.1228 500	+0.4475 250	+0.6961 500	-0.0401 625

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.700	+0.0193 375	-0.1228 500	+0.4475 250	+0.6961 500	-0.0401 625
01	.0192 971	.1225 665	.4461 175	.6972 762	.0401 242
02	.0192 565	.1222 819	.4447 089	.6984 020	.0400 855
03	.0192 157	.1219 963	.4432 994	.6995 274	.0400 462
04	.0191 747	.1217 096	.4418 888	.7006 525	.0400 064
0.705	+0.0191 335	-0.1214 219	+0.4404 772	+0.7017 773	-0.0399 661
06	.0190 921	.1211 331	.4390 646	.7029 016	.0399 252
07	.0190 506	.1208 433	.4376 510	.7040 256	.0398 839
08	.0190 088	.1205 524	.4362 364	.7051 492	.0398 420
09	.0189 669	.1202 606	.4348 208	.7062 725	.0397 996
0.710	+0.0189 248	-0.1199 676	+0.4334 042	+0.7073 954	-0.0397 567
11	.0188 825	.1196 737	.4319 866	.7085 178	.0397 133
12	.0188 400	.1193 787	.4305 681	.7096 399	.0396 693
13	.0187 973	.1190 827	.4291 485	.7107 616	.0396 248
14	.0187 545	.1187 856	.4277 280	.7118 829	.0395 798
0.715	+0.0187 114	-0.1184 875	+0.4263 065	+0.7130 039	-0.0395 343
16	.0186 682	.1181 884	.4248 840	.7141 244	.0394 882
17	.0186 248	.1178 883	.4234 606	.7152 444	.0394 416
18	.0185 812	.1175 871	.4220 362	.7163 641	.0393 944
19	.0185 374	.1172 850	.4206 109	.7174 834	.0393 468
0.720	+0.0184 934	-0.1169 818	+0.4191 846	+0.7186 022	-0.0392 986
21	.0184 493	.1166 776	.4177 574	.7197 207	.0392 498
22	.0184 050	.1163 723	.4163 293	.7208 387	.0392 005
23	.0183 604	.1160 661	.4149 002	.7219 562	.0391 507
24	.0183 157	.1157 588	.4134 701	.7230 733	.0391 004
0.725	+0.0182 708	-0.1154 506	+0.4120 392	+0.7241 900	-0.0390 495
26	.0182 258	.1151 413	.4106 073	.7253 063	.0389 980
27	.0181 805	.1148 311	.4091 745	.7264 221	.0389 460
28	.0181 351	.1145 198	.4077 408	.7275 374	.0388 935
29	.0180 895	.1142 075	.4063 061	.7286 523	.0388 404
0.730	+0.0180 437	-0.1138 942	+0.4048 706	+0.7297 668	-0.0387 868
31	.0179 977	.1135 800	.4034 342	.7308 807	.0387 326
32	.0179 515	.1132 647	.4019 968	.7319 942	.0386 779
33	.0179 052	.1129 484	.4005 586	.7331 073	.0386 227
34	.0178 587	.1126 312	.3991 195	.7342 198	.0385 668
0.735	+0.0178 120	-0.1123 129	+0.3976 795	+0.7353 319	-0.0385 105
36	.0177 651	.1119 937	.3962 386	.7364 435	.0384 535
37	.0177 180	.1116 735	.3947 969	.7375 546	.0383 961
38	.0176 708	.1113 523	.3933 543	.7386 653	.0383 380
39	.0176 233	.1110 301	.3919 108	.7397 754	.0382 794
0.740	+0.0175 757	-0.1107 070	+0.3904 664	+0.7408 850	-0.0382 203
41	.0175 280	.1103 828	.3890 212	.7419 942	.0381 605
42	.0174 800	.1100 577	.3875 752	.7431 028	.0381 003
43	.0174 318	.1097 316	.3861 283	.7442 109	.0380 394
44	.0173 835	.1094 046	.3846 805	.7453 185	.0379 780
0.745	+0.0173 350	-0.1090 765	+0.3832 319	+0.7464 256	-0.0379 160
46	.0172 863	.1087 475	.3817 825	.7475 322	.0378 535
47	.0172 375	.1084 176	.3803 323	.7486 382	.0377 904
48	.0171 884	.1080 866	.3788 812	.7497 437	.0377 267
49	.0171 392	.1077 547	.3774 293	.7508 487	.0376 625
0.750	+0.0170 898	-0.1074 219	+0.3759 766	+0.7519 531	-0.0375 977

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.750	+0.0170 898	-0.1074 219	+0.3759 766	+0.7519 531	-0.0375 977
51	.0170 403	.1070 881	.3745 230	.7530 570	.0375 323
52	.0169 905	.1067 533	.3730 687	.7541 604	.0374 663
53	.0169 406	.1064 176	.3716 136	.7552 632	.0373 998
54	.0168 905	.1060 809	.3701 576	.7563 654	.0373 326
0.755	+0.0168 402	-0.1057 433	+0.3687 009	+0.7574 671	-0.0372 649
56	.0167 898	.1054 047	.3672 433	.7585 682	.0371 967
57	.0167 392	.1050 652	.3657 850	.7596 688	.0371 278
58	.0166 884	.1047 247	.3643 259	.7607 688	.0370 584
59	.0166 374	.1043 833	.3628 661	.7618 682	.0369 884
0.760	+0.0165 862	-0.1040 410	+0.3614 054	+0.7629 670	-0.0369 178
61	.0165 349	.1036 977	.3599 440	.7640 653	.0368 466
62	.0164 834	.1033 535	.3584 819	.7651 630	.0367 748
63	.0164 317	.1030 083	.3570 189	.7662 601	.0367 024
64	.0163 799	.1026 622	.3555 553	.7673 565	.0366 295
0.765	+0.0163 279	-0.1023 152	+0.3540 908	+0.7684 524	-0.0365 559
66	.0162 757	.1019 673	.3526 257	.7695 477	.0364 818
67	.0162 233	.1016 184	.3511 597	.7706 424	.0364 071
68	.0161 708	.1012 686	.3496 931	.7717 365	.0363 317
69	.0161 181	.1009 179	.3482 257	.7728 299	.0362 558
0.770	+0.0160 652	-0.1005 662	+0.3467 576	+0.7739 228	-0.0361 793
71	.0160 121	.1002 137	.3452 888	.7750 150	.0361 022
72	.0159 589	.0998 602	.3438 192	.7761 066	.0360 245
73	.0159 055	.0995 058	.3423 490	.7771 975	.0359 462
74	.0158 519	.0991 505	.3408 780	.7782 879	.0358 673
0.775	+0.0157 982	-0.0987 943	+0.3394 063	+0.7793 775	-0.0357 877
76	.0157 443	.0984 372	.3379 340	.7804 666	.0357 076
77	.0156 902	.0980 792	.3364 609	.7815 550	.0356 269
78	.0156 360	.0977 203	.3349 872	.7826 427	.0355 456
79	.0155 815	.0973 605	.3335 127	.7837 298	.0354 636
0.780	+0.0155 269	-0.0969 998	+0.3320 376	+0.7848 162	-0.0353 811
81	.0154 722	.0966 381	.3305 619	.7859 020	.0352 979
82	.0154 173	.0962 756	.3290 854	.7869 871	.0352 141
83	.0153 622	.0959 122	.3276 083	.7880 715	.0351 297
84	.0153 069	.0955 479	.3261 305	.7891 553	.0350 447
0.785	+0.0152 515	-0.0951 828	+0.3246 521	+0.7902 384	-0.0349 591
86	.0151 959	.0948 167	.3231 730	.7913 207	.0348 729
87	.0151 401	.0944 498	.3216 932	.7924 024	.0347 860
88	.0150 842	.0940 819	.3202 129	.7934 834	.0346 985
89	.0150 281	.0937 132	.3187 318	.7945 638	.0346 104
0.790	+0.0149 718	-0.0933 436	+0.3172 502	+0.7956 434	-0.0345 217
91	.0149 153	.0929 732	.3157 679	.7967 223	.0344 324
92	.0148 587	.0926 018	.3142 850	.7978 005	.0343 424
93	.0148 020	.0922 296	.3128 015	.7988 779	.0342 518
94	.0147 450	.0918 566	.3113 174	.7999 547	.0341 606
0.795	+0.0146 880	-0.0914 826	+0.3098 327	+0.8010 308	-0.0340 687
96	.0146 307	.0911 078	.3083 473	.8021 061	.0339 763
97	.0145 733	.0907 322	.3068 614	.8031 807	.0338 831
98	.0145 157	.0903 556	.3053 748	.8042 545	.0337 894
99	.0144 579	.0899 782	.3038 877	.8053 276	.0336 950
0.800	+0.0144 000	-0.0896 000	+0.3024 000	+0.8064 000	-0.0336 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.800	+0.0144 000	-0.0896 000	+0.3024 000	+0.8064 000	-0.0336 000
01	.0143 419	.0892 209	.3009 117	.8074 716	.0335 043
02	.0142 837	.0888 410	.2994 228	.8085 425	.0334 081
03	.0142 253	.0884 602	.2979 334	.8096 126	.0333 111
04	.0141 667	.0880 785	.2964 434	.8106 820	.0332 136
0.805	+0.0141 080	-0.0876 960	+0.2949 529	+0.8117 506	-0.0331 153
06	.0140 491	.0873 127	.2934 617	.8128 184	.0330 165
07	.0139 900	.0869 286	.2919 701	.8138 855	.0329 170
08	.0139 308	.0865 435	.2904 779	.8149 517	.0328 168
09	.0138 714	.0861 577	.2889 851	.8160 172	.0327 161
0.810	+0.0138 119	-0.0857 710	+0.2874 918	+0.8170 820	-0.0326 146
11	.0137 522	.0853 835	.2859 980	.8181 459	.0325 125
12	.0136 923	.0849 952	.2845 036	.8192 090	.0324 098
13	.0136 323	.0846 060	.2830 088	.8202 714	.0323 064
14	.0135 721	.0842 161	.2815 134	.8213 329	.0322 023
0.815	+0.0135 118	-0.0838 252	+0.2800 175	+0.8223 936	-0.0320 976
16	.0134 513	.0834 336	.2785 211	.8234 535	.0319 923
17	.0133 906	.0830 412	.2770 241	.8245 127	.0318 863
18	.0133 298	.0826 479	.2755 267	.8255 709	.0317 796
19	.0132 689	.0822 538	.2740 288	.8266 284	.0316 723
0.820	+0.0132 077	-0.0818 590	+0.2725 304	+0.8276 850	-0.0315 643
21	.0131 465	.0814 633	.2710 316	.8287 408	.0314 556
22	.0130 850	.0810 668	.2695 322	.8297 958	.0313 463
23	.0130 234	.0806 695	.2680 324	.8308 499	.0312 363
24	.0129 617	.0802 714	.2665 321	.8319 032	.0311 256
0.825	+0.0128 998	-0.0798 725	+0.2650 313	+0.8329 557	-0.0310 143
26	.0128 377	.0794 728	.2635 301	.8340 072	.0309 023
27	.0127 755	.0790 723	.2620 285	.8350 580	.0307 896
28	.0127 131	.0786 710	.2605 264	.8361 078	.0306 763
29	.0126 506	.0782 689	.2590 238	.8371 568	.0305 623
0.830	+0.0125 879	-0.0778 660	+0.2575 208	+0.8382 050	-0.0304 476
31	.0125 250	.0774 624	.2560 174	.8392 522	.0303 323
32	.0124 621	.0770 579	.2545 135	.8402 986	.0302 162
33	.0123 989	.0766 527	.2530 092	.8413 441	.0300 995
34	.0123 356	.0762 467	.2515 045	.8423 887	.0299 821
0.835	+0.0122 722	-0.0758 400	+0.2499 994	+0.8434 324	-0.0298 640
36	.0122 086	.0754 324	.2484 939	.8444 752	.0297 453
37	.0121 448	.0750 241	.2469 880	.8455 171	.0296 258
38	.0120 809	.0746 150	.2454 817	.8465 581	.0295 057
39	.0120 168	.0742 052	.2439 749	.8475 983	.0293 849
0.840	+0.0119 526	-0.0737 946	+0.2424 678	+0.8486 374	-0.0292 634
41	.0118 883	.0733 832	.2409 604	.8496 757	.0291 412
42	.0118 238	.0729 710	.2394 525	.8507 131	.0290 183
43	.0117 591	.0725 581	.2379 443	.8517 495	.0288 947
44	.0116 943	.0721 445	.2364 356	.8527 850	.0287 704
0.845	+0.0116 294	-0.0717 301	+0.2349 267	+0.8538 195	-0.0286 455
46	.0115 642	.0713 149	.2334 173	.8548 531	.0285 198
47	.0114 990	.0708 990	.2319 077	.8558 858	.0283 934
48	.0114 336	.0704 824	.2303 976	.8569 175	.0282 664
49	.0113 680	.0700 650	.2288 873	.8579 483	.0281 386
0.850	+0.0113 023	-0.0696 469	+0.2273 766	+0.8589 781	-0.0280 102

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.850	+0.0113 023	-0.0696 469	+0.2273 766	+0.8589 781	-0.0280 102
51	.0112 365	.0692 280	.2258 655	.8600 070	.0278 810
52	.0111 705	.0688 084	.2243 542	.8610 349	.0277 511
53	.0111 044	.0683 881	.2228 425	.8620 618	.0276 205
54	.0110 381	.0679 670	.2213 305	.8630 877	.0274 893
0.855	+0.0109 717	-0.0675 452	+0.2198 181	+0.8641 127	-0.0273 573
56	.0109 051	.0671 227	.2183 055	.8651 367	.0272 246
57	.0108 384	.0666 994	.2167 926	.8661 596	.0270 912
58	.0107 715	.0662 755	.2152 794	.8671 816	.0269 570
59	.0107 045	.0658 508	.2137 658	.8682 026	.0268 222
0.860	+0.0106 373	-0.0654 254	+0.2122 520	+0.8692 226	-0.0266 867
61	.0105 700	.0649 992	.2107 380	.8702 416	.0265 504
62	.0105 026	.0645 724	.2092 236	.8712 596	.0264 134
63	.0104 350	.0641 449	.2077 090	.8722 766	.0262 757
64	.0103 673	.0637 166	.2061 941	.8732 925	.0261 373
0.865	+0.0102 994	-0.0632 877	+0.2046 789	+0.8743 075	-0.0259 981
66	.0102 314	.0628 580	.2031 635	.8753 214	.0258 583
67	.0101 633	.0624 277	.2016 478	.8763 342	.0257 177
68	.0100 950	.0619 966	.2001 319	.8773 460	.0255 763
69	.0100 266	.0615 649	.1986 158	.8783 568	.0254 343
0.870	+0.0099 580	-0.0611 324	+0.1970 994	+0.8793 666	-0.0252 915
71	.0098 893	.0606 993	.1955 828	.8803 753	.0251 480
72	.0098 204	.0602 655	.1940 660	.8813 829	.0250 038
73	.0097 514	.0598 310	.1925 489	.8823 895	.0248 588
74	.0096 823	.0593 958	.1910 316	.8833 950	.0247 131
0.875	+0.0096 130	-0.0589 600	+0.1895 142	+0.8843 994	-0.0245 667
76	.0095 436	.0585 234	.1879 965	.8854 028	.0244 195
77	.0094 741	.0580 862	.1864 786	.8864 051	.0242 716
78	.0094 044	.0576 483	.1849 605	.8874 063	.0241 229
79	.0093 346	.0572 098	.1834 423	.8884 064	.0239 735
0.880	+0.0092 646	-0.0567 706	+0.1819 238	+0.8894 054	-0.0238 234
81	.0091 946	.0563 307	.1804 052	.8904 034	.0236 725
82	.0091 243	.0558 901	.1788 864	.8914 002	.0235 208
83	.0090 540	.0554 489	.1773 675	.8923 959	.0233 685
84	.0089 835	.0550 071	.1758 484	.8933 906	.0232 153
0.885	+0.0089 128	-0.0545 645	+0.1743 291	+0.8943 841	-0.0230 615
86	.0088 421	.0541 214	.1728 097	.8953 765	.0229 068
87	.0087 712	.0536 776	.1712 901	.8963 677	.0227 515
88	.0087 001	.0532 331	.1697 704	.8973 579	.0225 953
89	.0086 290	.0527 880	.1682 506	.8983 469	.0224 385
0.890	+0.0085 577	-0.0523 422	+0.1667 306	+0.8993 348	-0.0222 808
91	.0084 863	.0518 958	.1652 105	.9003 215	.0221 224
92	.0084 147	.0514 488	.1636 903	.9013 071	.0219 633
93	.0083 430	.0510 012	.1621 700	.9022 915	.0218 033
94	.0082 712	.0505 529	.1606 495	.9032 748	.0216 427
0.895	+0.0081 992	-0.0501 039	+0.1591 290	+0.9042 569	-0.0214 812
96	.0081 271	.0496 544	.1576 084	.9052 379	.0213 190
97	.0080 549	.0492 042	.1560 877	.9062 177	.0211 560
98	.0079 826	.0487 534	.1545 669	.9071 963	.0209 923
99	.0079 101	.0483 020	.1530 460	.9081 737	.0208 278
0.900	+0.0078 375	-0.0478 500	+0.1515 250	+0.9091 500	-0.0206 625

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.900	+0.0078 375	-0.0478 500	+0.1515 250	+0.9091 500	-0.0206 625
01	.0077 648	.0473 974	.1500 040	.9101 251	.0204 964
02	.0076 919	.0469 441	.1484 829	.9110 990	.0203 296
03	.0076 189	.0464 903	.1469 617	.9120 716	.0201 620
04	.0075 458	.0460 358	.1454 405	.9130 431	.0199 936
0.905	+0.0074 726	-0.0455 807	+0.1439 192	+0.9140 134	-0.0198 245
06	.0073 992	.0451 251	.1423 979	.9149 825	.0196 546
07	.0073 257	.0446 688	.1408 766	.9159 503	.0194 838
08	.0072 521	.0442 119	.1393 552	.9169 170	.0193 124
09	.0071 784	.0437 545	.1378 338	.9178 824	.0191 401
0.910	+0.0071 045	-0.0432 964	+0.1363 124	+0.9188 466	-0.0189 670
11	.0070 305	.0428 378	.1347 910	.9198 095	.0187 932
12	.0069 564	.0423 786	.1332 695	.9207 712	.0186 185
13	.0068 821	.0419 188	.1317 481	.9217 317	.0184 431
14	.0068 078	.0414 584	.1302 266	.9226 909	.0182 669
0.915	+0.0067 333	-0.0409 975	+0.1287 052	+0.9236 480	-0.0180 899
16	.0066 587	.0405 359	.1271 837	.9246 056	.0179 121
17	.0065 840	.0400 738	.1256 623	.9255 611	.0177 335
18	.0065 091	.0396 112	.1241 409	.9265 153	.0175 541
19	.0064 341	.0391 480	.1226 196	.9274 682	.0173 739
0.920	+0.0063 590	-0.0386 842	+0.1210 982	+0.9284 198	-0.0171 930
21	.0062 838	.0382 198	.1195 769	.9293 702	.0170 112
22	.0062 085	.0377 549	.1180 557	.9303 193	.0168 286
23	.0061 330	.0372 894	.1165 345	.9312 671	.0166 452
24	.0060 575	.0368 234	.1150 134	.9322 136	.0164 610
0.925	+0.0059 818	-0.0363 568	+0.1134 923	+0.9331 588	-0.0162 760
26	.0059 060	.0358 897	.1119 713	.9341 027	.0160 902
27	.0058 301	.0354 221	.1104 503	.9350 453	.0159 036
28	.0057 540	.0349 539	.1089 295	.9359 866	.0157 162
29	.0056 779	.0344 851	.1074 087	.9369 265	.0155 280
0.930	+0.0056 016	-0.0340 158	+0.1058 880	+0.9378 652	-0.0153 389
31	.0055 252	.0335 460	.1043 674	.9388 025	.0151 491
32	.0054 487	.0330 757	.1028 469	.9397 385	.0149 584
33	.0053 721	.0326 048	.1013 265	.9406 731	.0147 669
34	.0052 953	.0321 334	.0998 062	.9416 064	.0145 746
0.935	+0.0052 185	-0.0316 615	+0.0982 861	+0.9425 384	-0.0143 815
36	.0051 415	.0311 891	.0967 661	.9434 690	.0141 875
37	.0050 644	.0307 161	.0952 461	.9443 982	.0139 927
38	.0049 873	.0302 426	.0937 264	.9453 261	.0137 971
39	.0049 100	.0297 687	.0922 067	.9462 527	.0136 007
0.940	+0.0048 325	-0.0292 942	+0.0906 872	+0.9471 778	-0.0134 035
41	.0047 550	.0288 192	.0891 679	.9481 016	.0132 054
42	.0046 774	.0283 437	.0876 487	.9490 240	.0130 065
43	.0045 996	.0278 677	.0861 297	.9499 451	.0128 067
44	.0045 218	.0273 912	.0846 108	.9508 647	.0126 062
0.945	+0.0044 438	-0.0269 142	+0.0830 922	+0.9517 830	-0.0124 048
46	.0043 657	.0264 367	.0815 737	.9526 998	.0122 025
47	.0042 875	.0259 587	.0800 553	.9536 152	.0119 994
48	.0042 093	.0254 802	.0785 372	.9545 293	.0117 955
49	.0041 309	.0250 013	.0770 193	.9554 419	.0115 908
0.950	+0.0040 523	-0.0245 219	+0.0755 016	+0.9563 531	-0.0113 852

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
0.950	+0.0040 523	-0.0245 219	+0.0755 016	+0.9563 531	-0.0113 852
51	.0039 737	.0240 420	.0739 840	.9572 629	.0111 787
52	.0038 950	.0235 616	.0724 667	.9581 713	.0109 714
53	.0038 162	.0230 807	.0709 496	.9590 782	.0107 633
54	.0037 372	.0225 994	.0694 328	.9599 837	.0105 543
0.955	+0.0036 582	-0.0221 176	+0.0679 162	+0.9608 878	-0.0103 445
56	.0035 791	.0216 354	.0663 998	.9617 904	.0101 338
57	.0034 998	.0211 526	.0648 836	.9626 915	.0099 223
58	.0034 205	.0206 695	.0633 677	.9635 912	.0097 099
59	.0033 410	.0201 858	.0618 520	.9644 895	.0094 967
0.960	+0.0032 614	-0.0197 018	+0.0603 366	+0.9653 862	-0.0092 826
61	.0031 818	.0192 172	.0588 215	.9662 815	.0090 676
62	.0031 020	.0187 322	.0573 066	.9671 754	.0088 518
63	.0030 221	.0182 468	.0557 921	.9680 677	.0086 351
64	.0029 422	.0177 610	.0542 778	.9689 586	.0084 176
0.965	+0.0028 621	-0.0172 746	+0.0527 638	+0.9698 480	-0.0081 992
66	.0027 819	.0167 879	.0512 500	.9707 359	.0079 799
67	.0027 017	.0163 007	.0497 366	.9716 222	.0077 598
68	.0026 213	.0158 131	.0482 235	.9725 071	.0075 388
69	.0025 408	.0153 251	.0467 107	.9733 905	.0073 170
0.970	+0.0024 603	-0.0148 366	+0.0451 982	+0.9742 724	-0.0070 942
71	.0023 796	.0143 478	.0436 860	.9751 527	.0068 706
72	.0022 989	.0138 585	.0421 742	.9760 315	.0066 461
73	.0022 180	.0133 687	.0406 627	.9769 088	.0064 208
74	.0021 371	.0128 786	.0391 515	.9777 846	.0061 945
0.975	+0.0020 560	-0.0123 881	+0.0376 407	+0.9786 588	-0.0059 674
76	.0019 749	.0118 971	.0361 303	.9795 315	.0057 394
77	.0018 936	.0114 058	.0346 202	.9804 026	.0055 106
78	.0018 123	.0109 140	.0331 104	.9812 722	.0052 808
79	.0017 309	.0104 219	.0316 010	.9821 402	.0050 502
0.980	+0.0016 493	-0.0099 294	+0.0300 920	+0.9830 066	-0.0048 187
81	.0015 677	.0094 364	.0285 834	.9838 715	.0045 863
82	.0014 860	.0089 431	.0270 752	.9847 348	.0043 530
83	.0014 042	.0084 494	.0255 674	.9855 966	.0041 188
84	.0013 223	.0079 553	.0240 599	.9864 567	.0038 837
0.985	+0.0012 403	-0.0074 608	+0.0225 529	+0.9873 153	-0.0036 477
86	.0011 583	.0069 660	.0210 463	.9881 723	.0034 109
87	.0010 761	.0064 707	.0195 401	.9890 277	.0031 731
88	.0009 939	.0059 751	.0180 343	.9898 814	.0029 344
89	.0009 115	.0054 792	.0165 289	.9907 336	.0026 949
0.990	+0.0008 291	-0.0049 828	+0.0150 240	+0.9915 842	-0.0024 544
91	.0007 466	.0044 861	.0135 195	.9924 331	.0022 131
92	.0006 640	.0039 891	.0120 155	.9932 804	.0019 708
93	.0005 813	.0034 917	.0105 119	.9941 261	.0017 276
94	.0004 985	.0029 939	.0090 088	.9949 702	.0014 836
0.995	+0.0004 156	-0.0024 958	+0.0075 061	+0.9958 126	-0.0012 386
96	.0003 327	.0019 973	.0060 039	.9966 534	.0009 927
97	.0002 496	.0014 985	.0045 022	.9974 925	.0007 459
98	.0001 665	.0009 993	.0030 010	.9983 300	.0004 982
99	.0000 833	.0004 998	.0015 002	+0.9991 658	.0002 495
1.000	+0.0000 000	-0.0000 000	+0.0000 000	+1.0000 000	-0.0000 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.000	-0.0000 000	+0.0000 000	-0.0000 000	+1.0000 000	+0.0000 000
01	.0000 834	.0005 002	.0014 997	.0008 325	.0002 505
02	.0001 668	.0010 007	.0029 990	.0016 633	.0005 018
03	.0002 504	.0015 015	.0044 977	.0024 925	.0007 541
04	.0003 340	.0020 026	.0059 959	.0033 199	.0010 073
1.005	-0.0004 177	+0.0025 041	-0.0074 936	+1.0041 457	+0.0012 615
06	.0005 015	.0030 059	.0089 908	.0049 698	.0015 166
07	.0005 853	.0035 080	.0104 874	.0057 922	.0017 725
08	.0006 693	.0040 104	.0119 835	.0066 129	.0020 295
09	.0007 533	.0045 131	.0134 790	.0074 319	.0022 873
1.010	-0.0008 374	+0.0050 162	-0.0149 740	+1.0082 492	+0.0025 461
11	.0009 216	.0055 195	.0164 684	.0090 647	.0028 058
12	.0010 059	.0060 231	.0179 623	.0098 786	.0030 664
13	.0010 902	.0065 271	.0194 555	.0106 907	.0033 280
14	.0011 746	.0070 313	.0209 482	.0115 010	.0035 905
1.015	-0.0012 591	+0.0075 358	-0.0224 404	+1.0123 097	+0.0038 540
16	.0013 437	.0080 406	.0239 319	.0131 166	.0041 184
17	.0014 283	.0085 457	.0254 228	.0139 217	.0043 837
18	.0015 130	.0090 511	.0269 131	.0147 251	.0046 500
19	.0015 978	.0095 567	.0284 029	.0155 268	.0049 172
1.020	-0.0016 827	+0.0100 626	-0.0298 920	+1.0163 266	+0.0051 853
21	.0017 676	.0105 688	.0313 804	.0171 248	.0054 544
22	.0018 526	.0110 753	.0328 683	.0179 211	.0057 245
23	.0019 377	.0115 820	.0343 555	.0187 156	.0059 955
24	.0020 228	.0120 890	.0358 421	.0195 084	.0062 675
1.025	-0.0021 081	+0.0125 963	-0.0373 280	+1.0202 994	+0.0065 404
26	.0021 933	.0131 038	.0388 133	.0210 886	.0068 142
27	.0022 787	.0136 116	.0402 979	.0218 760	.0070 891
28	.0023 641	.0141 196	.0417 819	.0226 616	.0073 648
29	.0024 496	.0146 279	.0432 652	.0234 454	.0076 416
1.030	-0.0025 352	+0.0151 364	-0.0447 478	+1.0242 274	+0.0079 193
31	.0026 209	.0156 451	.0462 297	.0250 075	.0081 979
32	.0027 066	.0161 541	.0477 110	.0257 859	.0084 776
33	.0027 923	.0166 633	.0491 915	.0265 624	.0087 582
34	.0028 782	.0171 728	.0506 714	.0273 370	.0090 397
1.035	-0.0029 641	+0.0176 825	-0.0521 505	+1.0281 099	+0.0093 222
36	.0030 500	.0181 924	.0536 289	.0288 808	.0096 057
37	.0031 361	.0187 025	.0551 066	.0296 500	.0098 902
38	.0032 222	.0192 129	.0565 836	.0304 173	.0101 756
39	.0033 083	.0197 235	.0580 599	.0311 827	.0104 621
1.040	-0.0033 946	+0.0202 342	-0.0595 354	+1.0319 462	+0.0107 494
41	.0034 808	.0207 452	.0610 101	.0327 079	.0110 378
42	.0035 672	.0212 564	.0624 841	.0334 677	.0113 272
43	.0036 536	.0217 678	.0639 574	.0342 257	.0116 175
44	.0037 401	.0222 794	.0654 299	.0349 817	.0119 088
1.045	-0.0038 266	+0.0227 913	-0.0669 016	+1.0357 359	+0.0122 011
46	.0039 132	.0233 033	.0683 725	.0364 881	.0124 944
47	.0039 999	.0238 154	.0698 427	.0372 385	.0127 886
48	.0040 866	.0243 278	.0713 121	.0379 870	.0130 839
49	.0041 733	.0248 404	.0727 807	.0387 335	.0133 801
1.050	-0.0042 602	+0.0253 531	-0.0742 484	+1.0394 781	+0.0136 773

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.050	-0.0042 602	+0.0253 531	-0.0742 484	+1.0394 781	+0.0136 773
51	.0043 470	.0258 660	.0757 154	.0402 208	.0139 756
52	.0044 340	.0263 791	.0771 816	.0409 616	.0142 748
53	.0045 210	.0268 924	.0786 469	.0417 005	.0145 750
54	.0046 080	.0274 059	.0801 114	.0424 374	.0148 762
1.055	-0.0046 951	+0.0279 195	-0.0815 751	+1.0431 723	+0.0151 784
56	.0047 823	.0284 332	.0830 379	.0439 053	.0154 816
57	.0048 695	.0289 471	.0844 999	.0446 364	.0157 859
58	.0049 568	.0294 612	.0859 611	.0453 655	.0160 911
59	.0050 441	.0299 755	.0874 213	.0460 927	.0163 973
1.060	-0.0051 315	+0.0304 898	-0.0888 808	+1.0468 178	+0.0167 045
61	.0052 189	.0310 044	.0903 393	.0475 410	.0170 128
62	.0053 064	.0315 190	.0917 970	.0482 623	.0173 220
63	.0053 939	.0320 339	.0932 538	.0489 815	.0176 323
64	.0054 815	.0325 488	.0947 097	.0496 988	.0179 436
1.065	-0.0055 691	+0.0330 639	-0.0961 647	+1.0504 140	+0.0182 559
66	.0056 568	.0335 791	.0976 188	.0511 273	.0185 692
67	.0057 445	.0340 944	.0990 719	.0518 385	.0188 835
68	.0058 322	.0346 099	.1005 242	.0525 477	.0191 988
69	.0059 201	.0351 255	.1019 756	.0532 550	.0195 152
1.070	-0.0060 079	+0.0356 412	-0.1034 260	+1.0539 602	+0.0198 326
71	.0060 958	.0361 570	.1048 755	.0546 633	.0201 510
72	.0061 838	.0366 729	.1063 240	.0553 645	.0204 704
73	.0062 718	.0371 889	.1077 716	.0560 636	.0207 909
74	.0063 598	.0377 051	.1092 183	.0567 606	.0211 124
1.075	-0.0064 479	+0.0382 213	-0.1106 640	+1.0574 557	+0.0214 349
76	.0065 360	.0387 376	.1121 087	.0581 486	.0217 585
77	.0066 242	.0392 540	.1135 524	.0588 395	.0220 831
78	.0067 124	.0397 706	.1149 952	.0595 284	.0224 087
79	.0068 007	.0402 872	.1164 370	.0602 151	.0227 353
1.080	-0.0068 890	+0.0408 038	-0.1178 778	+1.0608 998	+0.0230 630
81	.0069 773	.0413 206	.1193 175	.0615 825	.0233 918
82	.0070 657	.0418 374	.1207 563	.0622 630	.0237 216
83	.0071 541	.0423 544	.1221 941	.0629 414	.0240 524
84	.0072 425	.0428 714	.1236 308	.0636 178	.0243 843
1.085	-0.0073 310	+0.0433 884	-0.1250 666	+1.0642 920	+0.0247 172
86	.0074 195	.0439 055	.1265 013	.0649 642	.0250 511
87	.0075 081	.0444 227	.1279 349	.0656 342	.0253 861
88	.0075 967	.0449 399	.1293 675	.0663 021	.0257 222
89	.0076 853	.0454 572	.1307 991	.0669 679	.0260 593
1.090	-0.0077 740	+0.0459 746	-0.1322 296	+1.0676 316	+0.0263 975
91	.0078 627	.0464 920	.1336 590	.0682 931	.0267 367
92	.0079 515	.0470 094	.1350 874	.0689 525	.0270 770
93	.0080 402	.0475 269	.1365 147	.0696 097	.0274 183
94	.0081 290	.0480 444	.1379 409	.0702 648	.0277 607
1.095	-0.0082 179	+0.0485 619	-0.1393 660	+1.0709 178	+0.0281 042
96	.0083 067	.0490 795	.1407 900	.0715 686	.0284 487
97	.0083 956	.0495 971	.1422 129	.0722 172	.0287 943
98	.0084 846	.0501 147	.1436 347	.0728 636	.0291 410
99	.0085 735	.0506 323	.1450 554	.0735 079	.0294 887
1.100	-0.0086 625	+0.0511 500	-0.1464 750	+1.0741 500	+0.0298 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.100	-0.0086 625	+0.0511 500	-0.1464 750	+1.0741 500	+0.0298 375
01	.0087 515	.0516 677	.1478 934	.0747 899	.0301 874
02	.0088 406	.0521 854	.1493 107	.0754 276	.0305 383
03	.0089 296	.0527 030	.1507 269	.0760 631	.0308 903
04	.0090 187	.0532 207	.1521 419	.0766 964	.0312 434
1.105	-0.0091 078	+0.0537 384	-0.1535 557	+1.0773 276	+0.0315 976
06	.0091 970	.0542 561	.1549 684	.0779 564	.0319 528
07	.0092 862	.0547 738	.1563 799	.0785 831	.0323 092
08	.0093 754	.0552 915	.1577 903	.0792 076	.0326 666
09	.0094 646	.0558 091	.1591 994	.0798 298	.0330 251
1.110	-0.0095 538	+0.0563 268	-0.1606 074	+1.0804 498	+0.0333 847
11	.0096 431	.0568 444	.1620 142	.0810 675	.0337 454
12	.0097 324	.0573 620	.1634 197	.0816 830	.0341 071
13	.0098 217	.0578 795	.1648 241	.0822 962	.0344 700
14	.0099 110	.0583 971	.1662 272	.0829 072	.0348 339
1.115	-0.0100 003	+0.0589 146	-0.1676 291	+1.0835 160	+0.0351 990
16	.0100 897	.0594 320	.1690 298	.0841 224	.0355 651
17	.0101 791	.0599 495	.1704 293	.0847 266	.0359 323
18	.0102 685	.0604 668	.1718 275	.0853 285	.0363 007
19	.0103 579	.0609 842	.1732 245	.0859 281	.0366 701
1.120	-0.0104 474	+0.0615 014	-0.1746 202	+1.0865 254	+0.0370 406
21	.0105 368	.0620 187	.1760 146	.0871 205	.0374 123
22	.0106 263	.0625 358	.1774 078	.0877 132	.0377 850
23	.0107 158	.0630 529	.1787 997	.0883 036	.0381 589
24	.0108 053	.0635 700	.1801 903	.0888 917	.0385 338
1.125	-0.0108 948	+0.0640 869	-0.1815 796	+1.0894 775	+0.0389 099
26	.0109 843	.0646 038	.1829 676	.0900 610	.0392 871
27	.0110 738	.0651 206	.1843 543	.0906 422	.0396 654
28	.0111 634	.0656 374	.1857 397	.0912 210	.0400 448
29	.0112 529	.0661 540	.1871 238	.0917 974	.0404 253
1.130	-0.0113 425	+0.0666 706	-0.1885 066	+1.0923 716	+0.0408 070
31	.0114 321	.0671 870	.1898 880	.0929 433	.0411 898
32	.0115 217	.0677 034	.1912 681	.0935 128	.0415 736
33	.0116 113	.0682 197	.1926 469	.0940 798	.0419 587
34	.0117 009	.0687 359	.1940 243	.0946 445	.0423 448
1.135	-0.0117 905	+0.0692 520	-0.1954 003	+1.0952 068	+0.0427 321
36	.0118 801	.0697 679	.1967 750	.0957 668	.0431 205
37	.0119 698	.0702 838	.1981 483	.0963 243	.0435 100
38	.0120 594	.0707 995	.1995 203	.0968 795	.0439 006
39	.0121 490	.0713 151	.2008 908	.0974 323	.0442 924
1.140	-0.0122 387	+0.0718 306	-0.2022 600	+1.0979 826	+0.0446 853
41	.0123 283	.0723 460	.2036 277	.0985 306	.0450 794
42	.0124 180	.0728 613	.2049 941	.0990 762	.0454 746
43	.0125 076	.0733 764	.2063 590	.0996 193	.0458 709
44	.0125 973	.0738 913	.2077 225	.1001 600	.0462 684
1.145	-0.0126 869	+0.0744 062	-0.2090 846	+1.1006 983	+0.0466 670
46	.0127 766	.0749 209	.2104 453	.1012 342	.0470 668
47	.0128 662	.0754 354	.2118 045	.1017 676	.0474 677
48	.0129 559	.0759 498	.2131 623	.1022 985	.0478 698
49	.0130 455	.0764 640	.2145 186	.1028 271	.0482 730
1.150	-0.0131 352	+0.0769 781	-0.2158 734	+1.1033 531	+0.0486 773

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.150	-0.0131 352	+0.0769 781	-0.2158 734	+1.1033 531	+0.0486 773
51	.0132 248	.0774 920	.2172 268	.1038 767	.0490 829
52	.0133 144	.0780 058	.2185 787	.1043 979	.0494 895
53	.0134 041	.0785 194	.2199 292	.1049 165	.0498 974
54	.0134 937	.0790 328	.2212 781	.1054 327	.0503 063
1.155	-0.0135 833	+0.0795 460	-0.2226 256	+1.1059 464	+0.0507 165
56	.0136 730	.0800 591	.2239 715	.1064 576	.0511 278
57	.0137 626	.0805 720	.2253 160	.1069 663	.0515 402
58	.0138 522	.0810 846	.2266 589	.1074 725	.0519 539
59	.0139 418	.0815 971	.2280 003	.1079 762	.0523 687
1.160	-0.0140 314	+0.0821 094	-0.2293 402	+1.1084 774	+0.0527 846
61	.0141 209	.0826 215	.2306 785	.1089 761	.0532 018
62	.0142 105	.0831 334	.2320 153	.1094 723	.0536 201
63	.0143 001	.0836 451	.2333 505	.1099 659	.0540 396.
64	.0143 896	.0841 566	.2346 842	.1104 570	.0544 602
1.165	-0.0144 791	+0.0846 679	-0.2360 163	+1.1109 455	+0.0548 820
66	.0145 687	.0851 790	.2373 469	.1114 315	.0553 050
67	.0146 582	.0856 898	.2386 758	.1119 150	.0557 292
68	.0147 477	.0862 004	.2400 032	.1123 959	.0561 546
69	.0148 372	.0867 108	.2413 290	.1128 742	.0565 811
1.170	-0.0149 266	+0.0872 210	-0.2426 532	+1.1133 500	+0.0570 089
71	.0150 161	.0877 309	.2439 758	.1138 232	.0574 378
72	.0151 055	.0882 406	.2452 967	.1142 938	.0578 679
73	.0151 949	.0887 500	.2466 161	.1147 618	.0582 992
74	.0152 843	.0892 592	.2479 338	.1152 272	.0587 317
1.175	-0.0153 737	+0.0897 682	-0.2492 499	+1.1156 900	+0.0591 654
76	.0154 630	.0902 769	.2505 643	.1161 503	.0596 003
77	.0155 524	.0907 853	.2518 771	.1166 079	.0600 365
78	.0156 417	.0912 935	.2531 883	.1170 629	.0604 736
79	.0157 310	.0918 014	.2544 978	.1175 153	.0609 121
1.180	-0.0158 203	+0.0923 090	-0.2558 056	+1.1179 650	+0.0613 517
81	.0159 095	.0928 164	.2571 117	.1184 122	.0617 926
82	.0159 987	.0933 235	.2584 161	.1188 567	.0622 347
83	.0160 879	.0938 303	.2597 189	.1192 985	.0626 780
84	.0161 771	.0943 369	.2610 199	.1197 377	.0631 225
1.185	-0.0162 663	+0.0948 431	-0.2623 193	+1.1201 743	+0.0635 682
86	.0163 554	.0953 491	.2636 169	.1206 081	.0640 151
87	.0164 445	.0958 548	.2649 128	.1210 394	.0644 632
88	.0165 336	.0963 601	.2662 070	.1214 679	.0649 126
89	.0166 226	.0968 652	.2674 995	.1218 938	.0653 631
1.190	-0.0167 116	+0.0973 700	-0.2687 902	+1.1223 170	+0.0658 149
91	.0168 006	.0978 744	.2700 792	.1227 375	.0662 679
92	.0168 896	.0983 786	.2713 664	.1231 553	.0667 221
93	.0169 785	.0988 824	.2726 518	.1235 704	.0671 775
94	.0170 674	.0993 859	.2739 355	.1239 828	.0676 342
1.195	-0.0171 562	+0.0998 891	-0.2752 174	+1.1243 925	+0.0680 921
96	.0172 450	.1003 919	.2764 975	.1247 994	.0685 512
97	.0173 338	.1008 945	.2777 758	.1252 037	.0690 116
98	.0174 226	.1013 966	.2790 524	.1256 052	.0694 731
99	.0175 113	.1018 985	.2803 271	.1260 040	.0699 360
1.200	-0.0176 000	+0.1024 000	-0.2816 000	+1.1264 000	+0.0704 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.200	-0.0176 000	+0.1024 000	-0.2816 000	+1.1264 000	+0.0704 000
01	.0176 886	.1029 012	.2828 711	.1267 933	.0708 653
02	.0177 773	.1034 020	.2841 404	.1271 838	.0713 318
03	.0178 658	.1039 024	.2854 078	.1275 716	.0717 996
04	.0179 544	.1044 025	.2866 734	.1279 566	.0722 686
1.205	-0.0180 429	+0.1049 023	-0.2879 371	+1.1283 389	+0.0727 388
06	.0181 313	.1054 016	.2891 990	.1287 184	.0732 103
07	.0182 197	.1059 006	.2904 590	.1290 950	.0736 831
08	.0183 081	.1063 992	.2917 171	.1294 689	.0741 571
09	.0183 964	.1068 975	.2929 734	.1298 401	.0746 323
1.210	-0.0184 847	+0.1073 954	-0.2942 278	+1.1302 084	+0.0751 088
11	.0185 730	.1078 928	.2954 803	.1305 739	.0755 865
12	.0186 612	.1083 899	.2967 309	.1309 366	.0760 655
13	.0187 493	.1088 866	.2979 796	.1312 964	.0765 458
14	.0188 374	.1093 829	.2992 263	.1316 535	.0770 273
1.215	-0.0189 255	+0.1098 789	-0.3004 712	+1.1320 077	+0.0775 101
16	.0190 135	.1103 744	.3017 141	.1323 591	.0779 941
17	.0191 015	.1108 694	.3029 551	.1327 077	.0784 794
18	.0191 894	.1113 641	.3041 941	.1330 534	.0789 660
19	.0192 772	.1118 584	.3054 312	.1333 962	.0794 538
1.220	-0.0193 651	+0.1123 522	-0.3066 664	+1.1337 362	+0.0799 429
21	.0194 528	.1128 457	.3078 995	.1340 734	.0804 333
22	.0195 405	.1133 387	.3091 307	.1344 076	.0809 250
23	.0196 282	.1138 312	.3103 599	.1347 390	.0814 179
24	.0197 158	.1143 233	.3115 872	.1350 675	.0819 121
1.225	-0.0198 034	+0.1148 150	-0.3128 124	+1.1353 932	+0.0824 076
26	.0198 909	.1153 063	.3140 356	.1357 159	.0829 043
27	.0199 783	.1157 971	.3152 569	.1360 357	.0834 024
28	.0200 657	.1162 874	.3164 761	.1363 527	.0839 017
29	.0201 530	.1167 773	.3176 932	.1366 667	.0844 023
1.230	-0.0202 403	+0.1172 668	-0.3189 084	+1.1369 778	+0.0849 042
31	.0203 275	.1177 557	.3201 215	.1372 859	.0854 074
32	.0204 147	.1182 442	.3213 326	.1375 912	.0859 118
33	.0205 018	.1187 323	.3225 416	.1378 935	.0864 176
34	.0205 888	.1192 198	.3237 485	.1381 929	.0869 247
1.235	-0.0206 758	+0.1197 069	-0.3249 534	+1.1384 893	+0.0874 330
36	.0207 627	.1201 935	.3261 562	.1387 828	.0879 426
37	.0208 496	.1206 796	.3273 570	.1390 733	.0884 536
38	.0209 364	.1211 653	.3285 556	.1393 608	.0889 658
39	.0210 231	.1216 504	.3297 521	.1396 454	.0894 794
1.240	-0.0211 098	+0.1221 350	-0.3309 466	+1.1399 270	+0.0899 942
41	.0211 964	.1226 192	.3321 389	.1402 057	.0905 104
42	.0212 829	.1231 028	.3333 291	.1404 813	.0910 279
43	.0213 694	.1235 859	.3345 171	.1407 539	.0915 466
44	.0214 557	.1240 685	.3357 031	.1410 236	.0920 667
1.245	-0.0215 421	+0.1245 506	-0.3368 869	+1.1412 902	+0.0925 881
46	.0216 283	.1250 322	.3380 685	.1415 539	.0931 108
47	.0217 145	.1255 132	.3392 480	.1418 145	.0936 349
48	.0218 006	.1259 937	.3404 253	.1420 720	.0941 602
49	.0218 867	.1264 737	.3416 005	.1423 266	.0946 869
1.250	-0.0219 727	+0.1269 531	-0.3427 734	+1.1425 781	+0.0952 148

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.250	-0.0219 727	+0.1269 531	-0.3427 734	+1.1425 781	+0.0952 148
51	.0220 586	.1274 320	.3439 442	.1428 266	.0957 442
52	.0221 444	.1279 104	.3451 128	.1430 720	.0962 748
53	.0222 301	.1283 882	.3462 792	.1433 144	.0968 067
54	.0223 158	.1288 654	.3474 434	.1435 537	.0973 400
1.255	-0.0224 014	+0.1293 421	-0.3486 053	+1.1437 900	+0.0978 746
56	.0224 869	.1298 182	.3497 650	.1440 232	.0984 106
57	.0225 724	.1302 938	.3509 225	.1442 533	.0989 479
58	.0226 578	.1307 688	.3520 778	.1444 803	.0994 865
59	.0227 430	.1312 432	.3532 308	.1447 042	.1000 264
1.260	-0.0228 283	+0.1317 170	-0.3543 816	+1.1449 250	+0.1005 677
61	.0229 134	.1321 903	.3555 301	.1451 428	.1011 104
62	.0229 984	.1326 630	.3566 763	.1453 574	.1016 543
63	.0230 834	.1331 351	.3578 202	.1455 689	.1021 997
64	.0231 683	.1336 065	.3589 619	.1457 773	.1027 463
1.265	-0.0232 531	+0.1340 774	-0.3601 012	+1.1459 826	+0.1032 943
66	.0233 378	.1345 477	.3612 383	.1461 847	.1038 437
67	.0234 224	.1350 174	.3623 731	.1463 837	.1043 944
68	.0235 070	.1354 865	.3635 055	.1465 795	.1049 465
69	.0235 914	.1359 549	.3646 356	.1467 722	.1054 999
1.270	-0.0236 758	+0.1364 228	-0.3657 634	+1.1469 618	+0.1060 547
71	.0237 601	.1368 900	.3668 888	.1471 481	.1066 108
72	.0238 443	.1373 566	.3680 119	.1473 314	.1071 683
73	.0239 284	.1378 225	.3691 327	.1475 114	.1077 272
74	.0240 124	.1382 879	.3702 511	.1476 882	.1082 874
1.275	-0.0240 963	+0.1387 525	-0.3713 671	+1.1478 619	+0.1088 490
76	.0241 802	.1392 166	.3724 807	.1480 324	.1094 119
77	.0242 639	.1396 800	.3735 920	.1481 997	.1099 762
78	.0243 476	.1401 427	.3747 008	.1483 637	.1105 419
79	.0244 311	.1406 048	.3758 073	.1485 246	.1111 090
1.280	-0.0245 146	+0.1410 662	-0.3769 114	+1.1486 822	+0.1116 774
81	.0245 979	.1415 270	.3780 130	.1488 367	.1122 473
82	.0246 812	.1419 871	.3791 122	.1489 878	.1128 184
83	.0247 643	.1424 465	.3802 090	.1491 358	.1133 910
84	.0248 474	.1429 053	.3813 033	.1492 805	.1139 650
1.285	-0.0249 304	+0.1433 634	-0.3823 952	+1.1494 220	+0.1145 403
86	.0250 133	.1438 207	.3834 847	.1495 602	.1151 170
87	.0250 960	.1442 774	.3845 717	.1496 951	.1156 951
88	.0251 787	.1447 334	.3856 562	.1498 268	.1162 746
89	.0252 613	.1451 888	.3867 382	.1499 552	.1168 555
1.290	-0.0253 437	+0.1456 434	-0.3878 178	+1.1500 804	+0.1174 378
91	.0254 261	.1460 973	.3888 949	.1502 022	.1180 215
92	.0255 083	.1465 505	.3899 694	.1503 208	.1186 065
93	.0255 905	.1470 029	.3910 415	.1504 360	.1191 930
94	.0256 725	.1474 547	.3921 110	.1505 480	.1197 808
1.295	-0.0257 545	+0.1479 058	-0.3931 780	+1.1506 566	+0.1203 701
96	.0258 363	.1483 561	.3942 425	.1507 620	.1209 608
97	.0259 180	.1488 057	.3953 045	.1508 640	.1215 528
98	.0259 996	.1492 545	.3963 639	.1509 626	.1221 463
99	.0260 811	.1497 026	.3974 207	.1510 580	.1227 412
1.300	-0.0261 625	+0.1501 500	-0.3984 750	+1.1511 500	+0.1233 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.300	-0.0261 625	+0.1501 500	-0.3984 750	+1.1511 500	+0.1233 375
01	.0262 438	.1505 966	.3995 267	.1512 387	.1239 352
02	.0263 249	.1510 425	.4005 758	.1513 240	.1245 343
03	.0264 060	.1514 876	.4016 224	.1514 059	.1251 349
04	.0264 869	.1519 320	.4026 664	.1514 845	.1257 368
1.305	-0.0265 678	+0.1523 756	-0.4037 077	+1.1515 597	+0.1263 402
06	.0266 485	.1528 184	.4047 465	.1516 315	.1269 450
07	.0267 291	.1532 605	.4057 826	.1517 000	.1275 512
08	.0268 095	.1537 017	.4068 161	.1517 650	.1281 588
09	.0268 899	.1541 422	.4078 470	.1518 267	.1287 679
1.310	-0.0269 701	+0.1545 820	-0.4088 752	+1.1518 850	+0.1293 784
11	.0270 502	.1550 209	.4099 008	.1519 398	.1299 903
12	.0271 302	.1554 590	.4109 237	.1519 913	.1306 037
13	.0272 101	.1558 964	.4119 440	.1520 393	.1312 184
14	.0272 899	.1563 329	.4129 616	.1520 839	.1318 347
1.315	-0.0273 695	+0.1567 686	-0.4139 765	+1.1521 250	+0.1324 523
16	.0274 490	.1572 035	.4149 887	.1521 627	.1330 714
17	.0275 284	.1576 377	.4159 982	.1521 970	.1336 920
18	.0276 076	.1580 709	.4170 051	.1522 278	.1343 139
19	.0276 868	.1585 034	.4180 092	.1522 552	.1349 374
1.320	-0.0277 658	+0.1589 350	-0.4190 106	+1.1522 790	+0.1355 622
21	.0278 446	.1593 658	.4200 092	.1522 995	.1361 886
22	.0279 234	.1597 958	.4210 052	.1523 164	.1368 163
23	.0280 020	.1602 249	.4219 983	.1523 299	.1374 455
24	.0280 805	.1606 532	.4229 888	.1523 398	.1380 762
1.325	-0.0281 588	+0.1610 807	-0.4239 765	+1.1523 463	+0.1387 083
26	.0282 371	.1615 072	.4249 614	.1523 493	.1393 419
27	.0283 152	.1619 330	.4259 435	.1523 487	.1399 770
28	.0283 931	.1623 578	.4269 229	.1523 447	.1406 135
29	.0284 709	.1627 818	.4278 994	.1523 371	.1412 515
1.330	-0.0285 486	+0.1632 050	-0.4288 732	+1.1523 260	+0.1418 909
31	.0286 262	.1636 272	.4298 442	.1523 113	.1425 318
32	.0287 036	.1640 486	.4308 123	.1522 931	.1431 741
33	.0287 809	.1644 691	.4317 776	.1522 714	.1438 180
34	.0288 580	.1648 887	.4327 401	.1522 461	.1444 633
1.335	-0.0289 350	+0.1653 074	-0.4336 998	+1.1522 173	+0.1451 101
36	.0290 118	.1657 252	.4346 566	.1521 849	.1457 583
37	.0290 886	.1661 421	.4356 105	.1521 489	.1464 081
38	.0291 651	.1665 581	.4365 616	.1521 093	.1470 593
39	.0292 416	.1669 733	.4375 098	.1520 662	.1477 120
1.340	-0.0293 179	+0.1673 874	-0.4384 552	+1.1520 194	+0.1483 661
41	.0293 940	.1678 007	.4393 976	.1519 691	.1490 218
42	.0294 700	.1682 131	.4403 372	.1519 152	.1496 789
43	.0295 459	.1686 245	.4412 738	.1518 576	.1503 376
44	.0296 216	.1690 350	.4422 076	.1517 964	.1509 977
1.345	-0.0296 971	+0.1694 445	-0.4431 384	+1.1517 317	+0.1516 593
46	.0297 725	.1698 531	.4440 663	.1516 632	.1523 224
47	.0298 478	.1702 608	.4449 912	.1515 912	.1529 870
48	.0299 229	.1706 675	.4459 133	.1515 155	.1536 531
49	.0299 979	.1710 733	.4468 323	.1514 361	.1543 207
1.350	-0.0300 727	+0.1714 781	-0.4477 484	+1.1513 531	+0.1549 898

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.350	-0.0300 727	+0.1714 781	-0.4477 484	+1.1513 531	+0.1549 898
51	.0301 473	.1718 820	.4486 616	.1512 665	.1556 604
52	.0302 218	.1722 849	.4495 717	.1511 761	.1563 326
53	.0302 962	.1726 868	.4504 789	.1510 821	.1570 062
54	.0303 703	.1730 877	.4513 831	.1509 844	.1576 813
1.355	-0.0304 444	+0.1734 877	-0.4522 843	+1.1508 831	+0.1583 579
56	.0305 182	.1738 867	.4531 825	.1507 780	.1590 361
57	.0305 920	.1742 846	.4540 776	.1506 692	.1597 158
58	.0306 655	.1746 816	.4549 698	.1505 567	.1603 969
59	.0307 389	.1750 776	.4558 589	.1504 405	.1610 796
1.360	-0.0308 122	+0.1754 726	-0.4567 450	+1.1503 206	+0.1617 638
61	.0308 852	.1758 666	.4576 280	.1501 970	.1624 496
62	.0309 582	.1762 596	.4585 079	.1500 696	.1631 368
63	.0310 309	.1766 516	.4593 848	.1499 385	.1638 256
64	.0311 035	.1770 425	.4602 587	.1498 037	.1645 159
1.365	-0.0311 759	+0.1774 325	-0.4611 294	+1.1496 651	+0.1652 078
66	.0312 482	.1778 214	.4619 971	.1495 227	.1659 011
67	.0313 203	.1782 092	.4628 616	.1493 766	.1665 961
68	.0313 922	.1785 960	.4637 231	.1492 267	.1672 925
69	.0314 639	.1789 818	.4645 814	.1490 730	.1679 905
1.370	-0.0315 355	+0.1793 666	-0.4654 366	+1.1489 156	+0.1686 900
71	.0316 069	.1797 503	.4662 887	.1487 543	.1693 910
72	.0316 782	.1801 329	.4671 376	.1485 893	.1700 936
73	.0317 492	.1805 145	.4679 834	.1484 204	.1707 978
74	.0318 201	.1808 950	.4688 261	.1482 478	.1715 035
1.375	-0.0318 909	+0.1812 744	-0.4696 655	+1.1480 713	+0.1722 107
76	.0319 614	.1816 528	.4705 018	.1478 910	.1729 195
77	.0320 318	.1820 301	.4713 350	.1477 069	.1736 298
78	.0321 020	.1824 063	.4721 649	.1475 189	.1743 417
79	.0321 720	.1827 814	.4729 916	.1473 271	.1750 551
1.380	-0.0322 419	+0.1831 554	-0.4738 152	+1.1471 314	+0.1757 701
81	.0323 115	.1835 284	.4746 355	.1469 319	.1764 867
82	.0323 810	.1839 002	.4754 526	.1467 286	.1772 048
83	.0324 503	.1842 709	.4762 664	.1465 213	.1779 245
84	.0325 194	.1846 406	.4770 771	.1463 102	.1786 457
1.385	-0.0325 884	+0.1850 091	-0.4778 845	+1.1460 952	+0.1793 686
86	.0326 571	.1853 765	.4786 886	.1458 763	.1800 929
87	.0327 257	.1857 427	.4794 895	.1456 535	.1808 189
88	.0327 941	.1861 079	.4802 871	.1454 269	.1815 464
89	.0328 623	.1864 719	.4810 814	.1451 963	.1822 755
1.390	-0.0329 303	+0.1868 348	-0.4818 724	+1.1449 618	+0.1830 062
91	.0329 981	.1871 965	.4826 601	.1447 233	.1837 384
92	.0330 658	.1875 571	.4834 446	.1444 810	.1844 723
93	.0331 332	.1879 165	.4842 257	.1442 347	.1852 077
94	.0332 005	.1882 748	.4850 035	.1439 845	.1859 447
1.395	-0.0332 676	+0.1886 319	-0.4857 779	+1.1437 303	+0.1866 833
96	.0333 344	.1889 879	.4865 490	.1434 722	.1874 234
97	.0334 011	.1893 427	.4873 168	.1432 101	.1881 652
98	.0334 676	.1896 963	.4880 813	.1429 440	.1889 085
99	.0335 339	.1900 487	.4888 423	.1426 740	.1896 535
1.400	-0.0336 000	+0.1904 000	-0.4896 000	+1.1424 000	+0.1904 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.400	-0.0336 000	+0.1904 000	-0.4896 000	+1.1424 000	+0.1904 000
01	.0336 659	.1907 501	.4903 543	.1421 220	.1911 481
02	.0337 316	.1910 990	.4911 052	.1418 400	.1918 979
03	.0337 971	.1914 466	.4918 528	.1415 540	.1926 492
04	.0338 624	.1917 931	.4925 969	.1412 640	.1934 021
1.405	-0.0339 275	+0.1921 384	-0.4933 376	+1.1409 700	+0.1941 567
06	.0339 924	.1924 825	.4940 749	.1406 720	.1949 128
07	.0340 571	.1928 253	.4948 087	.1403 699	.1956 706
08	.0341 216	.1931 670	.4955 391	.1400 639	.1964 299
09	.0341 859	.1935 074	.4962 661	.1397 537	.1971 909
1.410	-0.0342 500	+0.1938 466	-0.4969 896	+1.1394 396	+0.1979 535
11	.0343 139	.1941 845	.4977 096	.1391 213	.1987 177
12	.0343 776	.1945 212	.4984 262	.1387 991	.1994 835
13	.0344 411	.1948 567	.4991 393	.1384 727	.2002 509
14	.0345 043	.1951 909	.4998 489	.1381 423	.2010 200
1.415	-0.0345 674	+0.1955 239	-0.5005 550	+1.1378 078	+0.2017 907
16	.0346 302	.1958 556	.5012 576	.1374 692	.2025 630
17	.0346 928	.1961 861	.5019 567	.1371 265	.2033 369
18	.0347 553	.1965 153	.5026 522	.1367 797	.2041 125
19	.0348 175	.1968 432	.5033 443	.1364 288	.2048 897
1.420	-0.0348 795	+0.1971 698	-0.5040 328	+1.1360 738	+0.2056 685
21	.0349 412	.1974 952	.5047 177	.1357 147	.2064 490
22	.0350 028	.1978 193	.5053 991	.1353 515	.2072 311
23	.0350 641	.1981 421	.5060 769	.1349 841	.2080 148
24	.0351 253	.1984 636	.5067 511	.1346 126	.2088 002
1.425	-0.0351 862	+0.1987 838	-0.5074 218	+1.1342 369	+0.2095 873
26	.0352 469	.1991 027	.5080 888	.1338 571	.2103 759
27	.0353 073	.1994 203	.5087 523	.1334 731	.2111 662
28	.0353 676	.1997 366	.5094 122	.1330 850	.2119 582
29	.0354 276	.2000 515	.5100 684	.1326 927	.2127 518
1.430	-0.0354 874	+0.2003 652	-0.5107 210	+1.1322 962	+0.2135 471
31	.0355 470	.2006 775	.5113 700	.1318 955	.2143 440
32	.0356 063	.2009 885	.5120 153	.1314 906	.2151 426
33	.0356 655	.2012 981	.5126 570	.1310 815	.2159 428
34	.0357 244	.2016 064	.5132 950	.1306 682	.2167 447
1.435	-0.0357 830	+0.2019 134	-0.5139 293	+1.1302 508	+0.2175 483
36	.0358 415	.2022 190	.5145 600	.1298 290	.2183 535
37	.0358 997	.2025 232	.5151 870	.1294 031	.2191 604
38	.0359 577	.2028 261	.5158 103	.1289 729	.2199 689
39	.0360 154	.2031 277	.5164 299	.1285 385	.2207 791
1.440	-0.0360 730	+0.2034 278	-0.5170 456	+1.1280 998	+0.2215 910
41	.0361 302	.2037 266	.5176 579	.1276 569	.2224 046
42	.0361 873	.2040 240	.5182 663	.1272 097	.2232 199
43	.0362 441	.2043 201	.5188 710	.1267 583	.2240 368
44	.0363 007	.2046 147	.5194 720	.1263 026	.2248 554
1.445	-0.0363 570	+0.2049 080	-0.5200 691	+1.1258 426	+0.2256 757
46	.0364 131	.2051 998	.5206 626	.1253 783	.2264 976
47	.0364 690	.2054 902	.5212 522	.1249 097	.2273 213
48	.0365 246	.2057 793	.5218 381	.1244 368	.2281 466
49	.0365 800	.2060 669	.5224 202	.1239 596	.2289 736
1.450	-0.0366 352	+0.2063 531	-0.5229 984	+1.1234 781	+0.2298 023

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.450	-0.0366 352	+0.2063 531	-0.5229 984	+1.1234 781	+0.2298 023
51	.0366 901	.2066 379	.5235 729	.1229 923	.2306 328
52	.0367 447	.2069 213	.5241 436	.1225 021	.2314 649
53	.0367 991	.2072 032	.5247 104	.1220 076	.2322 987
54	.0368 533	.2074 837	.5252 734	.1215 088	.2331 342
1.455	-0.0369 072	+0.2077 628	-0.5258 325	+1.1210 056	+0.2339 714
56	.0369 609	.2080 404	.5263 878	.1204 981	.2348 103
57	.0370 143	.2083 165	.5269 393	.1199 862	.2356 509
58	.0370 675	.2085 912	.5274 869	.1194 699	.2364 932
59	.0371 204	.2088 645	.5280 306	.1189 493	.2373 372
1.460	-0.0371 731	+0.2091 362	-0.5285 704	+1.1184 242	+0.2381 829
61	.0372 255	.2094 065	.5291 063	.1178 948	.2390 304
62	.0372 776	.2096 754	.5296 383	.1173 610	.2398 795
63	.0373 296	.2099 427	.5301 664	.1168 228	.2407 304
64	.0373 812	.2102 086	.5306 906	.1162 802	.2415 830
1.465	-0.0374 326	+0.2104 730	-0.5312 108	+1.1157 331	+0.2424 373
66	.0374 837	.2107 359	.5317 271	.1151 816	.2432 934
67	.0375 346	.2109 972	.5322 395	.1146 257	.2441 511
68	.0375 853	.2112 571	.5327 479	.1140 654	.2450 106
69	.0376 356	.2115 155	.5332 523	.1135 006	.2458 718
1.470	-0.0376 857	+0.2117 724	-0.5337 528	+1.1129 314	+0.2467 348
71	.0377 356	.2120 277	.5342 493	.1123 577	.2475 995
72	.0377 851	.2122 815	.5347 418	.1117 795	.2484 659
73	.0378 344	.2125 338	.5352 303	.1111 969	.2493 340
74	.0378 835	.2127 846	.5357 147	.1106 098	.2502 039
1.475	-0.0379 323	+0.2130 338	-0.5361 952	+1.1100 182	+0.2510 755
76	.0379 808	.2132 815	.5366 717	.1094 221	.2519 489
77	.0380 290	.2135 276	.5371 441	.1088 215	.2528 240
78	.0380 770	.2137 722	.5376 124	.1082 164	.2537 009
79	.0381 247	.2140 152	.5380 767	.1076 068	.2545 795
1.480	-0.0381 722	+0.2142 566	-0.5385 370	+1.1069 926	+0.2554 598
81	.0382 193	.2144 965	.5389 931	.1063 740	.2563 419
82	.0382 662	.2147 348	.5394 452	.1057 508	.2572 258
83	.0383 128	.2149 716	.5398 932	.1051 231	.2581 114
84	.0383 592	.2152 067	.5403 371	.1044 908	.2589 988
1.485	-0.0384 052	+0.2154 403	-0.5407 769	+1.1038 539	+0.2598 879
86	.0384 510	.2156 723	.5412 126	.1032 125	.2607 788
87	.0384 966	.2159 027	.5416 442	.1025 666	.2616 715
88	.0385 418	.2161 314	.5420 716	.1019 160	.2625 659
89	.0385 867	.2163 586	.5424 949	.1012 609	.2634 621
1.490	-0.0386 314	+0.2165 842	-0.5429 140	+1.1006 012	+0.2643 601
91	.0386 758	.2168 081	.5433 290	.0999 368	.2652 598
92	.0387 199	.2170 304	.5437 398	.0992 679	.2661 613
93	.0387 637	.2172 511	.5441 464	.0985 944	.2670 646
94	.0388 073	.2174 702	.5445 488	.0979 163	.2679 697
1.495	-0.0388 505	+0.2176 876	-0.5449 471	+1.0972 335	+0.2688 765
96	.0388 935	.2179 034	.5453 411	.0965 461	.2697 851
97	.0389 362	.2181 175	.5457 309	.0958 540	.2706 956
98	.0389 786	.2183 300	.5461 165	.0951 573	.2716 077
99	.0390 207	.2185 408	.5464 979	.0944 560	.2725 217
1.500	-0.0390 625	+0.2187 500	-0.5468 750	+1.0937 500	+0.2734 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.500	-0.0390 625	+0.2187 500	-0.5468 750	+1.0937 500	+0.2734 375
01	.0391 040	.2189 575	.5472 479	.0930 393	.2743 551
02	.0391 452	.2191 633	.5476 165	.0923 240	.2752 744
03	.0391 862	.2193 675	.5479 808	.0916 040	.2761 956
04	.0392 268	.2195 699	.5483 409	.0908 793	.2771 185
1.505	-0.0392 672	+0.2197 707	-0.5486 967	+1.0901 499	+0.2780 433
06	.0393 072	.2199 698	.5490 482	.0894 157	.2789 698
07	.0393 470	.2201 672	.5493 954	.0886 769	.2798 982
08	.0393 864	.2203 629	.5497 382	.0879 334	.2808 283
09	.0394 256	.2205 569	.5500 768	.0871 851	.2817 603
1.510	-0.0394 644	+0.2207 492	-0.5504 110	+1.0864 322	+0.2826 941
11	.0395 030	.2209 397	.5507 409	.0856 744	.2836 297
12	.0395 412	.2211 286	.5510 664	.0849 120	.2845 671
13	.0395 792	.2213 157	.5513 876	.0841 448	.2855 063
14	.0396 168	.2215 010	.5517 044	.0833 728	.2864 473
1.515	-0.0396 541	+0.2216 847	-0.5520 168	+1.0825 961	+0.2873 902
16	.0396 911	.2218 666	.5523 248	.0818 145	.2883 349
17	.0397 279	.2220 467	.5526 285	.0810 283	.2892 814
18	.0397 643	.2222 251	.5529 277	.0802 372	.2902 297
19	.0398 004	.2224 018	.5532 226	.0794 413	.2911 799
1.520	-0.0398 362	+0.2225 766	-0.5535 130	+1.0786 406	+0.2921 318
21	.0398 716	.2227 498	.5537 989	.0778 352	.2930 857
22	.0399 068	.2229 211	.5540 805	.0770 249	.2940 413
23	.0399 416	.2230 906	.5543 576	.0762 098	.2949 988
24	.0399 762	.2232 584	.5546 302	.0753 898	.2959 581
1.525	-0.0400 104	+0.2234 244	-0.5548 983	+1.0745 650	+0.2969 193
26	.0400 443	.2235 886	.5551 620	.0737 354	.2978 823
27	.0400 779	.2237 510	.5554 212	.0729 009	.2988 471
28	.0401 111	.2239 116	.5556 759	.0720 616	.2998 138
29	.0401 441	.2240 704	.5559 261	.0712 174	.3007 824
1.530	-0.0401 767	+0.2242 274	-0.5561 718	+1.0703 684	+0.3017 528
31	.0402 090	.2243 825	.5564 130	.0695 144	.3027 250
32	.0402 410	.2245 359	.5566 496	.0686 556	.3036 991
33	.0402 726	.2246 874	.5568 817	.0677 919	.3046 751
34	.0403 040	.2248 370	.5571 092	.0669 233	.3056 529
1.535	-0.0403 350	+0.2249 849	-0.5573 322	+1.0660 497	+0.3066 326
36	.0403 657	.2251 308	.5575 506	.0651 713	.3076 141
37	.0403 960	.2252 750	.5577 644	.0642 879	.3085 975
38	.0404 260	.2254 173	.5579 737	.0633 996	.3095 828
39	.0404 557	.2255 577	.5581 783	.0625 064	.3105 699
1.540	-0.0404 851	+0.2256 962	-0.5583 784	+1.0616 082	+0.3115 589
41	.0405 141	.2258 329	.5585 738	.0607 051	.3125 498
42	.0405 428	.2259 677	.5587 646	.0597 970	.3135 426
43	.0405 711	.2261 007	.5589 508	.0588 840	.3145 372
44	.0405 991	.2262 317	.5591 323	.0579 660	.3155 337
1.545	-0.0406 268	+0.2263 609	-0.5593 092	+1.0570 430	+0.3165 321
46	.0406 542	.2264 881	.5594 814	.0561 150	.3175 324
47	.0406 812	2266 135	.5596 489	.0551 821	.3185 345
48	.0407 078	.2267 370	.5598 118	.0542 441	.3195 386
49	.0407 342	.2268 585	.5599 700	.0533 011	.3205 445
1.550	-0.0407 602	+0.2269 781	-0.5601 234	+1.0523 531	+0.3215 523

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TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.550	-0.0407 602	+0.2269 781	-0.5601 234	+1.0523 531	+0.3215 523
51	.0407 858	.2270 958	.5602 722	.0514 001	.3225 621
52	.0408 111	.2272 116	.5604 163	.0504 421	.3235 737
53	.0408 360	.2273 255	.5605 556	.0494 790	.3245 872
54	.0408 607	.2274 374	.5606 902	.0485 109	.3256 026
1.555	-0.0408 849	+0.2275 473	-0.5608 200	+1.0475 377	+0.3266 199
56	.0409 088	.2276 553	.5609 451	.0465 595	.3276 391
57	.0409 324	.2277 614	.5610 654	.0455 762	.3286 602
58	.0409 556	.2278 655	.5611 810	.0445 878	.3296 833
59	.0409 785	.2279 677	.5612 918	.0435 944	.3307 082
1.560	-0.0410 010	+0.2280 678	-0.5613 978	+1.0425 958	+0.3317 350
61	.0410 231	.2281 660	.5614 989	.0415 922	.3327 638
62	.0410 449	.2282 623	.5615 953	.0405 835	.3337 945
63	.0410 664	.2283 565	.5616 869	.0395 697	.3348 271
64	.0410 874	.2284 488	.5617 736	.0385 507	.3358 616
1.565	-0.0411 082	+0.2285 390	-0.5618 555	+1.0375 266	+0.3368 980
66	.0411 285	.2286 273	.5619 325	.0364 974	.3379 364
67	.0411 486	.2287 135	.5620 047	.0354 631	.3389 767
68	.0411 682	.2287 977	.5620 720	.0344 236	.3400 189
69	.0411 875	.2288 800	.5621 344	.0333 790	.3410 630
1.570	-0.0412 064	+0.2289 602	-0.5621 920	+1.0323 292	+0.3421 091
71	.0412 250	2290 383	.5622 447	.0312 742	.3431 571
72	.0412 432	.2291 145	.5622 924	.0302 141	.3442 070
73	.0412 610	.2291 886	.5623 352	.0291 487	.3452 589
74	.0412 785	.2292 606	.5623 732	.0280 782	.3463 127
1.575	-0.0412 956	+0.2293 307	-0.5624 062	+1.0270 025	+0.3473 685
76	.0413 123	.2293 986	.5624 342	.0259 216	.3484 262
77	.0413 286	.2294 645	.5624 573	.0248 355	.3494 859
78	.0413 446	.2295 284	.5624 754	.0237 442	.3505 475
79	.0413 602	.2295 901	.5624 886	.0226 476	.3516 110
1.580	-0.0413 755	+0.2296 498	-0.5624 968	+1.0215 458	+0.3526 765
81	.0413 903	.2297 075	.5625 000	.0204 388	.3537 440
82	.0414 048	.2297 630	.5624 981	.0193 265	.3548 134
83	.0414 189	.2298 164	.5624 913	.0182 090	.3558 848
84	.0414 326	.2298 678	.5624 795	.0170 862	.3569 581
1.585	-0.0414 460	+0.2299 170	-0.5624 626	+1.0159 582	+0.3580 334
86	.0414 590	.2299 642	.5624 407	.0148 248	.3591 107
87	.0414 715	.2300 092	.5624 138	.0136 862	.3601 900
88	.0414 838	.2300 521	.5623 818	.0125 423	.3612 712
89	.0414 956	.2300 929	.5623 447	.0113 931	.3623 543
1.590	-0.0415 070	+0.2301 316	-0.5623 026	+1.0102 386	+0.3634 395
91	.0415 181	.2301 681	.5622 554	.0090 787	.3645 266
92	.0415 287	.2302 025	.5622 031	.0079 136	.3656 157
93	.0415 390	.2302 347	.5621 456	.0067 431	.3667 068
94	.0415 489	.2302 648	.5620 831	.0055 673	.3677 999
1.595	-0.0415 584	+0.2302 928	-0.5620 154	+1.0043 862	+0.3688 949
96	.0415 675	.2303 186	.5619 427	.0031 997	.3699 920
97	.0415 762	.2303 422	.5618 647	.0020 078	.3710 910
98	.0415 845	.2303 636	.5617 817	+1.0008 106	.3721 920
99	.0415 925	.2303 829	.5616 934	+0.9996 080	.3732 950
1.600	-0.0416 000	+0.2304 000	-0.5616 000	+0.9984 000	+0.3744 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.600	-0.0416 000	+0.2304 000	-0.5616 000	+0.9984 000	+0.3744 000
01	.0416 071	.2304 149	.5615 014	.9971 866	.3755 070
02	.0416 139	.2304 276	.5613 976	.9959 679	.3766 160
03	.0416 202	.2304 381	.5612 886	.9947 437	.3777 270
04	.0416 261	.2304 464	.5611 745	.9935 142	.3788 400
1.605	-0.0416 317	+0.2304 526	-0.5610 550	+0.9922 792	+0.3799 550
06	.0416 368	.2304 564	.5609 304	.9910 388	.3810 720
07	.0416 416	.2304 581	.5608 005	.9897 929	.3821 910
08	.0416 459	.2304 576	.5606 654	.9885 417	.3833 121
09	.0416 498	.2304 548	.5605 250	.9872 849	.3844 351
1.610	-0.0416 533	+0.2304 498	-0.5603 794	+0.9860 228	+0.3855 602
11	.0416 564	.2304 425	.5602 285	.9847 551	.3866 873
12	.0416 591	.2304 330	.5600 723	.9834 820	.3878 164
13	.0416 614	.2304 212	.5599 108	.9822 034	.3889 475
14	.0416 633	.2304 072	.5597 440	.9809 194	.3900 806
1.615	-0.0416 648	+0.2303 910	-0.5595 718	+0.9796 298	+0.3912 158
16	.0416 658	.2303 724	.5593 944	.9783 348	.3923 530
17	.0416 664	.2303 516	.5592 116	.9770 342	.3934 922
18	.0416 667	.2303 285	.5590 235	.9757 281	.3946 335
19	.0416 665	.2303 031	.5588 300	.9744 166	.3957 768
1.620	-0.0416 659	+0.2302 754	-0.5586 312	+0.9730 994	+0.3969 221
21	.0416 648	.2302 455	.5584 269	.9717 768	.3980 695
22	.0416 634	.2302 132	.5582 173	.9704 486	.3992 189
23	.0416 615	.2301 786	.5580 024	.9691 148	.4003 704
24	.0416 592	.2301 417	.5577 820	.9677 755	.4015 239
1.625	-0.0416 565	+0.2301 025	-0.5575 562	+0.9664 307	+0.4026 794
26	.0416 534	.2300 610	.5573 249	.9650 802	.4038 370
27	.0416 498	.2300 172	.5570 883	.9637 242	.4049 967
28	.0416 458	.2299 710	.5568 462	.9623 626	.4061 584
29	.0416 414	.2299 224	.5565 986	.9609 954	.4073 222
1.630	-0.0416 365	+0.2298 716	-0.5563 456	+0.9596 226	+0.4084 880
31	.0416 312	.2298 183	.5560 871	.9582 441	.4096 559
32	.0416 255	.2297 628	.5558 231	.9568 601	.4108 258
33	.0416 194	.2297 048	.5555 537	.9554 704	.4119 978
34	.0416 128	.2296 445	.5552 787	.9540 751	.4131 719
1.635	-0.0416 058	+0.2295 818	-0.5549 983	+0.9526 742	+0.4143 480
36	.0415 983	.2295 168	.5547 123	.9512 676	.4155 262
37	.0415 905	.2294 493	.5544 208	.9498 554	.4167 065
38	.0415 821	.2293 795	.5541 237	.9484 375	.4178 889
39	.0415 734	.2293 073	.5538 211	.9470 139	.4190 733
1.640	-0.0415 642	+0.2292 326	-0.5535 130	+0.9455 846	+0.4202 598
41	.0415 545	.2291 556	.5531 992	.9441 497	.4214 484
42	.0415 444	.2290 762	.5528 799	.9427 091	.4226 391
43	.0415 339	.2289 943	.5525 550	.9412 627	.4238 319
44	.0415 229	.2289 100	.5522 245	.9398 107	.4250 267
1.645	-0.0415 115	+0.2288 233	-0.5518 884	+0.9383 529	+0.4262 237
46	.0414 996	.2287 342	.5515 467	.9368 894	.4274 227
47	.0414 873	.2286 426	.5511 994	.9354 202	.4286 239
48	.0414 745	.2285 485	.5508 464	.9339 453	.4298 271
49	.0414 613	.2284 521	.5504 877	.9324 646	.4310 324
1.650	-0.0414 477	+0.2283 531	-0.5501 234	+0.9309 781	+0.4322 398

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.650	-0.0414 477	+0.2283 531	-0.5501 234	+0.9309 781	+0.4322 398
51	.0414 335	.2282 517	.5497 535	.9294 859	.4334 494
52	.0414 190	.2281 479	.5493 778	.9279 879	.4346 610
53	.0414 039	.2280 415	.5489 965	.9264 842	.4358 748
54	.0413 884	.2279 327	.5486 095	.9249 746	.4370 906
1.655	-0.0413 725	+0.2278 214	-0.5482 168	+0.9234 593	+0.4383 086
56	.0413 561	.2277 076	.5478 183	.9219 381	.4395 286
57	.0413 392	.2275 913	.5474 141	.9204 112	.4407 508
58	.0413 219	.2274 725	.5470 042	.9188 784	.4419 752
59	.0413 041	.2273 512	.5465 886	.9173 399	.4432 016
1.660	-0.0412 859	+0.2272 274	-0.5461 672	+0.9157 954	+0.4444 301
61	.0412 671	.2271 011	.5457 400	.9142 452	.4456 608
62	.0412 480	.2269 723	.5453 070	.9126 891	.4468 936
63	.0412 283	.2268 409	.5448 683	.9111 271	.4481 286
64	.0412 082	.2267 070	.5444 237	.9095 593	.4493 656
1.665	-0.0411 876	+0.2265 705	-0.5439 734	+0.9079 857	+0.4506 048
66	.0411 666	.2264 315	.5435 172	.9064 061	.4518 462
67	.0411 450	.2262 900	.5430 552	.9048 207	.4530 896
68	.0411 230	.2261 459	.5425 874	.9032 293	.4543 352
69	.0411 006	.2259 992	.5421 137	.9016 321	.4555 830
1.670	-0.0410 776	+0.2258 500	-0.5416 342	+0.9000 290	+0.4568 329
71	.0410 542	.2256 982	.5411 488	.8984 199	.4580 849
72	.0410 303	.2255 438	.5406 575	.8968 049	.4593 391
73	.0410 059	.2253 868	.5401 604	.8951 840	.4605 955
74	.0409 811	.2252 272	.5396 573	.8935 572	.4618 540
1.675	-0.0409 557	+0.2250 650	-0.5391 483	+0.8919 244	+0.4631 146
76	.0409 299	.2249 003	.5386 335	.8902 857	.4643 774
77	.0409 036	.2247 329	.5381 126	.8886 410	.4656 424
78	.0408 768	.2245 629	.5375 859	.8869 903	.4669 095
79	.0408 495	.2243 903	.5370 532	.8853 337	.4681 788
1.680	-0.0408 218	+0.2242 150	-0.5365 146	+0.8836 710	+0.4694 502
81	.0407 935	.2240 372	.5359 699	.8820 024	.4707 239
82	.0407 648	.2238 567	.5354 193	.8803 278	.4719 997
83	.0407 355	.2236 735	.5348 628	.8786 472	.4732 776
84	.0407 058	.2234 877	.5343 002	.8769 605	.4745 578
1.685	-0.0406 756	+0.2232 993	-0.5337 316	+0.8752 679	+0.4758 401
86	.0406 449	.2231 081	.5331 570	.8735 692	.4771 246
87	.0406 137	.2229 144	.5325 764	.8718 645	.4784 112
88	.0405 820	.2227 179	.5319 897	.8701 537	.4797 001
89	.0405 498	.2225 188	.5313 970	.8684 369	.4809 912
1.690	-0.0405 171	+0.2223 170	-0.5307 982	+0.8667 140	+0.4822 844
91	.0404 839	.2221 125	.5301 934	.8649 850	.4835 798
92	.0404 502	.2219 053	.5295 824	.8632 500	.4848 774
93	.0404 160	.2216 954	.5289 654	.8615 089	.4861 772
94	.0403 813	.2214 828	.5283 423	.8597 616	.4874 792
1.695	-0.0403 461	+0.2212 675	-0.5277 131	+0.8580 083	+0.4887 834
96	.0403 104	.2210 494	.5270 777	.8562 489	.4900 898
97	.0402 742	.2208 287	.5264 363	.8544 834	.4913 984
98	.0402 375	.2206 052	.5257 887	.8527 117	.4927 093
99	.0402 002	.2203 790	.5251 349	.8509 339	.4940 223
1.700	-0.0401 625	+0.2201 500	-0.5244 750	+0.8491 500	+0.4953 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.700	-0.0401 625	+0.2201 500	-0.5244 750	+0.8491 500	+0.4953 375
01	.0401 242	.2199 183	.5238 089	.8473 599	.4966 549
02	.0400 855	.2196 838	.5231 366	.8455 637	.4979 746
03	.0400 462	.2194 466	.5224 582	.8437 613	.4992 965
04	.0400 064	.2192 066	.5217 735	.8419 527	.5006 206
1.705	-0.0399 661	+0.2189 639	-0.5210 827	+0.8401 380	+0.5019 469
06	.0399 252	.2187 184	.5203 856	.8383 171	.5032 754
07	.0398 839	.2184 700	.5196 823	.8364 900	.5046 061
08	.0398 420	.2182 189	.5189 727	.8346 566	.5059 391
09	.0397 996	.2179 651	.5182 569	.8328 171	.5072 743
1.710	-0.0397 567	+0.2177 084	-0.5175 348	+0.8309 714	+0.5086 118
11	.0397 133	.2174 489	.5168 064	.8291 194	.5099 515
12	.0396 693	.2171 866	.5160 718	.8272 612	.5112 934
13	.0396 248	.2169 214	.5153 309	.8253 967	.5126 375
14	.0395 798	.2166 535	.5145 837	.8235 261	.5139 839
1.715	-0.0395 343	+0.2163 827	-0.5138 301	+0.8216 491	+0.5153 326
16	.0394 882	.2161 091	.5130 703	.8197 659	.5166 834
17	.0394 416	.2158 327	.5123 041	.8178 764	.5180 366
18	.0393 944	.2155 534	.5115 315	.8159 806	.5193 919
19	.0393 468	.2152 712	.5107 526	.8140 786	.5207 496
1.720	-0.0392 986	+0.2149 862	-0.5099 674	+0.8121 702	+0.5221 094
21	.0392 498	.2146 984	.5091 757	.8102 556	.5234 716
22	.0392 005	.2144 076	.5083 777	.8083 346	.5248 360
23	.0391 507	.2141 140	.5075 733	.8064 073	.5262 026
24	.0391 004	.2138 175	.5067 625	.8044 737	.5275 715
1.725	-0.0390 495	+0.2135 182	-0.5059 452	+0.8025 338	+0.5289 427
26	.0389 980	.2132 159	.5051 215	.8005 875	.5303 162
27	.0389 460	.2129 107	.5042 914	.7986 349	.5316 919
28	.0388 935	.2126 027	.5034 549	.7966 759	.5330 699
29	.0388 404	.2122 917	.5026 119	.7947 105	.5344 501
1.730	-0.0387 868	+0.2119 778	-0.5017 624	+0.7927 388	+0.5358 327
31	.0387 326	.2116 609	.5009 064	.7907 606	.5372 175
32	.0386 779	.2113 412	.5000 440	.7887 761	.5386 046
33	.0386 227	.2110 185	.4991 750	.7867 852	.5399 940
34	.0385 668	.2106 929	.4982 996	.7847 879	.5413 856
1.735	-0.0385 105	+0.2103 643	-0.4974 176	+0.7827 842	+0.5427 796
36	.0384 535	.2100 328	.4965 291	.7807 740	.5441 758
37	.0383 961	.2096 983	.4956 341	.7787 574	.5455 744
38	.0383 380	.2093 608	.4947 325	.7767 344	.5469 752
39	.0382 794	.2090 204	.4938 243	.7747 050	.5483 783
1.740	-0.0382 203	+0.2086 770	-0.4929 096	+0.7726 690	+0.5497 837
41	.0381 605	.2083 307	.4919 882	.7706 267	.5511 915
42	.0381 003	.2079 813	.4910 603	.7685 778	.5526 015
43	.0380 394	.2076 289	.4901 258	.7665 225	.5540 138
44	.0379 780	.2072 736	.4891 847	.7644 607	.5554 285
1.745	-0.0379 160	+0.2069 152	-0.4882 369	+0.7623 924	+0.5568 454
46	.0378 535	.2065 539	.4872 825	.7603 175	.5582 647
47	.0377 904	.2061 895	.4863 215	.7582 362	.5596 862
48	.0377 267	.2058 220	.4853 538	.7561 484	.5611 101
49	.0376 625	.2054 516	.4843 795	.7540 540	.5625 363
1.750	-0.0375 977	+0.2050 781	-0.4833 984	+0.7519 531	+0.5639 648

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p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.750	-0.0375 977	+0.2050 781	-0.4833 984	+0.7519 531	+0.5639 648
51	.0375 323	.2047 016	.4824 107	.7498 457	.5653 957
52	.0374 663	.2043 220	.4814 163	.7477 317	.5668 289
53	.0373 998	.2039 394	.4804 152	.7456 112	.5682 644
54	.0373 326	.2035 537	.4794 073	.7434 840	.5697 022
1.755	-0.0372 649	+0.2031 650	-0.4783 927	+0.7413 504	+0.5711 424
56	.0371 967	.2027 732	.4773 714	.7392 101	.5725 849
57	.0371 278	.2023 783	.4763 434	.7370 632	.5740 297
58	.0370 584	.2019 803	.4753 085	.7349 098	.5754 769
59	.0369 884	.2015 792	.4742 669	.7327 497	.5769 264
1.760	-0.0369 178	+0.2011 750	-0.4732 186	+0.7305 830	+0.5783 782
61	.0368 466	.2007 678	.4721 634	.7284 097	.5798 324
62	.0367 748	.2003 574	.4711 014	.7262 298	.5812 890
63	.0367 024	.1999 439	.4700 326	.7240 433	.5827 479
64	.0366 295	.1995 273	.4689 570	.7218 500	.5842 091
1.765	-0.0365 559	+0.1991 076	-0.4678 745	+0.7196 502	+0.5856 728
66	.0364 818	.1986 847	.4667 853	.7174 437	.5871 387
67	.0364 071	.1982 587	.4656 891	.7152 305	.5886 070
68	.0363 317	.1978 295	.4645 861	.7130 106	.5900 777
69	.0362 558	.1973 972	.4634 762	.7107 840	.5915 508
1.770	-0.0361 793	+0.1969 618	-0.4623 594	+0.7085 508	+0.5930 262
71	.0361 022	.1965 231	.4612 357	.7063 108	.5945 040
72	.0360 245	.1960 814	.4601 051	.7040 641	.5959 841
73	.0359 462	.1956 364	.4589 676	.7018 108	.5974 666
74	.0358 673	.1951 882	.4578 232	.6995 506	.5989 515
1.775	-0.0357 877	+0.1947 369	-0.4566 718	+0.6972 838	+0.6004 388
76	.0357 076	.1942 824	.4555 134	.6950 102	.6019 285
77	.0356 269	.1938 247	.4543 481	.6927 299	.6034 205
78	.0355 456	.1933 637	.4531 759	.6904 428	.6049 149
79	.0354 636	.1928 996	.4519 966	.6881 489	.6064 117
1.780	-0.0353 811	+0.1924 322	-0.4508 104	+0.6858 482	+0.6079 109
81	.0352 979	.1919 617	.4496 171	.6835 408	.6094 125
82	.0352 141	.1914 878	.4484 168	.6812 266	.6109 165
83	.0351 297	.1910 108	.4472 095	.6789 056	.6124 229
84	.0350 447	.1905 305	.4459 952	.6765 777	.6139 317
1.785	-0.0349 591	+0.1900 470	-0.4447 738	+0.6742 431	+0.6154 428
86	.0348 729	.1895 602	.4435 454	.6719 016	.6169 564
87	.0347 860	.1890 701	.4423 099	.6695 533	.6184 724
88	.0346 985	.1885 768	.4410 673	.6671 982	.6199 908
89	.0346 104	.1880 802	.4398 176	.6648 362	.6215 116
1.790	-0.0345 217	+0.1875 804	-0.4385 608	+0.6624 674	+0.6230 348
91	.0344 324	.1870 772	.4372 969	.6600 917	.6245 604
92	.0343 424	.1865 708	.4360 259	.6577 091	.6260 884
93	.0342 518	.1860 610	.4347 477	.6553 196	.6276 189
94	.0341 606	.1855 480	.4334 624	.6529 233	.6291 518
1.795	-0.0340 687	+0.1850 316	-0.4321 700	+0.6505 200	+0.6306 871
96	.0339 763	.1845 120	.4308 704	.6481 098	.6322 248
97	.0338 831	.1839 890	.4295 636	.6456 928	.6337 650
98	.0337 894	.1834 626	.4282 496	.6432 688	.6353 075
99	.0336 950	.1829 330	.4269 284	.6408 379	.6368 526
1.800	-0.0336 000	+0.1824 000	-0.4256 000	+0.6384 000	+0.6384 000

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.800	-0.0336 000	+0.1824 000	-0.4256 000	+0.6384 000	+0.6384 000
01	.0335 043	.1818 637	.4242 644	.6359 552	.6399 499
02	.0334 081	.1813 240	.4229 215	.6335 034	.6415 022
03	.0333 111	.1807 809	.4215 715	.6310 447	.6430 570
04	.0332 136	.1802 345	.4202 141	.6285 790	.6446 142
1.805	-0.0331 153	+0.1796 847	-0.4188 495	+0.6261 063	+0.6461 738
06	.0330 165	.1791 315	.4174 777	.6236 267	.6477 359
07	.0329 170	.1785 750	.4160 985	.6211 400	.6493 005
08	.0328 168	.1780 150	.4147 120	.6186 463	.6508 675
09	.0327 161	.1774 517	.4133 183	.6161 457	.6524 370
1.810	-0.0326 146	+0.1768 850	-0.4119 172	+0.6136 380	+0.6540 089
11	.0325 125	.1763 148	.4105 088	.6111 232	.6555 833
12	.0324 098	.1757 413	.4090 930	.6086 015	.6571 601
13	.0323 064	.1751 643	.4076 699	.6060 727	.6587 394
14	.0322 023	.1745 839	.4062 395	.6035 368	.6603 212
1.815	-0.0320 976	+0.1740 000	-0.4048 017	+0.6009 939	+0.6619 054
16	.0319 923	.1734 127	.4033 565	.5984 439	.6634 921
17	.0318 863	.1728 220	.4019 038	.5958 868	.6650 813
18	.0317 796	.1722 278	.4004 438	.5933 227	.6666 730
19	.0316 723	.1716 302	.3989 764	.5907 514	.6682 671
1.820	-0.0315 643	+0.1710 290	-0.3975 016	+0.5881 730	+0.6698 637
21	.0314 556	.1704 245	.3960 193	.5855 876	.6714 628
22	.0313 463	.1698 164	.3945 295	.5829 950	.6730 644
23	.0312 363	.1692 049	.3930 323	.5803 953	.6746 685
24	.0311 256	.1685 898	.3915 277	.5777 884	.6762 751
1.825	-0.0310 143	+0.1679 713	-0.3900 155	+0.5751 744	+0.6778 841
26	.0309 023	.1673 493	.3884 959	.5725 533	.6794 957
27	.0307 896	.1667 237	.3869 687	.5699 250	.6811 097
28	.0306 763	.1660 947	.3854 341	.5672 895	.6827 263
29	.0305 623	.1654 621	.3838 919	.5646 468	.6843 453
1.830	-0.0304 476	+0.1648 260	-0.3823 422	+0.5619 970	+0.6859 669
31	.0303 323	.1641 863	.3807 849	.5593 399	.6875 909
32	.0302 162	.1635 431	.3792 201	.5566 757	.6892 175
33	.0300 995	.1628 964	.3776 477	.5540 042	.6908 466
34	.0299 821	.1622 461	.3760 678	.5513 256	.6924 782
1.835	-0.0298 640	+0.1615 923	-0.3744 802	+0.5486 397	+0.6941 123
36	.0297 453	.1609 349	.3728 850	.5459 465	.6957 489
37	.0296 258	.1602 739	.3712 823	.5432 461	.6973 881
38	.0295 057	.1596 093	.3696 719	.5405 385	.6990 297
39	.0293 849	.1589 412	.3680 538	.5378 236	.7006 739
1.840	-0.0292 634	+0.1582 694	-0.3664 282	+0.5351 014	+0.7023 206
41	.0291 412	.1575 941	.3647 948	.5323 720	.7039 699
42	.0290 183	.1569 152	.3631 538	.5296 353	.7056 217
43	.0288 947	.1562 326	.3615 051	.5268 912	.7072 760
44	.0287 704	.1555 464	.3598 488	.5241 399	.7089 328
1.845	-0.0286 455	+0.1548 567	-0.3581 847	+0.5213 813	+0.7105 922
46	.0285 198	.1541 632	.3565 129	.5186 153	.7122 542
47	.0283 934	.1534 662	.3548 334	.5158 420	.7139 186
48	.0282 664	.1527 655	.3531 462	.5130 614	.7155 857
49	.0281 386	.1520 611	.3514 512	.5102 734	.7172 552
1.850	-0.0280 102	+0.1513 531	-0.3497 484	+0.5074 781	+0.7189 273

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.850	-0.0280 102	+0.1513 531	-0.3497 484	+0.5074 781	+0.7189 273
51	.0278 810	.1506 415	.3480 379	.5046 754	.7206 020
52	.0277 511	.1499 261	.3463 196	.5018 654	.7222 792
53	.0276 205	.1492 071	.3445 936	.4990 480	.7239 590
54	.0274 893	.1484 844	.3428 597	.4962 231	.7256 414
1.855	-0.0273 573	+0.1477 581	-0.3411 180	+0.4933 909	+0.7273 263
56	.0272 246	.1470 280	.3393 685	.4905 513	.7290 138
57	.0270 912	.1462 942	.3376 111	.4877 043	.7307 038
58	.0269 570	.1455 567	.3358 459	.4848 498	.7323 964
59	.0268 222	.1448 155	.3340 729	.4819 880	.7340 916
1.860	-0.0266 867	+0.1440 706	-0.3322 920	+0.4791 186	+0.7357 893
61	.0265 504	.1433 220	.3305 032	.4762 419	.7374 897
62	.0264 134	.1425 696	.3287 065	.4733 577	.7391 926
63	.0262 757	.1418 135	.3269 019	.4704 660	.7408 981
64	.0261 373	.1410 537	.3250 894	.4675 668	.7426 061
1.865	-0.0259 981	+0.1402 901	-0.3232 690	+0.4646 602	+0.7443 168
66	.0258 583	.1395 227	.3214 406	.4617 461	.7460 301
67	.0257 177	.1387 516	.3196 043	.4588 245	.7477 459
68	.0255 763	.1379 767	.3177 600	.4558 954	.7494 643
69	.0254 343	.1371 980	.3159 078	.4529 587	.7511 854
1.870	-0.0252 915	+0.1364 156	-0.3140 476	+0.4500 146	+0.7529 090
71	.0251 480	.1356 293	.3121 794	.4470 629	.7546 352
72	.0250 038	.1348 393	.3103 032	.4441 037	.7563 640
73	.0248 588	.1340 454	.3084 190	.4411 369	.7580 955
74	.0247 131	.1332 478	.3065 267	.4381 626	.7598 295
1.875	-0.0245 667	+0.1324 463	-0.3046 265	+0.4351 807	+0.7615 662
76	.0244 195	.1316 410	.3027 181	.4321 912	.7633 054
77	.0242 716	.1308 319	.3008 018	.4291 942	.7650 473
78	.0241 229	.1300 189	.2988 773	.4261 895	.7667 918
79	.0239 735	.1292 021	.2969 448	.4231 773	.7685 389
1.880	-0.0238 234	+0.1283 814	-0.2950 042	+0.4201 574	+0.7702 886
81	.0236 725	.1275 569	.2930 554	.4171 300	.7720 410
82	.0235 208	.1267 286	.2910 986	.4140 949	.7737 960
83	.0233 685	.1258 963	.2891 336	.4110 522	.7755 536
84	.0232 153	.1250 602	.2871 605	.4080 018	.7773 138
1.885	-0.0230 615	+0.1242 202	-0.2851 793	+0.4049 438	+0.7790 767
86	.0229 068	.1233 763	.2831 899	.4018 782	.7808 422
87	.0227 515	.1225 285	.2811 923	.3988 048	.7826 104
88	.0225 953	.1216 769	.2791 865	.3957 238	.7843 812
89	.0224 385	.1208 213	.2771 726	.3926 351	.7861 546
1.890	-0.0222 808	+0.1199 618	-0.2751 504	+0.3895 388	+0.7879 307
91	.0221 224	.1190 983	.2731 200	.3864 347	.7897 094
92	.0219 633	.1182 310	.2710 814	.3833 229	.7914 908
93	.0218 033	.1173 597	.2690 346	.3802 034	.7932 748
94	.0216 427	.1164 845	.2669 795	.3770 761	.7950 615
1.895	-0.0214 812	+0.1156 053	-0.2649 161	+0.3739 412	+0.7968 508
96	.0213 190	.1147 222	.2628 445	.3707 985	.7986 428
97	.0211 560	.1138 351	.2607 646	.3676 480	.8004 375
98	.0209 923	.1129 440	.2586 764	.3644 898	.8022 348
99	.0208 278	.1120 490	.2565 798	.3613 238	.8040 348
1.900	-0.0206 625	+0.1111 500	-0.2544 750	+0.3581 500	+0.8058 375

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.900	-0.0206 625	+0.1111 500	-0.2544 750	+0.3581 500	+0.8058 375
01	.0204 964	.1102 470	.2523 618	.3549 684	.8076 428
02	.0203 296	.1093 400	.2502 403	.3517 791	.8094 508
03	.0201 620	.1084 290	.2481 105	.3485 819	.8112 615
04	.0199 936	.1075 140	.2459 722	.3453 769	.8130 749
1.905	-0.0198 245	+0.1065 950	-0.2438 256	+0.3421 642	+0.8148 909
06	.0196 546	.1056 720	.2416 706	.3389 435	.8167 097
07	.0194 838	.1047 449	.2395 073	.3357 151	.8185 311
08	.0193 124	.1038 139	.2373 355	.3324 788	.8203 552
09	.0191 401	.1028 787	.2351 552	.3292 346	.8221 820
1.910	-0.0189 670	+0.1019 396	-0.2329 666	+0.3259 826	+0.8240 115
11	.0187 932	.1009 963	.2307 695	.3227 227	.8258 437
12	.0186 185	.1000 491	.2285 640	.3194 549	.8276 786
13	.0184 431	.0990 977	.2263 499	.3161 792	.8295 161
14	.0182 669	.0981 423	.2241 274	.3128 956	.8313 564
1.915	-0.0180 899	+0.0971 828	-0.2218 964	+0.3096 042	+0.8331 994
16	.0179 121	.0962 192	.2196 570	.3063 048	.8350 451
17	.0177 335	.0952 515	.2174 090	.3029 974	.8368 935
18	.0175 541	.0942 797	.2151 524	.2996 822	.8387 446
19	.0173 739	.0933 038	.2128 874	.2963 590	.8405 985
1.920	-0.0171 930	+0.0923 238	-0.2106 138	+0.2930 278	+0.8424 550
21	.0170 112	.0913 397	.2083 316	.2896 887	.8443 143
22	.0168 286	.0903 515	.2060 409	.2863 417	.8461 763
23	.0166 452	.0893 591	.2037 415	.2829 866	.8480 410
24	.0164 610	.0883 626	.2014 336	.2796 236	.8499 085
1.925	-0.0162 760	+0.0873 619	-0.1991 171	+0.2762 525	+0.8517 787
26	.0160 902	.0863 571	.1967 920	.2728 735	.8536 516
27	.0159 036	.0853 481	.1944 582	.2694 865	.8555 272
28	.0157 162	.0843 350	.1921 158	.2660 914	.8574 056
29	.0155 280	.0833 177	.1897 647	.2626 883	.8592 867
1.930	-0.0153 389	+0.0822 962	-0.1874 050	+0.2592 772	+0.8611 706
31	.0151 491	.0812 705	.1850 366	.2558 580	.8630 572
32	.0149 584	.0802 406	.1826 595	.2524 308	.8649 465
33	.0147 669	.0792 065	.1802 737	.2489 955	.8668 386
34	.0145 746	.0781 682	.1778 792	.2455 521	.8687 335
1.935	-0.0143 815	+0.0771 258	-0.1754 760	+0.2421 006	+0.8706 311
36	.0141 875	.0760 790	.1730 641	.2386 411	.8725 315
37	.0139 927	.0750 281	.1706 434	.2351 734	.8744 346
38	.0137 971	.0739 729	.1682 140	.2316 977	.8763 405
39	.0136 007	.0729 135	.1657 758	.2282 138	.8782 491
1.940	-0.0134 035	+0.0718 498	-0.1633 288	+0.2247 218	+0.8801 605
41	.0132 054	.0707 819	.1608 730	.2212 217	.8820 747
42	.0130 065	.0697 097	.1584 084	.2177 134	.8839 917
43	.0128 067	.0686 333	.1559 350	.2141 970	.8859 114
44	.0126 062	.0675 526	.1534 528	.2106 725	.8878 339
1.945	-0.0124 048	+0.0664 676	-0.1509 617	+0.2071 397	+0.8897 592
46	.0122 025	.0653 783	.1484 618	.2035 988	.8916 872
47	.0119 994	.0642 847	.1459 530	.2000 497	.8936 181
48	.0117 955	.0631 868	.1434 354	.1964 924	.8955 517
49	.0115 908	.0620 846	.1409 089	.1929 268	.8974 881
1.950	-0.0113 852	+0.0609 781	-0.1383 734	+0.1893 531	+0.8994 273

TABLE OF LAGRANGEAN INTERPOLATION COEFFICIENTS

p	A_{-2}	A_{-1}	A_0	A_1	A_2
1.950	-0.0113 852	+0.0609 781	-0.1383 734	+0.1893 531	+0.8994 273
51	.0111 787	.0598 673	.1358 291	.1857 712	.9013 693
52	.0109 714	.0587 521	.1332 759	.1821 810	.9033 141
53	.0107 633	.0576 326	.1307 137	.1785 826	.9052 617
54	.0105 543	.0565 088	.1281 426	.1749 759	.9072 121
1.955	-0.0103 445	+0.0553 806	-0.1255 625	+0.1713 610	+0.9091 653
56	.0101 338	.0542 481	.1229 734	.1677 378	.9111 213
57	.0099 223	.0531 112	.1203 754	.1641 064	.9130 801
58	.0097 099	.0519 699	.1177 684	.1604 666	.9150 418
59	.0094 967	.0508 243	.1151 524	.1568 186	.9170 062
1.960	-0.0092 826	+0.0496 742	-0.1125 274	+0.1531 622	+0.9189 734
61	.0090 676	.0485 198	.1098 933	.1494 976	.9209 435
62	.0088 518	.0473 610	.1072 502	.1458 246	.9229 164
63	.0086 351	.0461 978	.1045 981	.1421 433	.9248 921
64	.0084 176	.0450 302	.1019 369	.1384 537	.9268 706
1.965	-0.0081 992	+0.0438 581	-0.0992 666	+0.1347 557	+0.9288 520
66	.0079 799	.0426 816	.0965 873	.1310 494	.9308 362
67	.0077 598	.0415 007	.0938 988	.1273 347	.9328 232
68	.0075 388	.0403 154	.0912 013	.1236 116	.9348 131
69	.0073 170	.0391 256	.0884 946	.1198 802	.9368 058
1.970	-0.0070 942	+0.0379 314	-0.0857 788	+0.1161 404	+0.9388 013
71	.0068 706	.0367 327	.0830 539	.1123 921	.9407 997
72	.0066 461	.0355 295	.0803 198	.1086 355	.9428 009
73	.0064 208	.0343 219	.0775 765	.1048 704	.9448 049
74	.0061 945	.0331 098	.0748 240	.1010 970	.9468 119
1.975	-0.0059 674	+0.0318 932	-0.0720 624	+0.0973 150	+0.9488 216
76	.0057 394	.0306 721	.0692 916	.0935 247	.9508 343
77	.0055 106	.0294 465	.0665 115	.0897 259	.9528 497
78	.0052 808	.0282 164	.0637 222	.0859 186	.9548 681
79	.0050 502	.0269 818	.0609 237	.0821 029	.9568 893
1.980	-0.0048 187	+0.0257 426	-0.0581 160	+0.0782 786	+0.9589 133
81	.0045 863	.0244 990	.0552 989	.0744 459	.9609 403
82	.0043 530	.0232 508	.0524 726	.0706 047	.9629 701
83	.0041 188	.0219 981	.0496 371	.0667 550	.9650 027
84	.0038 837	.0207 408	.0467 922	.0628 968	.9670 383
1.985	-0.0036 477	+0.0194 789	-0.0439 380	+0.0590 301	+0.9690 767
86	.0034 109	.0182 125	.0410 745	.0551 548	.9711 180
87	.0031 731	.0169 416	.0382 016	.0512 710	.9731 622
88	.0029 344	.0156 660	.0353 195	.0473 786	.9752 093
89	.0026 949	.0143 859	.0324 279	.0434 777	.9772 592
1.990	-0.0024 544	+0.0131 012	-0.0295 270	+0.0395 682	+0.9793 121
91	.0022 131	.0118 118	.0266 167	.0356 501	.9813 678
92	.0019 708	.0105 179	.0236 970	.0317 234	.9834 265
93	.0017 276	.0092 194	.0207 679	.0277 882	.9854 880
94	.0014 836	.0079 163	.0178 294	.0238 443	.9875 524
1.995	-0.0012 386	+0.0066 085	-0.0148 815	+0.0198 919	+0.9896 197
96	.0009 927	.0052 961	.0119 241	.0159 308	.9916 900
97	.0007 459	.0039 790	.0089 573	.0119 610	.9937 631
98	.0004 982	.0026 573	.0059 810	.0079 827	.9958 392
99	.0002 495	.0013 310	.0029 953	.0039 957	+0.9979 181
2.000	-0.0000 000	+0.0000 000	-0.0000 000	+0.0000 000	+1.0000 000

MARCHANT METHODS

APPROXIMATING POLYNOMIAL FROM DIFFERENCE ARRAY (STIRLING METHOD)

REMARKS: *It is often desired to obtain an algebraic expression for a function that is determined by the relation that a series of tabulated amounts bears to corresponding values of the independent variable. When values of the latter are taken at equidistant points so that an array of differences may be set up, an equation in the form of an Approximating Polynomial may be readily obtained. If the nth difference of the array is constant, the Approximating Polynomial will represent the function correctly provided differences up to, and including those of the nth order are taken into account. If there are differences in the array which are of higher order than those taken into account, the Approximating Polynomial will approximate the function insofar as it can be done by a polynomial of degree "n".*

Obtaining an Approximating Polynomial by means outlined herein provides the most rapid method of fitting an equation to non-periodic tabulated data of scientific and statistical computations. It is assumed that the data are "smoothed"; that is to say, the obvious errors of observation are eliminated as is the case when the tabulated values are taken from a curve or determined by least-squares methods. If functions appear in periodic form, the Approximating Polynomial found by the method herein is generally suitable only for showing one quarter-period (approximately) of the periodic function. Fourier Series analysis is generally employed for obtaining equations of periodic functions.

OUTLINE: The Approximating Polynomial described herein has the form

$$(1)\quad y = a_0 + a_1u + a_2u^2 + a_3u^3 \;\ldots\ldots\; + a_nu^n$$

in which the "a" values are coefficients to be determined, and "u" represents the independent variable reduced to the initial condition that $u = 0$ when $y = a_0$ and that the difference between tabulated values of the independent variable, in terms of "u", is "1". For example, in the table showing the relation between x and y, below, the values of "u" are shown in the middle column assuming that the 0 point of "u" is to be at $x = 0.3$. It is obvious that if an Approximating Polynomial in the form of (1) is obtained, it is easy to convert it to one that shows y as a function of x. This simple transformation is not discussed herein.

EXAMPLE:

Function			Differences				
x	u	y	1st	2nd	3rd	4th	5th
0.	-3	1.00000					
			-4865				
0.1	-2	0.95135		1547			
			-3318		-299		
0.2	-1	0.91817		1248		100	
			-2070 $(d'_{-\frac{1}{2}})$		-199 $(d'''_{-\frac{1}{2}})$		-32 $(d^{v}_{-\frac{1}{2}})$
0.3	0	0.89747		1049 (d''_0)		68 (d''''_0)	
			-1021 $(d'_{\frac{1}{2}})$		-131 $(d'''_{\frac{1}{2}})$		-32 $(d^{v}_{\frac{1}{2}})$
0.4	+1	0.88726		918		36	
			- 103		- 95		
0.5	+2	0.88623		823			
			720				
0.6	+3	0.89343					

The method exemplified herein will be applied to the tabulated values listed on the previous page which are shown with differences. An Approximating Polynomial is to be obtained in such form that it will show optimum accuracy in the vicinity of $x = 0.3$. This value is then chosen as the base point for obtaining the "a" coefficients, so "u" is set at 0 when $x = 0.3$.

The formula used is that of Stirling and is chosen because it is the easiest to apply. The Bessel formula* gives somewhat more accurate results in the region that is half-way between the equidistant tabular values. This difference, however, is exceedingly slight so that rarely will it be advisable to go to this refinement. The Newton formula* is useful for obtaining an Approximating Polynomial when only the values at the top of a table are obtainable. However, even in this case the Stirling Method may be used if it is satisfactory to extrapolate probable differences upward from the known differences.

In the Approximating Polynomial (1), Page 1, the Stirling formulas for coefficients, up to consideration of 5th differences, are below:

(2) $a_1 = \frac{1}{2}\,(d'_{-\frac{1}{2}} + d'_{\frac{1}{2}}) - 1/12\,(d'''_{-\frac{1}{2}} + d'''_{\frac{1}{2}}) + 1/60\,(d^{v}_{-\frac{1}{2}} + d^{v}_{\frac{1}{2}})$

(3) $a_2 = \frac{1}{2}\,d''_0 - 1/24\,d''''_0$

(4) $a_3 = 1/12\,(d'''_{-\frac{1}{2}} + d'''_{\frac{1}{2}}) - 1/48\,(d^{v}_{-\frac{1}{2}} + d^{v}_{\frac{1}{2}})$

(5) $a_4 = 1/24\,d''''_0$

(6) $a_5 = 1/240\,(d^{v}_{-\frac{1}{2}} + d^{v}_{\frac{1}{2}})$

The terms up to 4th differences appear in Scarborough: Numerical Mathematical Analysis, 1930 edition, Page 80. Those for 5th differences were supplied by courtesy of Dr. Raymond T. Birge, Professor of Physics, University of California, to whom we are also indebted for other helpful data in connection with this process.

The nomenclature of equations (2) to (6), inclusive, applies to the preceding difference array and is further explained in Marchant Method MM-189a. It will be noted that certain factors are repeated or bear simple ratios to others.

For ordinary computing, any terms that do not affect the final result in one place at the right of the one that is to be retained would be omitted. If values of the polynomial are desired close to the centering point, it is often possible to shorten the work if advantage is taken of this principle. In this case, it is not possible to do this if 5-place accuracy is desired without uncertainty within the range $u = -1$ to $u = +1$, because the maximum effect of the 5th difference is noted in coefficient a_3 as 0.000 013 3 (see below) so it would affect 6th place by 13.

If accuracy to the number of places of the tabulated values is desired up to the limits of the values from which differences are taken; i.e., the extreme range of the tabulated values, the higher-order coefficients must be taken to more places than those of lower order. For example, at the extreme range of the table, $u = \pm 3$. As the coefficient a_5 multiplies u^5, or 243, it is evident that a_5 must be carried to a sufficient number of places so that the error of its right-hand digit when multiplied by 243 will not affect 6th place.

(*) Scarborough: Numerical Mathematical Analysis, 1930 edition, the Johns Hopkins Press, Pages 80-81, gives coefficients for Newton, Bessel, and Stirling formulas up to and including 4th differences. Values including those due to 5th differences will be supplied upon application.

Using these principles, the coefficients are obtained as follows:

$a_1 = -0.015\ 455 - (-0.000\ 275) + (-0.000\ 011) = -0.015\ 191$

$a_2 = +0.005\ 245 - 0.000\ 028\ 3 = +0.005\ 216\ 7$

$a_3 = -0.000\ 275 - (-0.000\ 013\ 33) = -0.000\ 261\ 67$

$a_4 = +0.000\ 028\ 33$

$a_5 = -0.000\ 002\ 667$

The Approximating Polynomial, accordingly, is

$$(7)\quad y = 0.89747 - 0.015191u + 0.0052167u^2 - 0.00026167u^3 + 0.000028333u^4 - 0.000002667u^5$$

To show how closely this approximates the tabulated function, when u varies from -3 to +3, its values are computed to six places.

x	u	y Computed to 6 places	Tabulated y	6th place error
0	-3	1.000 000	1.000 00	0
0.1	-2	0.951 351	.951 35	1
0.2	-1	0.918 170	.918 17	0
0.3	0			
0.4	+1	0.887 260	.887 26	0
0.5	+2	0.886 229	.886 23	1
0.6	+3	0.893 429	.893 43	1

MARCHANT CALCULATOR APPLICATION

No exemplification of the details of Marchant application to this work is given because it embodies the simplest of calculator manipulation. Because work of this sort is usually infrequently done and because some of the factors of equations (2) to (6) inclusive are repetitions or bear simple numerical ratios to others, it is usually advisable to evaluate each factor individually, copying the amounts to work sheet and summing them afterwards. For these reasons, the accumulation of partial products is not recommended, though this procedure should undoubtedly be followed if there is a great volume of the work to be done.

In nearly all cases, except where it is desired to obtain an empirical formula (see below), it is usually satisfactory to use the function in terms of "u"; thus, for direct interpolation and related work the "x" is converted to the corresponding "u" before applying the formula, and in cases of inverse interpolation and the like, the "x" is obtained after the "u" has been found.

APPROXIMATION OF FUNCTIONS BY POLYNOMIALS

Polynomials of the type considered herein for the representation of a tabulated function have not been given the consideration in mathematical literature that their importance warrants. It is believed that this is due to the usual comparatively laborious process of setting them up by solving systems of linear equations, which has long been the conventional method of converting n tabulated values into a power series of degree n. Now that it is recognized that they are much more easily obtained from their difference arrays, more and more uses are certain to be found for them.

One principal use of these polynomials is to provide means of handling complicated analytical or transcendental functions in which substitution is difficult owing to the complexity of the terms. Equidistant values are established, sufficient to determine the Approximation Polynomial. A few intermediate values are obtained for a later check of the error. The Polynomial is then used in

(over)

place of the function for which it is a substitution. When given the "u" value, the "y" is obtainable by direct substitution in the polynomial. When given the "y" value, the "u" is easily obtained by the Birge-Vieta Method (see Marchant Method MM-225).

The above-described procedure is particularly helpful in cases where the differential or integral values of a complicated function are desired. Many of these cannot be integrated directly and differentiation is often difficult. If the expression is approximated as a polynomial, however, it is a simple matter to obtain successive differential or integral forms and without the discontinuities which use of the original expression might entail. A characteristic of the Approximation Polynomial is that its graph has minimum curvature.

The use of these polynomials in cases of large volume of interpolation, such as in table preparation, is obvious, though in such instances the procedure of Marchant Method MM-189 should be compared. Inverse interpolation is easily handled by using the Birge-Vieta Method for solving for "u". (Compare also Marchant Methods MM-209, 220, and 221).

The polynomials readily lend themselves to extrapolation provided it is understood that the uncertainty of the result increases (sometimes rapidly) as one leaves the region contained between the extreme values from which the differences are tabulated. This effect becomes increasingly serious as the degree of the polynomial increases.

The polynomials also provide a way of exploring the effect of the powers of the independent variable in cases of experimental tabulated data, thus leading directly to an empirical formula to express the relationships. Obviously, if the coefficient of, say, the third power of x (not of u) is large and those of other powers are negligible, the experimenter will be on the lookout for influences that vary according to the cube of the independent variable. Care must be taken not to accept too literally the significance of the polynomial as a working formula, however, because an empirical formula should, if possible, have some physical meaning or reasonable basis for being in the form used.

If the polynomial shows comparatively large coefficients of x (not of u) for certain powers, an empirical formula, however, may generally be set up using those coefficients and powers only. The values of y corresponding to the tabulated x's may then be computed from this new polynomial and compared with the original tabulated values. The residuals then may be considered as n values of another new polynomial containing only the powers that are to be retained in the proposed empirical formula. By solving these as a system of linear equations, applying least squares methods, a modification is obtained of the coefficients of the powers that are to be retained in the empirical formula. This modified formula then becomes the improved empirical formula.*

The above-described procedure is the usual one of taking advantage of an approximating polynomial (power series) as a base for an empirical formula. Another case in which such a polynomial may be converted into a simplified empirical formula is that in which the successive coefficients follow a definite law, indicating a convergent series, which represents some other function such as an exponential, trigonometric, etc.

(*) Steinmetz: Engineering Mathematics, 3rd Edition, McGraw-Hill Publ. Co., Pages 215-16. See also Marchant Method MM-183 and Marchant Method MM-182 (Page 7, Section 6). These relate to the Crout Method for solving such systems of equations.

MARCHANT METHODS

Note: The following article is reproduced for inclusion in the Marchant-Method series because it describes the proper computing procedure for obtaining the Probable Error or Standard Deviation of a function when the Probable Errors or Standard Deviations of its elements are known. Functions that differ from the ordinary formula-type, as shown herein, are handled according to the principles used in this example, but care must be taken in applying them. Information will be given to cover such cases if we are supplied with the type of functions involved.

Marchant Calculating Machine Company

HOW MANY FIGURES FOR THE ANSWER?

Business figures usually involve money and consequently must be exact to "nearest cent." With scientific figures, on the other hand, it is sometimes not so simple to express the degree of accuracy of the result. Unless the Marchanter understands this, he is at a disadvantage when he tries to present authoritative figure information to those who handle this class of work. The object of this article is to outline means of caring for a most frequently encountered case of this sort; viz., how should the answer be written when the Probable Error of each of its factors is known?

The solution for such cases is somewhat time-consuming. However, because of the fact that most calculating follows a "pattern" in which the factors of the successive problems do not vary greatly from each other, it is usually advisable to study a few cases from any series in order to know the error trend. Many instances require the computation of each case because without it the results are of doubtful value.

For example, suppose that all factors of the following problem, except the Constant (pi = 3.14159) represent values obtained from observations or experiment, each having the Probable Error of Col. (b) as tabulated below. What is the P. E. of the answer, and how may the answer be expressed?

$$\frac{3.14159 \times 4.962^3 \times 0.0343}{1.623 \times 3.050^2} = ?$$

A ten-place answer, obviously ridiculously "correct" is 0.8719582822.
Though there are short-cuts to the P. E. in cases like this, the principle is shown below:

	(a) Factor	(b) Probable Error of (a)	(c) Relative P. E. (b) ÷ (a)	(d) Square Col. (c)	(e) (d) x Square of Exponents
(1)	4.962	.018	.00363	.000013	.000117 (mult. by 3^2)
(2)	.0343	.0012	.03499	.001224	.001224
(3)	1.623	.024	.01479	.000219	.000219
(4)	3.050	.011	.00361	.000013	.000052 (mult. by 2^2)
				Total of Col. (e)	.001612

All constants are omitted.

Relative P. E. of answer is square root of 0.001612, or 0.040.
Actual P. E. of answer is 0.040 x 0.8719 . . ., or 0.035.
Answer is, therefore, written as 0.872 ± 0.035 P. E.

It follows from the definition of Probable Error that it is as even a chance that the true answer lies within the limits of 0.837 and 0.907 as that it lies outside of these limits.

To avoid intrusion of errors due to rounding, it is important that the amounts which have Probable Errors be written to the same number of decimals as shown in the P. E. For a similar reason, it is important that any Constants be expressed to a sufficient number of places so that rounding of their right-hand digits will not introduce an error approaching the magnitude of the right-hand digit of the P. E. to the number of figures expressed. Rounding errors do not follow the normal Probability Curve. In this instance, it is obvious that the loss of the 6th decimal in the Constant, which could not exceed 5 and is actually 3, creates a relative error of the Constant of such a small amount that it could not affect the P. E. of the answer.

The same computing plan applies if the original errors are given in terms of Standard Deviations. In such a case, the final error figure is the Standard Deviation of the answer. As both Probable Error and Standard Deviation are based upon the same Probability Curve, they have a constant relationship. The Probable Error is 0.6745 x Standard Deviation, so when one is known the other is easily found.

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(3)	1.623	.024	.01479	.000219	.000219
(4)	3.050	.011	.00361	.000013	.000052 (mult. by 2^2)
				Total of Col. (e)	.001612

All constants are omitted.

Relative P. E. of answer is square root of 0.001612, or 0.040.

Actual P. E. of answer is 0.040 x 0.8719 . . ., or 0.035.

Answer is, therefore, written as 0.872 ± 0.035 P. E.

It follows from the definition of Probable Error that it is as even a chance that the true answer lies within the limits of 0.837 and 0.907 as that it lies outside of these limits.

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MARCHANT METHODS

INVERSE CURVILINEAR INTERPOLATION

THE COMRIE "TWO-CALCULATOR" METHOD

REMARKS: *The method described herein was introduced by Dr. L. J. Comrie at the British Nautical Almanac Office in 1934. As applied to lever-set calculators, the method was described in the British Nautical Almanac for 1937. The principles that govern this method are subject to numerous variations. Practice with the method will disclose them to the skilled computer if he will observe the hints given in the Notes which follow the description of the method.*

OUTLINE: The Bessel interpolation formula for cases in which 4th differences do not exceed 1000 may be expressed in the form

$$(1) \quad f_n = f_0 + nd'_{½} + B''(M''_0 + M''_1) + B''' d'''_{½}$$

in which $(M''_0 + M''_1)$ represents $(d''_0 + d''_1) - 0.184\ (d''''_0 + d''''_1)$ according to the Comrie Throw-Back, as described in Marchant Method MM-189, and with nomenclature as described in MM-189a.

For purposes of inverse interpolation, this formula is put in the form

$$(2) \quad f_n - B''(M''_0 + M''_1) - B''' d'''_{½} = f_0 + nd'_{½}$$

From this, it will be seen that all values are known except "n," B" and B'''.

Inasmuch as B" and B''' vary according to "n," it is seen that if each side of the above equation is evaluated so that they are equal, and if in doing so the B" and B''' values of the left-hand side correspond to the "n" of the right-hand side, the value of "n" that satisfies this condition will be the desired "n" for the inverse interpolation.

Because of the dependence of B" and B''' upon "n", the former cannot be had until "n" is known. However, if we start with an approximate "n" and apply corresponding approximate B" and B''' to the left-hand side, and then revise B" and B''' and their consequent effects on the left-hand side to correspond to the necessary variation of "n" of the right-hand side to produce equality, we are able by this converging process to establish a final value of "n" that satisfies the conditions stated in the previous paragraph.

It will be seen in the following example, how this may be done by the use of two Marchants, side by side. The left-hand Marchant develops the value of the left-hand side of equation (2), and the right-hand Marchant develops the right-hand side of the equation. The desired "n" appears in the Upper Dial of the right-hand Marchant upon completion of the work.

The exact process of solving the example used herein applies to cases in which 4th differences do not exceed 1000. If they are greater than this amount, proceed as in Note E.

EXAMPLE: Given: the function with differences tabulated as on next page.

(over)

		Differences			
		1st	2nd	3rd	4th
0.2	$+0.91817 (f_0)$		$+1248 (d''_0)$		$+100 (d''''_0)$
		$-2070 (d'_{1/2})$		$-199 (d'''_{1/2})$	
0.3	$+0.89747 (f_1)$		$+1049 (d''_1)$		$+68 (d''''_1)$

$$M''_0 + M''_1 = 0.02297 - (0.184 \times 0.00168) = +0.02266.$$

Find: x when y = 0.91546. As y has 5 places, x will not be obtained to more than 5 places at most, or "n" taken to 4 significant figures. Influences that do not affect 5th place in "n" will, therefore, be disregarded.

OPERATIONS: Decimals on each Marchant: Upper Dial 6, Middle Dial 11, Keyboard Dial 5. Upper Green Shift Key down on left-hand Marchant (LHM) and Non-Shift Key down on right-hand Marchant (RHM). It is assumed that some form of M model Marchant is used, though the method is easily adapted to the D models. Inasmuch as values of y decrease as x increases, move Manual Counter Control on RHM toward operator.

(1) On RHM, set up f_0 (0.91817) in Keyboard Dial and, with carriage in 6th position, touch Add Bar. Clear Upper Dial.

(2) On RHM set up $d'_{1/2}$ (0.02070) in Keyboard Dial and reverse multiply by such two-figure amount as will cause Middle Dial to read as close as possible to f_n (0.91546).

First approx. "n" (0.13) appears in Upper Dial.
Corresponding approx. f_n (0.915479) appears in Middle Dial.
From curve or table, corresponding B" is -0.0283 and B‴ is +0.0070.

(3) On LHM set up f_n (0.91546) and, with carriage in 6th position, touch Add Bar. Clear Upper Dial.

(4) On LHM set up $d'''_{1/2}$ (0.00199) and multiply by B‴ (0.0070) in such manner as will form product $B''' d'''_{1/2}$ and simultaneously deduct it from f_n. Inasmuch as $d'''_{1/2}$ is minus and B‴ is plus, their product is minus, so because deduction is required, a direct multiplication is made (in case of a "reverse" multiplication move Manual Counter Control toward the operator). See Note C.

First approx. $f_n - B''' d'''_{1/2}$ (0.91547393) appears in Middle Dial.

(5) On LHM clear Upper and Keyboard Dials, set up $M''_0 + M''_1$ (0.02266) in Keyboard Dial and multiply by B" (0.0283) in such a way as to form the product $B''(M''_0 + M''_1)$ and simultaneously deduct it from Middle Dial amount. Inasmuch as $M''_0 + M''_1$ is plus, and B" is minus, their product is minus so because deduction is required, a direct multiplication is made.

First approx. of left side of Eq. 2 (0.916 115 208) appears in Middle Dial.

(6) On RHM, without clearing any dials, multiply by such amount as will show an Upper Dial reading in three figures (n), and Middle Dial reading as close as possible to that of LHM, as developed in Step 5.

Second approx. "n" (0.099) appears in Upper Dial.
Corresponding $f_0 + nd'_{1/2}$ (0.916 120 70) appears in Middle Dial.
Corresponding B" is -0.0223 and B‴ is +0.0060.

(7) On LHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads the B" value referred to in last line of Step (6) (0.0223), thus reflecting the effect of decrease of B" from first approximation to second approximation.

Middle Dial reads 0.915 979 248.

(8) On RHM, without clearing any dials, multiply by such amount as will show an Upper Dial reading to five places (n) and Middle Dial as close as possible to that of LHM as developed in Step 7.

Third approx. "n" (0.10583) appears in Upper Dial.
Corresponding $f_0 + nd'_{\frac{1}{2}}$ (0.915 979 319) appears in Middle Dial.
Corresponding B" is - 0.0237 and B"' is +0.0062.

(9) On LHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads the new B" (0.0237).

Middle Dial reads 0.916 010 972

(10) On RHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads as close as possible to that of LHM as developed in Step 9.

Fourth approx. "n" (0.10430) appears in Upper Dial.
Corresponding $f_0 + nd'_{\frac{1}{2}}$ (0.916 010 99) appears in Middle Dial.
Corresponding B" is - 0.0234 and B"' is +0.0062.

(11) On LHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads the new B" (0.0234).

Middle Dial reads 0.916 004 174.

(12) On RHM, without clearing any dials, adjust Upper Dial by direct or reverse multiplication until it reads as close as possible to that of LHM as developed in Step 11.

Fifth approx. "n" (0.10463) appears in Upper Dial (See Note A).
Corresponding $f_0 + nd'_{\frac{1}{2}}$ (0.916 004 159) appears in Middle Dial.
"n" is taken to four places as 0.1046; or, x = 0.21046.

It will now be noted that the corresponding B" does not change from that of Step 10, and similarly for B"'. Before the work may be regarded as complete, however, it is necessary to see whether or not the alteration of B"' from its value in Step 4 (0.0070) to its value in Step 10 (0.0062) will affect 5th place of adjusted f_n in LHM. Inasmuch as the adjustment is 0.0008 x 0.00199, it will be seen that 5th place is not affected. If it were affected, it would be necessary at this point to clear Upper and Keyboard Dials of LHM, set up 0.00199 and multiply by the difference between the value of B"' used in Step 4 and its final value, and then proceed as in Step 12 on RHM, then clear Upper and Keyboard Dials on LHM, set up 0.02266, and multiply by the <u>difference</u> between the last B" used (0.0234) and its new value found after these adjustments are made. RHM must then be adjusted to produce what will doubtless be a final "n". The necessity of doing this greatly slows the process so to prevent the possibility of its happening it is important to see that the original B"' used in Step 4 is so close to the final value that no further adjustment of it is necessary. This subject is fully discussed in Note C.

(over)

N O T E S

(A) It will be observed that the tabulated function has a rejected 5th difference of $d''''_1 - d''''_0 = -0.00032$. It is important to know if this will affect accuracy of "n" by more than $\frac{1}{2}$ in 4th place, and if so, how many places may be retained.

The largest 5th difference that may be retained without its affecting 4th place of "n" by more than $\frac{1}{2}$ is shown by the following formula:

(3) $\dfrac{500\ d'_{1/2}}{10^a}$ in which "a" is number of decimals in "n" not to be affected.

Applying it to this case, we have.

$$\frac{500 \times 0.02070}{10^4} = 0.0010 \text{ absolute value.}$$

Inasmuch as this is greater than the absolute 5th difference of 0.00032, it is evident that "n" may be taken to 4 places, or n = 0.1046, or x = 0.21046.

The coefficients of Eq. (3) applying to rejected differences, other than the 5th, are

3rd, 60; Mean 4th, 20; Mean 6th, 100.

(B) According to the conventions of the mathematics of inverse interpolation, it is assumed that the values of y are exact; that is to say, their final digits do not represent rounded figures. In practice, however, such amounts are rounded and at times this fact materially affects the number of figures that may be retained in "n". The rule is to take the number of places of the change in x to no more than the number of significant figures in the first difference of the y, and if the latter begins with 1, 2, or 3 to take one less place. For example, in this case there are four significant figures in the $d'_{1/2}$ and it begins with 2. The number of figures of the change in x that may be taken is, therefore, 3. Inasmuch as the change of argument is "n" (0.1046), this may be taken only as 0.105 from which, if rounding errors are to be considered, x should not be indicated to more places than x = 0.2105.

(C) The reason that the first multiplication in LHM is by B''' instead of B'' is because in most cases it is not necessary to make subsequent adjustments of the effect of B''' though adjustments of B'' are nearly always required. The work is greatly simplified if reasonable assurance is had at this point that no further adjustment need be made because of the convergence of B''' to its final value. For example, in this case d''' is 0.00199 and B''' is 0.0070 for the first approximation of "n". Inspection of the Chart of values of B''' shows that a wide variation of "n" (from .10 to .20) may be made with a variation of B''' of only 0.001. Inasmuch as such a variation would cause only 0.000002 in the adjusted f_n in Middle Dial, it will be seen that quite wide variation of "n" would not affect fifth place of the adjusted f_n. The computer, therefore, has reasonable assurance that f_n will not require further adjustment (as referred to in Note at end of Step 12) and, therefore, the first approximation of B''' (0.0070) may be used at this step.

However, if the function has a large third difference so the above relationship does not hold, it is better to make an estimate of about what the final B''' is likely to be and to use such estimated value as the multiplier in Step 4, rather than to use the value which corresponds to the first approximation of "n". Such an estimated B''' could be obtained in this case, if desired, as follows: With aid of slide rule $(M''_0 + M''_1)$ is multiplied by first approx B'' and found to be about -0.00064. By the reasoning stated in Step 5, a modified "n" in RHM may be obtained by adjusting Upper Dial of RHM until its Middle Dial equals about 0.91548 plus 0.00064, or 0.91612, yet not using more than two places in the approx. "n". Performing this operation on RHM shows approx. "n" of 0.10 and reference to table shows corresponding B‴ to be 0.006, which value would be used in Step 4. Steps 5 and 6 would then be performed though the values found would differ somewhat from those stated herein, because with respect to RHM a great part of the adjustment of Step 6 has already been done in the intermediate step discussed above.

(D) In cases where d"" is so small as not to be taken into account (apply Note A to 4th differences), method may be somewhat simplified by omitting the "throw-back" from 4th difference to produce $M''_0 + M''_1$, thus using 2nd differences as written. However, the modification of 2nd differences by the "throw-back" is such an easy thing to do that its omission is not recommended, because in some instances doing so improves the precision in border-line cases of accumulated products.

(E) In cases where d"" exceeds 1000, the left-hand side of the equation uses 2nd and 4th differences instead of a modified 2nd difference. The extra term is B"" x $(d''''_0 + d''''_1)$. Additional steps are required on LHM to compute the effect of this term and its subsequent adjustments. In certain instances of this sort it will be found impracticable to follow the process exactly, owing to slow convergence. This is likely to occur in cases where the x is to have many more places than exist in the tabulated y's, or where the differences of succeeding orders do not decrease materially from those of next lower order. In such cases, obtaining an approximate "n" using a portion of the figures of the y's gives the region in which the desired x is located. Sub-tabulating the y's to tenths will then enable a new difference array to be set up to which the usual methods may be applied.

In cases of large 4th differences, the 5th and higher differences may become important. There are means of throwing back these differences to those of lower order, so that the process need not be complicated by coefficients B^{v} and higher. Details as to this "throw-back" will be supplied upon application.

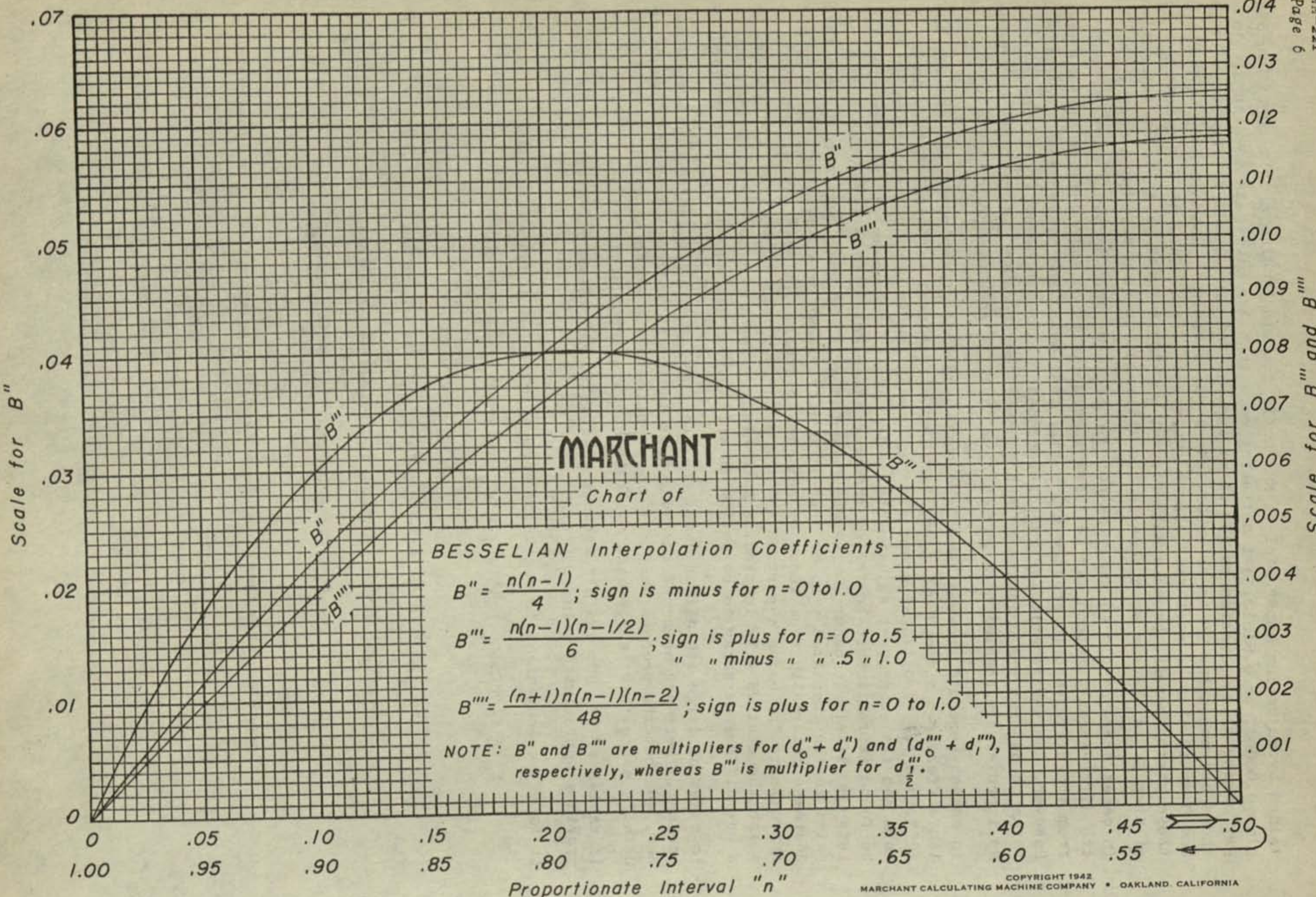
(F) If tables of B", B‴, and B"" are not available, the chart on reverse side hereof provides approximate values.

(G) It is often convenient to check the work by direct interpolation; thus,

$$f_n = f_0 + nd'_{\frac{1}{2}} + B''(M''_0 + M''_1) + B'''d'''_{\frac{1}{2}}, \text{ as follows}$$

$$\begin{aligned}
&& f_0 &= 0.91817 \\
nd'_{\frac{1}{2}} &= 0.10463 \times (-0.02070) & &= -0.002166 \\
B''(M''_0 + M''_1) &= -0.0234 \times 0.02266 & &= -0.000530 \\
B'''d''''_{\frac{1}{2}} &= 0.0062 \times (-0.00199) & &= \underline{-0.000012} \\
&& f_n &= 0.91546
\end{aligned}$$

In practice, this check would be performed by accumulative multiplication.


MARCHANT
Chart of
BESSELIAN Interpolation Coefficients
$B'' = \frac{n(n-1)}{4}$; sign is minus for n = 0 to 1.0
$B''' = \frac{n(n-1)(n-1/2)}{6}$; sign is plus for n = 0 to .5
" " minus " " .5 " 1.0
$B'''' = \frac{(n+1)n(n-1)(n-2)}{48}$; sign is plus for n = 0 to 1.0
NOTE: B'' and B'''' are multipliers for $(d_0'' + d_1'')$ and $(d_0'''' + d_1'''')$, respectively, whereas B''' is multiplier for $d_{\frac{1}{2}}'''$.
B''
B'''
B''''
Scale for B''
.07
.06
.05
.04
.03
.02
.01
0
Scale for B''' and B''''
.014
.013
.012
.011
.010
.009
.008
.007
.006
.005
.004
.003
.002
.001
0 .05 .10 .15 .20 .25 .30 .35 .40 .45 .50
1.00 .95 .90 .85 .80 .75 .70 .65 .60 .55
Proportionate Interval "n"

MARCHANT METHODS

SIMPLIFIED METHOD OF EXTRACTING CUBE ROOT

The method described below quickly obtains the Cube Root of any number to five significant figures. There is also described a simple method of extending this five-figure root to one of ten significant figures. This latter method was originated by L.J.Comrie, Ph.D., President and Managing Director Scientific Computing Service Co.,Ltd., London, W. C. I., formerly Superintendent H. M. Nautical Almanac Office, Greenwich. It appears in the Third Edition of Barlow's Tables.

TO OBTAIN A CUBE ROOT TO 5 SIGNIFICANT FIGURES

Finding Proper Number in Column A

Column A contains a single sequence of numbers from 100 to 1,000 and the decimal place may be set at any desired position in any of these numbers. In finding the Cube Root of any given number, the first step is to find in Column A the nearest number to the given number, in doing which the decimal place is to be set to correspond to that of the given number.

Example: For instance, if it is desired to find the Cube Root of 10.1357, the nearest number in Column A is found in the second line of the table in which it is recorded as 102, but is to be used in working the problem as 10.2.

Decimal Point Determines Selection of Column 1, 2, or 3

If the given number, of which the Cube Root is to be found, has more than three figures before the decimal place, it should be divided into groups of three figures each by commas, just as is ordinarily done in writing large numbers. If it has no figures before the decimal place, the figures after the decimal place should similarly be divided into groups of three by means of commas beginning counting from the decimal place. The number of significant figures appearing before the decimal point, or before the first comma that has significant figures preceding it, determines whether Column 1, Column 2, or Column 3 is to be used. If there is one figure before the decimal point or comma, Column 1 is used; if two figures, Column 2; and if three figures, Column 3.

Example: Thus, if the given number is 23,475,260, Column 2 is to be used, while if the given number is 0.004,152,27 Column 1 is to be used.

Calculation of Cube Root to 5 Significant Figures

To calculate the Cube Root of a given number, add the given number into the Marchant Middle Dial; find in Column A the nearest number to the given number, set it up in the Keyboard Dial and multiply by 2, considering it decimally as described above. The number now appearing in the product register is approximately three times the given number. Set up on the Keyboard Dial the number appearing in Column 1, 2 or 3, opposite the number taken from Column A, choosing the proper column as outlined above. With the carriage in the proper position with relation to the keyboard setup for performing division, depress the Automatic Division Key, whereupon the cube root, correct to at least five significant figures, will appear in the Upper Dial. (For locating decimal in root, see Illustrative Example on reverse side hereof.)

EXTENSION OF FIVE-FIGURE ROOT TO ONE OF TEN SIGNIFICANT FIGURES

Divide the number whose ten-figure root it is desired to obtain by the square of its

(over)

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five-figure root, noting the quotient to ten figures. Add twice the five-figure root to this quotient. One third of this sum will be the desired root accurate to ten significant figures.

ILLUSTRATIVE EXAMPLES

In determining 5-figure roots the decimal point of the root is determined after solution, so all calculations should be started with the carriage shifted to 9th position on 10-column models and to 7th position on 8-column models. All amounts are set up in the Keyboard Dial with left-hand digits in the 8th or 10th position, depending upon whether an 8 or 10-column Marchant is used.

EXAMPLE I: Find Cube Root of 2,865,637:

(a) With carriage shifted to the extreme right (lacking one), set up in Keyboard Dial 2,865,637 and touch Add Bar.

(b) Note from Table of Divisors (Column A) that 288 is the nearest number to 2865, etc. Set up 288 in Keyboard Dial directly below 286 that is in Middle Dial and multiply by 2 (depressing Add Bar and No. 2 Multiplier Key simultaneously on M Model Marchant).

Clear Upper and Keyboard Dials (the former on early models only) Middle Dial now shows 8625637.

(c) Note that 2,865,637. has one digit at left of the first comma, so select divisor 6.07272 from Column 1 corresponding to 288 in Column A. Set it up in the extreme left of Keyboard Dial and divide, stopping division after six digits of quotient are formed.

(d) Inasmuch as 2,865,637. has three groups in front of the decimal, there will be three digits of the root in front of the decimal.

The root correct to five figures is 142.04.

EXAMPLE II: Find Cube Root of .00094271:

The number is pointed off with commas thus: .000,942,71. Proceed same as above, setting it up as if it were 942,71. and adding it into the Middle Dial. The "nearest" number from Column A is 948, which is then added twice, and as the number of significant digits at the left of the first comma which has significant digits preceding it is three, the divisor is taken from Column 3 and is 289,508. Since the first significant digit of the original number is in the second group right of the decimal point, the first significant digit of the root is the second figure at right of the decimal point. The root is therefore .098053.

EXAMPLE III: Extend the Cube Root of Example 1 to 10 Significant Figures:

The square of its 5-figure root (142.04) is 20175.3616. Dividing 2,865,637 by this amount gives a 10-digit quotient of 142.0364629. Adding to this twice the 5-figure root (2 x 142.04, or 284.08) gives 426.1164629. Dividing this by 3 produces the 10-figure root of

142.0388210.

NOTE: In the two divisions of this process the Upper Dial may develop only 9 digits. In such cases the 10th digit is obtained by reference to the "remainder fraction" that appears in the Middle and Keyboard Dials (see Marchant Instruction Book, 1942 Edition, Page 38).

CUBE ROOT DIVISORS

A	Col. 1	Col. 2	Col. 3	A	Col. 1	Col. 2	Col. 3
100	3.00000	13.9248	64.6330	200	4.76220	22.1042	102.599
102	3.03987	14.1098	65.4920	203	4.80971	22.3247	103.622
104	3.07948	14.2937	66.3453	206	4.85698	22.5441	104.640
106	3.11883	14.4763	67.1932	209	4.90402	22.7624	105.654
108	3.15793	14.6579	68.0358	212	4.95083	22.9797	106.663
110	3.19681	14.8383	68.8731	215	4.99743	23.1960	107.666
112	3.23544	15.0176	69.7054	218	5.04381	23.4113	108.666
114	3.27384	15.1959	70.5328	221	5.08998	23.6256	109.660
116	3.31202	15.3731	71.3554	224	5.13594	23.8389	110.650
118	3.34999	15.5493	72.1732	227	5.18170	24.0513	111.636
120	3.38773	15.7244	72.9864	230	5.22725	24.2627	112.618
122	3.42527	15.8987	73.7952	233	5.27260	24.4733	113.595
124	3.46260	16.0720	74.5995	236	5.31777	24.6829	114.568
126	3.49974	16.2443	75.3995	240	5.37769	24.9610	115.859
128	3.53667	16.4158	76.1953	244	5.43728	25.2376	117.143
130	3.57342	16.5863	76.9869	248	5.49654	25.5127	118.419
132	3.60997	16.7560	77.7745	252	5.55548	25.7863	119.689
134	3.64635	16.9248	78.5581	256	5.61412	26.0584	120.952
136	3.68254	17.0928	79.3379	260	5.67244	26.3292	122.209
138	3.71855	17.2600	80.1138	264	5.73047	26.5985	123.459
140	3.75440	17.4264	80.8860	268	5.78821	26.8665	124.704
142	3.79007	17.5919	81.6545	272	5.84567	27.1332	125.941
144	3.82557	17.7567	82.4194	276	5.90284	27.3985	127.173
146	3.86091	17.9208	83.1808	280	5.95973	27.6626	128.399
148	3.89609	18.0841	83.9388	284	6.01635	27.9255	129.619
150	3.93111	18.2466	84.6933	288	6.07272	28.1870	130.833
152	3.96598	18.4085	85.4444	292	6.12882	28.4474	132.041
154	4.00069	18.5696	86.1923	296	6.18466	28.7067	133.245
156	4.03526	18.7300	86.9370	300	6.24025	28.9647	134.442
158	4.06967	18.8897	87.6784	304	6.29560	29.2216	135.634
160	4.10395	19.0488	88.4168	308	6.35070	29.4773	136.822
162	4.13807	19.2072	89.1520	312	6.40557	29.7320	138.004
164	4.17205	19.3650	89.8843	316	6.46020	29.9856	139.181
166	4.20591	19.5221	90.6136	320	6.51460	30.2381	140.353
168	4.23963	19.6786	91.3400	325	6.58229	30.5523	141.811
170	4.27321	19.8345	92.0634	330	6.64963	30.8649	143.262
173	4.32333	20.0671	93.1434	335	6.71663	31.1759	144.705
176	4.37317	20.2985	94.2171	340	6.78329	31.4853	146.142
179	4.42273	20.5285	95.2847	345	6.84964	31.7932	147.571
182	4.47200	20.7572	96.3466	350	6.91566	32.0996	148.993
185	4.52101	20.9847	97.4025	355	6.98136	32.4046	150.409
188	4.56976	21.2109	98.4525	360	7.04676	32.7082	151.818
191	4.61826	21.4360	99.4970	365	7.11186	33.0103	153.220
194	4.66648	21.6598	100.530	370	7.17666	33.3111	154.616
197	4.71446	21.8826	101.570	375	7.24117	33.6105	156.007

A	Col. 1	Col. 2	Col. 3	A	Col. 1	Col. 2	Col. 3
380	7.30540	33.9086	157.390	631	10.2441	47.5489	220.703
386	7.38208	34.2646	159.042	640	10.3413	48.0000	222.796
392	7.45839	34.6188	160.686	649	10.4380	48.4490	224.880
398	7.53431	34.9712	162.322	658	10.5343	48.8959	226.954
404	7.60984	35.3217	163.949	667	10.6301	49.3407	229.019
410	7.68500	35.6706	165.568	676	10.7254	49.7835	231.075
416	7.75979	36.0178	167.180	685	10.8205	50.2244	233.121
422	7.83423	36.3633	168.783	694	10.9151	50.6634	235.159
428	7.90831	36.7071	170.379	703	11.0093	51.1005	237.187
434	7.98205	37.0494	171.968	712	11.1030	51.5357	239.207
440	8.05545	37.3901	173.549	721	11.1964	51.9690	241.219
446	8.12851	37.7292	175.124	730	11.2894	52.4007	243.222
452	8.20125	38.0668	176.691	740	11.3922	52.8781	245.438
458	8.27367	38.4030	178.251	750	11.4947	53.3534	247.645
465	8.35776	38.7933	180.063	760	11.5966	53.8266	249.841
472	8.44143	39.1817	181.865	770	11.6981	54.2978	252.028
479	8.52468	39.5681	183.659	780	11.7991	54.7668	254.205
486	8.60753	39.9527	185.448	790	11.8998	55.2339	256.373
493	8.68999	40.3354	187.220	800	12.0000	55.6991	258.532
500	8.77206	40.7163	188.988	811	12.1098	56.2085	260.897
507	8.85374	41.0954	190.748	822	12.2190	56.7156	263.250
514	8.93505	41.4728	192.500	833	12.3278	57.2204	265.594
521	9.01598	41.8485	194.243	844	12.4361	57.7231	267.927
528	9.09656	42.2225	195.979	855	12.5439	58.2235	270.250
535	9.17678	42.5948	197.708	866	12.6512	58.7219	272.563
542	9.25666	42.9656	199.429	877	12.7581	59.2180	274.866
550	9.34752	43.3874	201.386	888	12.8646	59.7122	277.160
558	9.43794	43.8071	203.334	900	12.9803	60.2490	279.651
566	9.52794	44.2248	205.273	912	13.0954	60.7833	282.131
574	9.61751	44.6405	207.203	924	13.2100	61.3154	284.601
582	9.70666	45.0543	209.124	936	13.3241	61.8451	287.059
590	9.79541	45.4663	211.036	948	13.4378	62.3725	289.508
598	9.88376	45.8764	212.939	961	13.5603	62.9415	292.148
606	9.97171	46.2846	214.834	974	13.6823	63.5079	294.777
614	10.0593	46.6910	216.721	987	13.8038	64.0717	297.394
622	10.1465	47.0957	218.599	1000	13.9248	64.6330	300.000

Brief Specifications of MARCHANT *Silent Speed* Calculators

At left:

Model **ACR-8D** Capacity 8x9x16.

(Model **ACR-10-D** is same except capacity is 10x11x20)

Electric Short-Cut Multiplication.
Automatic Division, Addition and Subtraction.
Add and Subtract Bars separate from Multiplying Bars.
Complete Capacity Carry-Over.
True-Figure Dials for all 3 factors, **including** the keyboard factor.
Selective Automatic Clear-Return Carriage Tabulation.

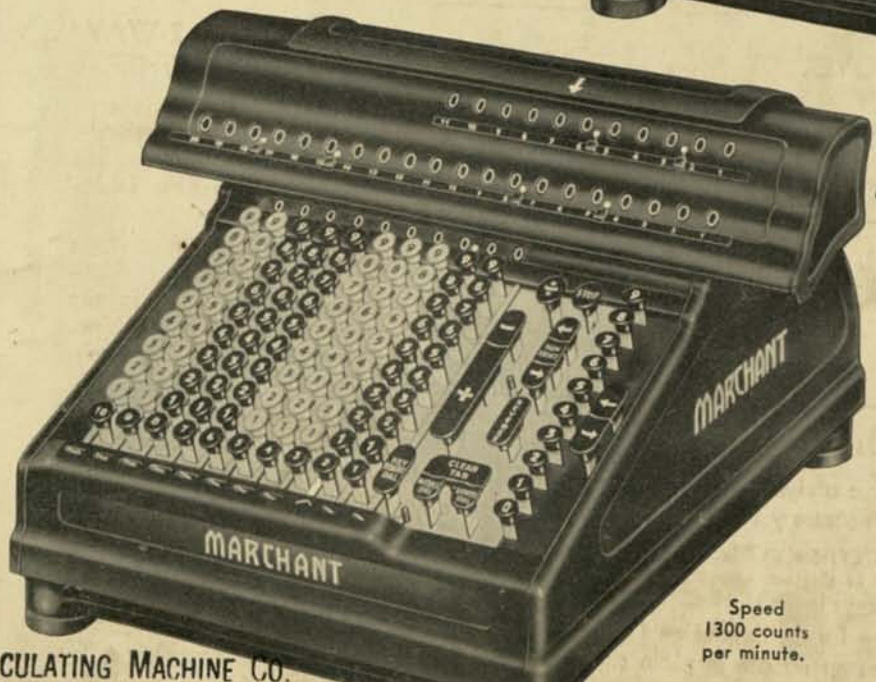
Speed 700 counts per minute.

At right:

Model **ACR-8M** Capacity 8x9x16.

Automatic Multiplication, Division, Addition and Subtraction.
Add and Subtract Bars separate from Multiplying Mechanism.
Complete Capacity Carry-Over.
True-Figure Dials for all 3 Factors **including** keyboard factor.
Selective Automatic Clear-Return Carriage Tabulation.

Speed 1300 counts per minute.



Speed 1300 counts per minute.

At left:

Model **ACT-10M** Capacity 10x11x20.

Automatic Multiplication, Division, Addition, and Subtraction.
Add and Subtract Bars separate from Multiplying Mechanism.
Complete Capacity Carry-Over.
True-Figure Dials for all 3 factors **including** the keyboard factor.
Instant and Selective Automatic Clear-Tab Carriage Tabulation in either direction from any position to any position.

MARCHANT CALCULATING MACHINE CO.
216 TREMONT STREET
BOSTON, MASS.

MARCHANT CALCULATING MACHINE COMPANY
Home Office: Oakland, California, U. S. A.
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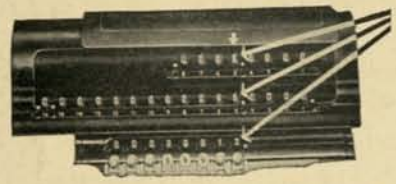
Operating Advantages of MARCHANT "Silent Speed" Calculators

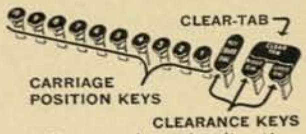
DIALS FOR ALL 3 FACTORS



(Including the Keyboard Factor)

Gives instant straight-line proof of both operator set-up and calculator-produced factors. Eliminates zig-zag search for hidden keyboard figures.

SELECTIVE AUTOMATIC CLEAR - RETURN OF CARRIAGE*

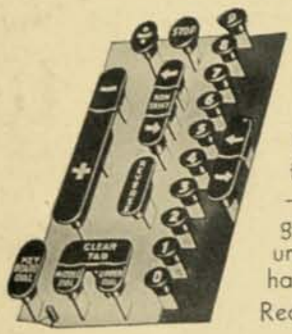


The carriage automatically returns to the next starting position after each division, simultaneously clearing dials for the entry of next amounts.

After each multiplication, carriage is electrically returned to any desired position, with or without simultaneous clearance of any or all dials.

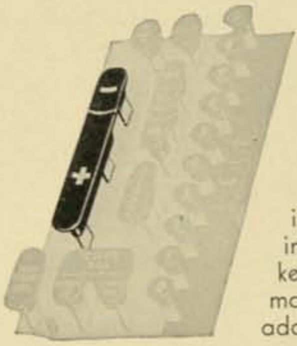
(*) Illustration shows ACT-10M. Other models somewhat similar.

ONE HAND KEYBOARD CONTROL



Every operation controlled from this keyboard—all keys conveniently grouped for simple operation under the fingertips of one hand — left as easy as right. Reduces hand and finger travel.

ADD AND SUBTRACT BARS



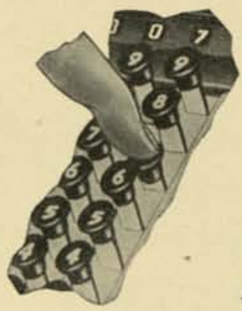
Separate from Multiplying Mechanism
(Marchant needs no repeat key)

A separate set of adding and subtracting bars is provided which is wholly independent of the multiplying mechanism. No repeat keys or levers to pre-set to make the bars suitable for adding.

POSITIVE ELECTRIC CLEARANCE

Conveniently grouped Clearance Keys permit instant and complete electrical clearance of the keyboard and all dials simultaneously or individually — by a single one-hand operation —with the carriage in any position. Partially cleared dials are impossible.

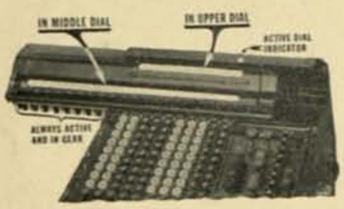
FLEXIBLE SINGLE KEY DEPRESSION



No keys will remain partially depressed. Not more than one key can be set in the same column.

Instant Dial-Proof is provided for each key as it is set.

COMPLETE CAPACITY CARRY-OVER



All dials active regardless of carriage position — no dead spot at left of carriage — no figures dropped. Accuracy to the limit by any method.

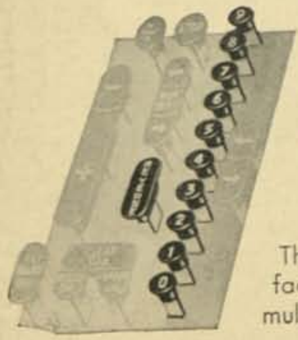
OPTIONAL 2-WAY CARRIAGE SHIFT*

In multiplication, the carriage glides in either direction — automatically controlled by green Directional Shift Keys.

Regardless of how the carriage is set to shift during multiplication, it always shifts in the proper direction during division.

(*) On ACR-8M and ACT-10M only.

AUTOMATIC SIMULTANEOUS MULTIPLICATION*



The extreme ease and simplicity of Marchant multiplication is shown by noting the few things that are necessary to do in order to complete any multiplication.

TO MULTIPLY: Set one factor in multiple keyboard—(easily checked in Keyboard Dial.) Enter other factor in single-row keyboard—(easily checked in Upper Dial.) The "Right Answer" automatically appears in the Middle Dial—IN A FLASH!

There is no waiting for multiplication to take place after the two factors are set up. Both factors and the answer remain in plain sight in straight-line dials upon completion of every multiplication. No factor disappears. **(*) On ACR-8M and ACT-10M only.**

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MARCHANT METHODS

MM-235
MATHEMATICS
Jan., 1943

NOGRADY METHOD FOR SOLUTION OF CUBIC EQUATIONS

REMARKS: *The application of the Birge-Vieta Method (See MM-225) to the solution of a cubic (third degree) equation gives the real root that is nearest to the first approximation. The work must then be repeated for other real roots. No imaginary roots are found. Special study has been given by Henry A. Nogrady* to the problem of obtaining all roots of such equations, both real and imaginary. Complete exposition of the method is given in his monograph, "A New Method for the Solution of Cubic Equations."* By aid of a table included in this book, the work is greatly simplified.*

The description herein exemplifies the use of the Marchant calculator when applied to the general cubic equation having three real roots, or having one real root and two conjugate complex roots. Modification to fit cases of two real roots, one real and two non-conjugate complex roots, and three complex roots, as well as tests for recognizing in what classification any equation comes, is fully covered in the Nogrady monograph, which is assumed to be in possession of the reader.

OUTLINE: The general cubic equation

(1) $ax^3 + bx^2 + cx + d = 0$

where a, b, c, and d are any numbers, is transformed into

(2) $y^3 + py + q = 0$

by substituting

(3) $\dfrac{3ac - b^2}{3a^2} = p$ and $\dfrac{2b}{27a^3} - \dfrac{bc}{3a^2} + \dfrac{d}{a} = q$

(2) then becomes

(4) $z^3 + nz + n = 0$

by substituting

(5) $\dfrac{p^3}{q^2} = n$

If n is real, eq. *(4)* has at least one real root. Its value is tabulated in the Nogrady monograph as z_1. By substitutions not outlined herein, the other roots of *(4)* are

(6) $z_2 = \dfrac{z_1}{2}\left(-1 + \sqrt{\dfrac{z_1 - 3}{z_1 + 1}}\right)$

(7) and $z_3 = \dfrac{z_1}{2}\left(-1 - \sqrt{\dfrac{z_1 - 3}{z_1 + 1}}\right)$

(*) "A New Method for the Solution of Cubic Equations" by Henry A. Nogrady, ~~29 - 18 Taylor Ave.,~~ Detroit, Michigan. For sale by the author, price $1.25 postpaid.
18987 SANTA BARBARA DRIVE

When z_1, z_2, and z_3 are found, the corresponding y's are found by multiplying the z's by the ratio q/p.

The corresponding x's are found by subtracting b/3a from the y's.

The computation is expedited if the following terms are evaluated in the order named:

$3a$, $3ac$, $3a^2$, $27a^3$, p, bc, q, b/3a, q^2, n, q/p, $(z_1 - 3)/(z_1 + 1)$. Extract root of previous amount, and then evaluate the y's and x's. This listing of elements of the computation does not comprise the bettering of the table value of z_1 (see Eq. 6).

EXAMPLE I

Find roots to 5 places of $x^3 + 2x^2 + 10x - 3 = 0$

By substitutions outlined above

$$y^3 + 8.66667y - 9.07407 = 0$$

and

$$z^3 + 7.90592z + 7.90592 = 0$$

From Nogrady Table, Page XXIV, the nearest n = 7.911462 for which the corresponding root z_1 is -0.906.

This value is improved to six figures by the following process:

(8) Six-figure value of $z_1 = \dfrac{2z_1^{\,3} - n}{3z_1^{\,2} + n}$

NOTE: A four-figure value requires only linear interpolation except at certain extremes of table.

in which $z_1 = -0.906$ and $n = 7.90592$; or Six-figure $z_1 = -0.905950$

from which $z_2 = .45298 + 2.91919\ i$ and $z_3 = .45298 - 2.91919\ i$

Multiplying these z's by q/p, we have

$$y_1 = 0.94854;\ y_2 = -0.47427 + 3.05642\ i;\ y_3 = -0.47427 - 3.05642\ i$$

Subtracting b/3a, we have

$$x_1 = 0.28187;\ x_2 = 1.14094 + 3.05642\ i;\ x_3 = 1.14094 - 3.05642\ i$$

The latter two roots, because of symmetry, are termed Conjugate Complex Roots. The symbol "i" indicates $\sqrt{-1}$.

OPERATIONS: Decimals; Upper Dial 6, Middle Dial 11, Keyboard Dial 5. Use any Marchant 8 or 10 column model. If "M" model is used, have Upper Green Shift Key down. The details below apply to Model M.

NOTE: Because the coefficients are simple integers, certain operations listed below normally would be omitted. For sake of completeness, however, they are listed. Whether a multiplication or division is positive or negative depends upon the sign of the factors and whether their product is to be added or subtracted. The procedure given below requires this obvious modification in the case of examples that have different signs from the equation considered herein.

(1) Set up in Keyboard Dial "a" (1.00000) and multiply by 3.

Copy "3a" (3.00000) from Middle Dial to Work Sheet.

(2) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "c" (10.00000).

Copy "3ac" (30.00000) from Middle Dial to Work Sheet.

(3) Clear Upper and Middle Dials, and multiply by "a" (1.00000).

Copy "$3a^2$" (3.00000) from Middle Dial to Work Sheet.

(4) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "a" (1.00000).

(5) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by 9.

Copy "$27a^3$" (27.00000) from Middle Dial to Work Sheet.

(6) Clear all dials, set up in Keyboard Dial "3ac" (30.00000), shift to 7th position, and depress Add Bar. Then depress Subtract Bar, set up "b" (2.00000) in Keyboard Dial, and reverse multiply by "b" (2.00000).

(7) Change Keyboard Dial to read "$3a^2$" (3.00000), and divide.

Copy "p" (8.66667) from Upper Dial to Work Sheet.

(8) Clear all dials, set up in Keyboard Dial "b" (2.00000), and multiply by "c" (10.00000).

Copy "bc" (20.00000) from Middle Dial to Work Sheet.

(9) Clear Upper and Middle Dials and multiply by "b" (2.00000).

(10) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "2b" (4.00000).

"$2b^3$" (16.00000) appears in Middle Dial, but it need not be copied to Work Sheet.

(11) Change Keyboard Dial to read "$27a^3$" (27.00000) and divide.

(12) Clear Keyboard and Middle Dials, set up in Keyboard Dial "bc" (20.00000), shift to 7th position, depress Add Bar and then depress Subtract Bar, change Keyboard Dial to read "$3a^2$" (3.00000), move Manual Counter Control toward the operator, and depress Division Key in the manner that will not cause Upper Dial to clear.

(13) Clear Keyboard and Middle Dials, set up in Keyboard Dial "d" (3.00000), shift to 7th position, depress Add Bar, and then depress Subtract Bar, change Keyboard Dial to read "a" (1.00000), and inasmuch as "d" is negative the Manual Counter Control will be left as it was in Step 12; i.e., toward the operator. Depress Division Key in the manner that will not cause Upper Dial to clear. Move Manual Counter Control away from operator.

NOTE: It will now be observed that Upper Dial shows a negative amount. This is evaluated as a positive amount and copied to Work Sheet as "q" (-9.07407).

(14) Clear all dials, set up "b" (2.00000), and, with carriage in 7th position, depress Add Bar.

(over)

(15) Change Keyboard Dial to read "3a" (3.00000) and divide.

Copy "b/3a" (0.66667) from Upper Dial to Work Sheet.

(16) Clear all dials, set up "q" (9.07407) and multiply by "q" (9.07407).

Copy "q^2" (82.33875) from Middle Dial to Work Sheet.

(17) Clear all dials, set up "p" (8.66667) and multiply by "p" (8.66667).

(18) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "p" (8.66667).

(19) Change Keyboard Dial to read "q^2" (82.33875) and divide.
Copy "n" (7.90592) from Upper Dial to Work Sheet.

(20) From Table of Nogrady Roots, Page XXIV, the nearest "n" is 7.911462 for which corresponding root "z_1" is -0.906.

NOTE: The computation for improving this root to o.905950 by formula 8 is obvious. It is taken to 5 places as -0.90595.

(21) Clear all dials, set up in Keyboard Dial "q" (9.07407) and, with carriage in 7th position, depress Add Bar. Change Keyboard Dial to read "O" (8.66667) and divide.

Copy "q/p" (1.04701) from Upper Dial to Work Sheet.

(22) Clear all dials, set up in Keyboard Dial "z_1-3" (3.90595) and with carriage in 7th position, depress Add Bar. Change Keyboard Dial to read "z_1 + 1" (0.09405) and divide.

Copy $(z_1-3)/(z_1+1)$ or (-41.53057) from Upper Dial to Work Sheet.

(23) Extract Square Root of -41.53057 by Marchant Table No.56, producing a five-figure root of 6.4444 which is expressed as 6.4444 i, indicating that it is the square root of a negative number.

NOTE: This square root may be improved, if desired, by the method on the reverse side of Table No. 56 to 6.44442 i.

(24) Clear all dials, set up in Keyboard Dial "$z_1/2$" (0.45298) and multiply by square root from Step 23 (6.44442).

Copy coefficient of i (2.91919) from Middle Dial to Work Sheet, thus completing all figures from z_2 and z_3.

(25) Clear all dials, set up in Keyboard Dial "q/p" (1.04701) and multiply by z_1 (0.90595) and the real and imaginary parts of z_2 and z_3 (0.45298) and (2.91919) producing

y_1 (0.94854); y_2 (-0.47427 + 3.05642 i);
and y_3 (-0.47427 - 3.05642 i).

(26) Clear all dials. With carriage in 7th position, set up y_1 (0.94854), and add. Set up "b/3a" (0.66667) and, with Non-Shift Key down, reverse multiply by 1.

x_1 (0.28187) appears in Middle Dial.

(27) Clear Middle Dial and touch Add Bar. Set up the real part of y_2 and y_3 (0.47427) and add, thus completing values for

x_2 (1.14094 + 3.05642 i)
x_3 (1.14094 - 3.05642 i)

EXAMPLE II

Find roots to 5 places of $x^3 - 7x + 6 = 0$

This is in the form of $y^3 + py + q = 0$, so the operations following Step No. 15 need only be done with certain obvious deletions. The outline is below:

$n = p^3/q^2 = - 343/36 = -9.52778.$

From Table, nearest "n" is -9.516913 for which z_1 is -1.169.

This value is improved by *(8)* to

$$z_1 = \frac{2\ (-1.169^3) + 9.52778}{3\ (-1.169^2) - 9.52778} = \frac{6.33276}{-5.42810} = -1.16667$$

$$q/p = 6/-7 = -0.85714$$

$$\sqrt{(z_1 - 3)/(z_1 + 1)} = \sqrt{25} = 5$$

$$z_2 = -0.58333 \cdot 4 = -2.33333$$

$$z_3 = -0.58333 \cdot -6 = 3.50000$$

$$x_1 = y_1 = -0.85714 \cdot -1.16667 = 1.$$

$$x_2 = y_2 = -0.85714 \cdot -2.33333 = 2.$$

$$x_3 = y_3 = -0.85714 \cdot 3.50000 = -3$$

The Marchant operations are similar to most of those following Step 16 of Example I.

MARCHANT — SILENT — METHODS

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If n is real, eq. *(4)* has at least one real root. Its value is tabulated in the Nogrady monograph as z_1. By substitutions not outlined herein, the other roots of *(4)* are

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(*) "A New Method for the Solution of Cubic Equations" by Henry A. Nogrady, 18987 Santa Barbara Drive, Detroit, Michigan. For sale by the author, price $1.15 postpaid.

(over)

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The corresponding x's are found by subtracting b/3a from the y's.

The computation is expedited if the following terms are evaluated in the order named:

$3a$, $3ac$, $3a^2$, $27a^3$, p, bc, q, b/3a, q^2, n, q/p, $(z_1 - 3)/(z_1 + 1)$. Extract root of previous amount, and then evaluate the y's and x's. This listing of elements of the computation does not comprise the bettering of the table value of z_1 (see Eq. 6).

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(8) Six-figure value of $z_1 = \dfrac{2z_1^3 - n}{3z_1^2 + n}$

NOTE: A four-figure value requires only linear interpolation except at certain extremes of table.

in which $z_1 = -0.906$ and $n = 7.90592$; or Six-figure $z_1 = -0.905950$

from which $z_2 = .45298 + 2.91919\ i$ and $z_3 = .45298 - 2.91919\ i$

Multiplying these z's by q/p, we have

$$y_1 = 0.94854;\ y_2 = -0.47427 + 3.05642\ i;\ y_3 = -0.47427 - 3.05642\ i$$

Subtracting b/3a, we have

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The latter two roots, because of symmetry, are termed Conjugate Complex Roots. The symbol "i" indicates $\sqrt{-1}$.

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(1) Set up in Keyboard Dial "a" (1.00000) and multiply by 3.
Copy "3a" (3.00000) from Middle Dial to Work Sheet.

(2) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "c" (10.00000).
Copy "3ac" (30.00000) from Middle Dial to Work Sheet.

(3) Clear Upper and Middle Dials, and multiply by "a" (1.00000).
Copy "$3a^2$" (3.00000) from Middle Dial to Work Sheet.

(4) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "a" (1.00000).

(5) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by 9.
Copy "$27a^3$" (27.00000) from Middle Dial to Work Sheet.

(6) Clear all dials, set up in Keyboard Dial "3ac" (30.00000), shift to 7th position, and depress Add Bar. Then depress Subtract Bar, set up "b" (2.00000) in Keyboard Dial, and reverse multiply by "b" (2.00000).

(7) Change Keyboard Dial to read "$3a^2$" (3.00000), and divide.
Copy "p" (8.66667) from Upper Dial to Work Sheet.

(8) Clear all dials, set up in Keyboard Dial "b" (2.00000), and multiply by "c" (10.00000).
Copy "bc" (20.00000) from Middle Dial to Work Sheet.

(9) Clear Upper and Middle Dials and multiply by "b" (2.00000).

(10) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "2b" (4.00000).

"$2b^3$" (16.00000) appears in Middle Dial, but it need not be copied to Work Sheet.

(11) Change Keyboard Dial to read "$27a^3$" (27.00000) and divide.

(12) Clear Keyboard and Middle Dials, set up in Keyboard Dial "bc" (20.00000), shift to 7th position, depress Add Bar and then depress Subtract Bar, change Keyboard Dial to read "$3a^2$" (3.00000), move Manual Counter Control toward the operator, and depress Division Key in the manner that will not cause Upper Dial to clear.

(13) Clear Keyboard and Middle Dials, set up in Keyboard Dial "d" (3.00000), shift to 7th position, depress Add Bar, and then depress Subtract Bar, change Keyboard Dial to read "a" (1.00000), and inasmuch as "d" is negative the Manual Counter Control will be left as it was in Step 12; i.e., toward the operator. Depress Division Key in the manner that will not cause Upper Dial to clear. Move Manual Counter Control away from operator.

NOTE: It will now be observed that Upper Dial shows a negative amount. This is evaluated as a positive amount and copied to Work Sheet as "q" (-9.07407).

(14) Clear all dials, set up "b" (2.00000), and, with carriage in 7th position, depress Add Bar.

(over)

(15) Change Keyboard Dial to read "3a" (3.00000) and divide.

Copy "b/3a" (0.66667) from Upper Dial to Work Sheet.

(16) Clear all dials, set up "q" (9.07407) and multiply by "q" (9.07407).

Copy "q^2" (82.33875) from Middle Dial to Work Sheet.

(17) Clear all dials, set up "p" (8.66667) and multiply by "p" (8.66667).

(18) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "p" (8.66667).

(19) Change Keyboard Dial to read "q^2" (82.33875) and divide.
Copy "n" (7.90592) from Upper Dial to Work Sheet.

(20) From Table of Nogrady Roots, Page XXIV, the nearest "n" is 7.911462 for which corresponding root "z_1" is -0.906.

NOTE: The computation for improving this root to 0.905950 by formula 8 is obvious. It is taken to 5 places as -0.90595.

(21) Clear all dials, set up in Keyboard Dial "q" (9.07407) and, with carriage in 7th position, depress Add Bar. Change Keyboard Dial to read "O" (8.66667) and divide.

Copy "q/p" (1.04701) from Upper Dial to Work Sheet.

(22) Clear all dials, set up in Keyboard Dial "z_1-3" (3.90595) and with carriage in 7th position, depress Add Bar. Change Keyboard Dial to read "z_1 + 1" (0.09405) and divide.

Copy $(z_1-3)/(z_1+1)$ or (-41.53057) from Upper Dial to Work Sheet.

(23) Extract Square Root of -41.53057 by Marchant Table No.56, producing a five-figure root of 6.4444 which is expressed as 6.4444 i, indicating that it is the square root of a negative number.

NOTE: This square root may be improved, if desired, by the method on the reverse side of Table No. 56 to 6.44442 i.

(24) Clear all dials, set up in Keyboard Dial "$z_1/2$" (0.45298) and multiply by square root from Step 23 (6.44442).

Copy coefficient of i (2.91919) from Middle Dial to Work Sheet, thus completing all figures from z_2 and z_3.

(25) Clear all dials, set up in Keyboard Dial "q/p" (1.04701) and multiply by z_1 (0.90595) and the real and imaginary parts of z_2 and z_3 (0.45298) and (2.91919) producing

y_1 (0.94854); y_2 (-0.47427 + 3.05642 i);
and y_3 (-0.47427 - 3.05642 i).

(26) Clear all dials. With carriage in 7th position, set up y_1 (0.94854), and add. Set up "b/3a" (0.66667) and, with Non-Shift Key down, reverse multiply by 1.

x_1 (0.28187) appears in Middle Dial.

(27) Clear Middle Dial and touch Add Bar. Set up the real part of y_2 and y_3 (0.47427) and add, thus completing values for

x_2 (1.14094 + 3.05642 i)
x_3 (1.14094 - 3.05642 i)

EXAMPLE II

Find roots to 5 places of $x^3 - 7x + 6 = 0$

This is in the form of $y^3 + py + q = 0$, so the operations following Step No. 15 need only be done with certain obvious deletions. The outline is below:

$n = p^3/q^2 = -343/36 = -9.52778.$

From Table, nearest "n" is -9.516913 for which z_1 is -1.169.

This value is improved by (8) to

$$z_1 = \frac{2\ (-1.169^3) + 9.52778}{3\ (-1.169^2) - 9.52778} = \frac{6.33276}{-5.42810} = -1.16667$$

$$q/p = 6/-7 = -0.85714$$

$$\sqrt{(z_1 - 3)/(z_1 + 1)} = \sqrt{25} = 5$$

$$z_2 = -0.58333 \cdot 4 = -2.33333$$

$$z_3 = -0.58333 \cdot -6 = 3.50000$$

$$x_1 = y_1 = -0.85714 \cdot -1.16667 = 1.$$

$$x_2 = y_2 = -0.85714 \cdot -2.33333 = 2.$$

$$x_3 = y_3 = -0.85714 \cdot 3.50000 = -3$$

The Marchant operations are similar to most of those following Step 16 of Example I.

MILNE METHOD OF STEP-BY-STEP DOUBLE INTEGRATION OF SECOND ORDER DIFFERENTIAL EQUATIONS IN WHICH FIRST DERIVATIVES ARE ABSENT

(A Supplement to Marchant Method MM-216)

REMARKS: *Dr. W. E. Milne* calls attention to the frequency of problems in dynamics and astronomy that are of the type discussed herein. For their rapid solution, he has published the formulas given below. Application of this method to the Marchant is shown in outline form. A knowledge of MM-216 is assumed.*

EXAMPLE: Integrate twice $d^2y/dx^2 = (x^2 - 1)y = u$, with initial value $y = 1$ and $dy/dx = 0$ when $x = 0$, and starting values with differences as below. Seven-place accuracy of "y" is desired.

x	y	u	1st	2nd	3rd	4th	5th
0	1.00000000	-1.00000000					
			1493764				
0.1	0.99501248	-0.98506236		2913400			
			4407164		-217263		
0.2	0.98019867	-0.94099072		2696137		-128177	
			7103301		-345440		23427
0.3	0.95599748	-0.86995771		2350697		-104750	
			9453998		-450190		
0.4	0.92311635	-0.77541773		1900507			
			11354505				
0.5	0.88249690	-0.66187268					

OUTLINE: The published Milne formulas applying to this work are below:

3-TERM FORMULAS, exact if third differences of u are constant (fourth differences vanish) and with known error-factor of $\delta y/13$ if fourth differences are constant (fifth differences vanish).

Open type for integrating ahead:

$$y_{n+1} = y_n + y_{n-2} - y_{n-3} + (h^2/4)\ (5u_n + 2u_{n-1} + 5u_{n-2}) \tag{1}$$

Closed type for recalculating by back-check:

$$y_n = 2y_{n-1} - y_{n-2} + (h^2/12)\ (u_n + 10u_{n-1} + u_{n-2}) \tag{2}$$

5-TERM FORMULAS, exact if fifth differences of u are constant (sixth differences vanish) and with known error-factor of $\delta y/26$ if sixth differences are constant (seventh differences vanish).

Open type for integrating ahead:

$$y_{n+1} = y_n + y_{n-4} - y_{n-5} + (h^2/48)\ (67u_n - 8u_{n-1} + 122u_{n-2} - 8u_{n-3} + 67u_{n-4}) \tag{3}$$

Closed type for recalculating by back-check:

$$y_n = y_{n-1} + y_{n-3} - y_{n-4} + (h^2/240)\ (17u_n + 232u_{n-1} + 222u_{n-2} + 232u_{n-3} + 17u_{n-4}) \tag{4}$$

(*) *W. E. Milne, On the Numerical Integration of Certain Differential Equations of the Second Order, Am. Math. Mo. 40:322-327 (1933), also National Research Council, No. 92, Numerical Integration of Differential Equations (Report of A. A. Bennett, W. E. Milne, H. Bateman) (1933).*

(over)

MARCHANT CALCULATING MACHINE COMPANY OAKLAND, CALIFORNIA
PRINTED IN U. S. A.

WILL THE INTERVAL CHOSEN FOR STARTING VALUES PROVIDE DESIRED ACCURACY?

The need of making an intelligent guess as to the interval (h) to be used for any desired accuracy of the result occurs in any work of this sort. This matter was only touched upon in Marchant Method MM-216, so an analysis of this case may be helpful in suggesting an approach to the problem.

(a) Inasmuch as the six tabulated values of "u" provide only a 5th difference, is it possible to infer what the higher differences might be? We can do so only in very broad terms, and inasmuch as reasoning with regard to these matters is often required, the subject is gone into at length.

(b) It is most unlikely that the function "u" has a constant 5th or 6th difference that would enable the 5-term formula to be used with results accurate to the number of places of the tabulated "u". We know this because a 7th or 8th degree power series (which would have constant 7th or 8th differences of f (x)) could not have its second derivative "u" in the algebraic form of "u". It is probable that "u" has higher differences and most certainly has a 6th and 7th difference.

(c) Inferences as to the size of 6th and 7th differences are made from the following. The descending differences show the following ratios:

Ratio	1st	difference	to	function	.015
"	2nd	"	"	1st diff.	.195
"	3rd	"	"	2nd "	.075
"	4th	"	"	3rd "	.590
"	5th	"	"	4th "	.181

The ascending differences similarly show the following ratios:

Ratio	1st	difference	to	function	.122
"	2nd	"	"	1st diff.	.245
"	3rd	"	"	2nd "	.146
"	4th	"	"	3rd "	.400
"	5th	"	"	4th "	.223

Plotting these ratios against orders of difference shows that these fluctuating differences show a trend, and that the amplitude of the fluctuations decreases as "x" increases. This is proved by the fact that the ascending differences have smaller amplitudes than the descending. If the trend of the tops and bottoms of these "swings" is plotted and extended to 6th and 7th orders of difference, it appears that the ratio of 6th ascending difference to 5th is about .47 and of 7th to 6th is about .33. Applying these ratios, it appears to be a good guess that the ascending 6th difference will not exceed .00011 and the ascending 7th difference will not exceed .00004.

(d) The Milne 5-Term Formulas supply an exact "y" if 6th differences in "u" vanish and use of the Milne Error Factors provide means of obtaining an exact "y" if 7th differences of "u" vanish. In functions of the type considered herein, however, neither 6th or 7th differences may be expected to vanish. A good guess as to the ascending 6th difference is made in the above as that it is 0.00011 at $x = 0.5$. What error in "y" may be expected from this difference?

(e) To estimate this, it is necessary to study the error formulas in the light of the derivatives of tabulated u's. These are:

(5) Error of (1) is $(17/240)\ h^6 u^{(4)}(x_p)$

(6) " " (2) " $(-1/240)\ h^6 u^{(4)}(x_q)$

(7) " " (3) " $(7870/120960)\ h^8 u^{(6)}(x_p)$

(8) " " (4) " $(-318/120960)\ h^8 u^{(6)}(x_q)$

In which $u^{(4)}(x_p)$ is the fourth derivative of u with respect to x when x is some value (x_p) that is within the range of all x's that determine the 4th central difference of u corresponding to the value of x for which the error is desired. For example, if it is desired to estimate the error of y at, say x_{-1}, the fourth central difference corresponding to x_{-1} is determined by the 5 values of x from x_{-3} to x_{+1} inclusive. The upper bound of error at point x_{-1} will thus be controlled by the greatest 4th derivative of u for <u>any</u> of the x's from x_{-3} to x_{+1} inclusive.

The expression $u^{(6)}(x_p)$ is similarly defined, and, likewise, for those with subscript q.

(f) As it is difficult to obtain the derivatives of u because y is involved implicitly, the derivatives may be estimated from the difference array of tabulated u, as follows:

(9) $u^{(4)}(x) = (1/h^4)\,(d^{\text{iv}}{}_0 - (1/6)\,d^{\text{VI}}{}_0 + (7/240)\,d^{\text{VIII}}{}_0 \ldots)$

(10) $u^{(6)}(x) = (1/h^6)\,(d^{\text{VI}}{}_0 - (1/4)\ d^{\text{VIII}}{}_0 + (13/240)\ d^{\text{X}} \ldots.)$

in which d^{iv}, d^{VI}, etc., are 4th, 6th, etc., central differences of u corresponding to any tabulated x, using the nomenclature of Marchant Method MM-189a.

(g) It is one of the beauties of the Milne Method that by correcting the result of the check formulas (2) or (4) by means of the Milne Error Correction $\delta y/13$ or $\delta y/26$, respectively, the error is eliminated if 5th and 7th differences, respectively, vanish. When there are differences higher than the 4th and 6th, respectively, however, the Milne Error Correction does not eliminate the entire error because the "x" value for the open-type formula is not necessarily the same as that for the corresponding closed-type formula. The derivatives, therefore, may differ so there is a small residual error that is not eliminated by applying the Milne Error correction to (2) or (4). Just what this is, in any case, has not been explored by us, and it appears that it would be difficult to determine because there is no means of knowing which of the values of x in the range of values is to be taken for determining the derivative that controls the error.

A working rule, which has as its support only the fact that it is satisfactory in the cases for which it has been used, is that if there are differences higher than the 4th and 6th when using the 3-term and 5-term formulas, it is satisfactory to regard the Milne Error Correction as eliminating the effect of the 4th or 6th difference, respectively, but not affecting the error due to differences of higher order. When the derivatives are expressed in terms of differences as in (9) and (10), this means that the error in (2) after applying the Milne Error Correction is perhaps of the order of

(11) $(-1/240)\ h^2\ (-(1/6)\,d^{\text{VI}}{}_0 + (7/240)\ d^{\text{VIII}}{}_0 \ldots.)$

and, similarly, in (4) it perhaps may be taken as

(12) $(-318/120960)\ h^2\ (-(1/4)\, d^{VIII}_0 + (13/240)\ d^X \ldots.)$*

the upper bound in either case being the largest value in the range of "x" values corresponding to the u's that determine d'''_0 in (9) or d^{VI}_0 in (10).

Applying this analysis to the problem at hand, it is noted that if when applying (1) and then applying the Milne Error Factor of $\delta y/13$, and the largest 6th difference of u is 0.00011, as above estimated, we have for the error by applying (11)

$$e = (-1/240)\ (.01)\,(-.00002) = .000000001 \text{ approx.}$$

(h) It will thus be observed that if only one value of y is desired beyond starting values, the 3-term formulas (1) and (2) would probably supply a result accurate to 8 places. However, the propagation of this error as additional values are computed results in an accumulation of errors. Exploration of this propagation of error in any case is not a difficult mathematical task, but even if we ignore the effect on the terms involving u, it is seen that each new y involves the addition of two previous y's and the subtraction of but one. When the computation has proceeded so that the starting values no longer enter the determination, the error in the example considered herein is at least 55e if we use (2) after 10 values subsequent to starting values. This assumes that all errors have same sign. We have no right to assume otherwise, based upon the available data at this point.

From this, it is seen that use of the 3-term formulas should be satisfactory to establish 7-place accuracy if only a few values are required, but the 5-term formulas should be used if 7-place accuracy is desired in the vicinity, roughly, of values beyond the 5th new value.

(i) By this reasoning, we decide to use the 5-term formula under the assumption that values up to and beyond those for x = 1.5 are desired. It now remains to be determined what interval should be used.

By reasoning similar to that in (c), we guess that a rough ascending 8th difference of tabulated u's is .000026, from which error of (4) after applying Milne Error Factor of $\delta y/26$, by using (12), is

$$(318/120960)\ (.01)\,(-.000026/4) = .00000000017 \text{ approx.}$$

When applied for ten values, it will be, roughly, .000000009, which assures 7-place accuracy with interval h = 0.1.

(j) We, therefore, conclude that the best procedure is to use interval h = 0.1 and the 5-term formulas (3) and (4). This will doubtless provide 8-place accuracy in the range up to x = 1.5, instead of the 7-place accuracy desired. Any reasonable attempt to cut the work to obtain 7-place accuracy up to x = 1.5 and no more, such as by using an odd interval, is not warranted.

(k) The next five values are determined by the method of MM-216, according to work sheets attached. Calculations are carried to 8 places, with final rounding to seven, developing the following values (Only the first computation is shown completely. The others are obvious as the subscripts advance by 1):

(*) If 5th differences of u from (2) or 7th differences of u from (4) are constant, (9) and (10) are not effective in producing a usable (11) and (12). In such cases, estimate derivatives from formulas similar to (9) and (10) obtained from ascending or descending differences, which contain both odd and even orders of difference. These*lead to formulas similar to (11) and (12), which contain only 5th and 7th differences, respectively.

x	Symbol	y to 8 places	Final y
0.6	1	0.83527021	0.8352702
.7	2	.78270454	0.7827045
.8	3	.72614903	0.7261490
.9	4	.66697680	0.6669768
1.0	5	.60653065	0.6065307

DISCUSSION

The computation shows that after applying the Milne Error Correction, the change was not enough to affect last place of the 8-place answer obtained from (4) for any of the values to $x = 1.0$. This might have been predicted and would indicate that if we wished to stop the calculation at $x = 1.0$, it might have been possible to use (3) without any back-check. We did not do so, as the supposition was that at least ten values beyond starting values would be obtained.

The actual differences of u, insofar as we can tabulate them, show them to be less than estimated; thus,

	As predicted	Highest Actual
6th diff. "u"	.00011	.000065
8th diff. "u"	.000026	.000012

The trend of these differences, however, shows the likelihood that there are larger ones than the above "highest actual" values that are in the complete range of values of x, for it must be realized that tabulation of the ten values of u supplies only four 6th differences and two 8th differences.

The reader is cautioned not to accept the fact that if the Milne Error Correction does not produce any alteration in the right-hand digit of y as obtained from (4), it is evidence that the result is thereby accurate to the number of places of y so computed. This is only true if 7th differences vanish if (4) is used. If such difference does not vanish, as stated in (g), there is a residual error in each y believed to be expressed by (12) which accumulates and compounds as the number of values of computed y's increases. In the present case, it is unlikely that 8th place would be affected by the <u>Error Correction</u>, even for five more values, but by considerations outlined in (h) and (i), we see that only 7-place accuracy of y can reasonably be expected.

The function used in this example is, of course, the second derivative of

$$y = e^{-\frac{x^2}{2}}$$

which is much used in Quantum Mechanics. A study of its 6th and 8th derivatives, after giving effect to eliminating the effect of 6th and 8th differences in those derivatives, as in (g), provides reasonable support for the conclusions reached above by the somewhat rough predictions made from the first five tabulated values.

FACTOR SHEET: MILNE 5-POINT (6 ORDINATE) FORMULA FOR DOUBLE INTEGRATION BY STEP-BY-STEP METHOD

ORIGINAL CALCULATION: $h = 0.1$ Length Factor: $h^2/48 = 0.01/48$

			- Column Multipliers -		
x	u	$u = (x^2 - 1)y$	67	-8	122
0	n = -5	-1.00000000			
0.1	-4	-0.98506236	-65.999178		
0.2	-3	-0.94099072	63.046378	+7.527926	
0.3	-2	-0.86995771	58.287167	6.959662	-106.134841
0.4	-1	-0.77541773	51.952988	6.203342	94.600963
0.5	0	-0.66187268	44.345470	5.294981	80.748467
0.6	+1	-0.53457293	35.816386	4.276583	65.217897
0.7	+2	-0.39917932	26.745014	3.193435	48.699877
0.8	+3	-0.26141366	17.514715	2.091309	31.892467
0.9	+4	-0.12672559	8.490615	1.013805	15.460522
1.0	+5	0	0	0	0

BACK-CHECK CALCULATION: $h = 0.1$ Length Factor: $h^2/240 = 0.01/240$

			17	232	222
0.2	-3	-0.94099072	-15.996842		
0.3	-2	-0.86995771	14.789281	-201.830189	
0.4	-1	-0.77541773	13.182101	179.896913	-172.142736
0.5	0	-0.66187268	11.251836	153.554462	146.935735
0.6	+1	-0.5345729~~1~~3	9.0877~~30~~40	124.020920	118.675190
0.7	+2	-0.3991793~~0~~2	6.786048	92.609602	88.617809
0.8	+3	-0.26141366	4.444032	60.647969	58.033833
0.9	+4	-0.126725~~60~~59	2.154335	29.400337	28.133081
1.0	+5	0	0	0	0

WORK SHEET: MILNE 5-POINT (6 ORDINATE) FORMULA FOR DOUBLE INTEGRATION BY STEP-BY-STEP METHOD

Example: $d^2y/dx^2 = (x^2 - 1)y$

FORMULA	TRIAL		CHECK	
$x = 0.6$	y_0	0.88249690	y_0	0.88249690
	y_{-4}	0.99501248	y_{-2}	0.95599748
	$-y_{-5}$	1.00000000	$-y_{-3}$	0.98019867
		0.87750938		0.85829571
$0.01/48\ (67u_{-4} - 8u_{-3} + 122u_{-2} - 8u_{-1} + 67u_{0})$		-0.04223921		
	y_{+1}	0.83527017		
	$-u_{+1}$	0.53457291		
$.01/240\ (17u_{-3} + 232u_{-2} + 222u_{-1} + 232u_{0} + 17u_{1})$				-0.02302550
	Final		y_{+1}	0.83527021
$\delta y = 4 \quad E = .00000004/26$			$-u_{+1}$	0.53457293

The remainder of the Work Sheet is similar except that all subscripts are advanced by 1, with values entered accordingly.

SOLUTION OF RIGHT TRIANGLES

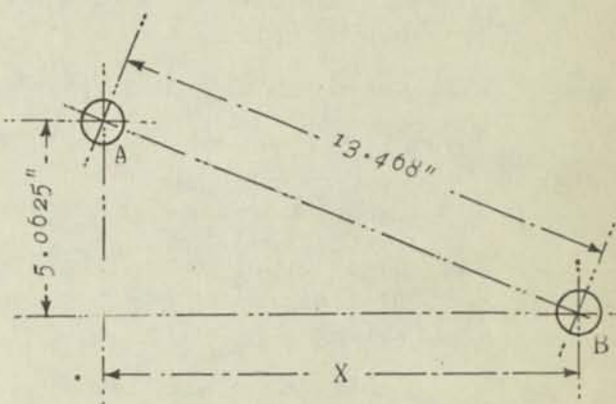
Location of Drilling Centers

REMARKS: *In machine shop layout work, particularly in connection with layout of drilling jigs, it is often necessary to locate centers of holes when they are not indicated on the drawing in the manner necessary for use with layout tools or boring machines. Obtaining the proper dimensions for such cases is easily done with the aid of a Marchant Calculator.*

NOTE: *In all cases as described below, the words "distance between holes" signify distance between their centers.*

CASE I: Linear Location

Given all dimensions shown, to find X:



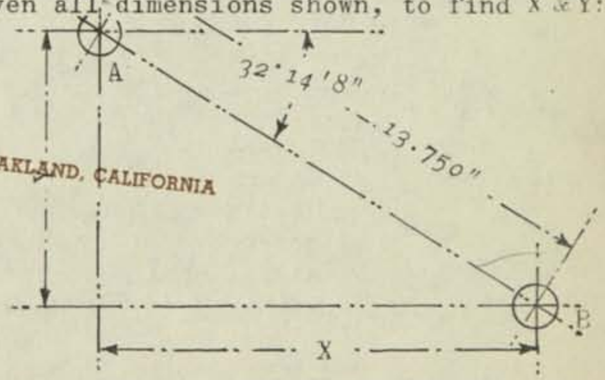
In sketch (not to scale), the location of hole A is known by dimensions that permit its center to be easily located on the work. Hole B, on the other hand, is located only by its vertical distance from A and its actual distance (on the slope) from A. For proper layout of the work with the facilities at hand, it is necessary to know the horizontal distance of hole B from hole A.

What is the horizontal distance of hole B from hole A to within ½ of 1/1000 inch?

Obviously, it is necessary to find the distance "X". This is done by solving the right triangle when given its hypotenuse (13.468) and the length of one side; thus, $X = \sqrt{13.468^2 - 5.0625^2}$

CASE II: Angular Location

Given all dimensions shown, to find X & Y:



This case is similar to Case I, except that hole B is located only by its actual distance (on the slope) from A, and by the angle that the center-line connecting both holes bears from the horizontal.

What are the horizontal and vertical distances of hole B from hole A to within ½ of 1/1000 inch?

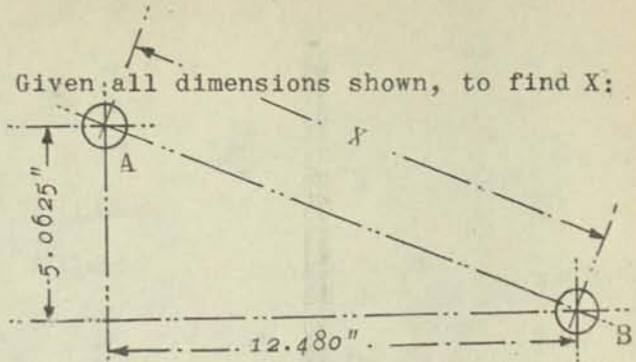
Obviously, it is necessary to find distances "X" and "Y", respectively; thus,

$$X = \sin\,(90 - 32°14'8") \times 13.750; \quad Y = \sin 32°14'8" \times 13.750$$

(over)

CASE III: Distance Between Holes

Given all dimensions shown, to find X:



In this case, the holes A and B are located with reference to each other by their vertical and horizontal distances.

What is the distance between the two holes to ½ of 1/1000 inch?

Obviously, it is necessary to find the hypotenuse X when given the two sides of the right triangle; thus, $X = \sqrt{12.480^2 + 5.0625^2}$

It is assumed that extracting square root by Marchant Method as outlined on Marchant Table No. 56 and obtaining angular functions from tables of Marchant Method MM-99 are understood.

MARCHANT METHOD FOR CASE I

Use any Marchant 8-column M or D model with decimals as follows: Upper Dial 6, Middle Dial 11, and Keyboard Dial 5. In all cases, set the decimals as far toward the left as will accomodate the factors in order that a 5 figure square root may be obtained without the necessity of re-setting any part of the problem.

(1) Set up 13.468 in Keyboard Dial and multiply by 13.468.

(2) Clear Upper and Keyboard Dials only, set up 5.0625 in Keyboard Dial, move Manual Counter Control toward the operator, and reverse multiply by 5.0625. Move Manual Counter Control away from operator.

(3) Clear Upper and Keyboard Dials only, shift to 7th position, set up in Keyboard Dial nearest number to 155.758, etc. that appears in Col. A of Table 56 (156.) and add.

Middle Dial now reads 311.758, etc., but it need not be separately noted.

(4) It should now be noted that the square root will contain two whole numbers (at left of its decimal), so carriage is shifted to 8th position so the root will appear properly pointed off by Upper Dial Decimal.

(5) Set up in Keyboard Dial the proper Square Root Divisor (2497999) that appears in Col. 1 adjacent to Col. A, so that left-hand figure of divisor is directly below left-hand figure of Middle Dial amount. The divisor from Col. 1 is selected because the number of which the root is desired (155.758 etc.) has one figure in the left-hand "period" when 155 is separated into periods of two figures beginning at decimal point; thus 1'55'. Depress Division Key.

Distance "X" (12.480) appears in Upper Dial. This value is correct to 5 places; that is to say, the error of 6th figure does not exceed 5, so it satisfies the conditions of the problem; viz., the amount is correct to within ½ of 1/1000 of an inch.

NOTE: Not more than 5 figures of Upper Dial amount may be used, but the 5th figure should be increased by "1" if the 6th figure is "5" or over. If a 5-figure root does not provide the desired accuracy, convert to a 9-figure root by method described on reverse side of Marchant Table 56.

MARCHANT METHOD FOR CASE II

Use any Marchant 8-column M or D model with decimals as follows: Upper Dial 4, Middle Dial 11, Keyboard Dial 7. The desired accuracy requires use of not more than 6 places of the trigonometric function, but inasmuch as the tables of Marchant Method MM-99 provide 7 places, such values will be used to avoid possibility of error in rounding off tabular values to six places.

(1) Set up in Keyboard Dial sin 32°14' from table of MM-99 (.5333685) and, with carriage in 5th position, add into Middle Dial.

(2) Set up in Keyboard Dial (at extreme right) the increment for 8 seconds corresponding to 32°15' (328), and add.

(3) Set up in Keyboard Dial the amount that appears in Middle Dial (.5334013), clear Upper and Middle Dials, and multiply by 13.750.

Distance Y (7.334...) appears in Middle Dial. Copy it as 7.334."

(4) Clear all dials, set up in Keyboard Dial 90° in terms of degrees, minutes, and seconds; thus, 8.9059060 (the decimal between 8 and 9 has no significance), shift to 1st position, and add.

(5) Similarly, set up 32°14'8" in Keyboard Dial; thus, 3.2014008 and subtract, thus producing complement of 32°14'8" (57°45'52").

(6) Clear all dials, shift to 5th position, set up in Keyboard Dial sin 57°46' (.8458830) and add.

(7) Set up in Keyboard Dial (at extreme right) the increment for 52 seconds corresponding to 57°45' (207) and subtract.

(8) Set up in Keyboard Dial the amount that appears in Middle Dial (.8458623), clear Upper and Middle Dials, and multiply by 13.750.

Distance X (11.6306...) appears in Middle Dial. Copy it as 11.631."

MARCHANT METHOD FOR CASE III

Same decimal setting as for Case I.

(1) Set up 12.480 in Keyboard Dial, and multiply by 12.480.

(2) Clear Upper and Keyboard Dials only, set up 5.0625 in Keyboard Dial, and multiply by 5.0625.

(3) Clear Upper and Keyboard Dials only, shift to 7th position, set up nearest number to 181.3... from Col. A of Table 56 (181), and add.

(4) Shift to 8th position, set up in Keyboard Dial the proper Square Root Divisor (2690725) so that left-hand figure of divisor is directly below left-hand figure of Middle Dial amount, and divide.

Distance "X" (13.4677...) appears in Upper Dial. Copy it as 13.468", as it is correct to five figures.

It is noted that this problem is the reverse of that of Case I.

MARCHANT — METHODS

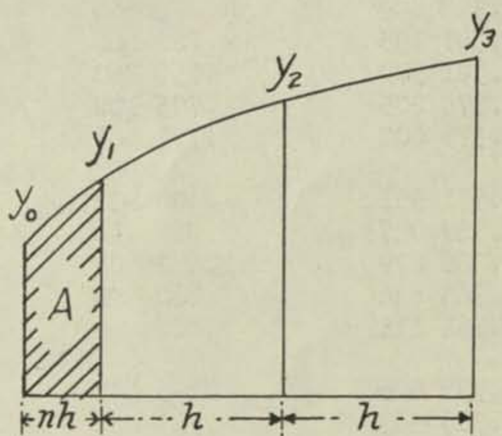
WATKINS METHOD FOR AREA BELOW CURVE WHEN END SECTION HAS DIFFERENT SPACING FROM THAT OF ADJACENT SECTIONS

REMARKS: *Marchant Method MM-215 provides means of solving certain problems involving fractionally-spaced ordinates. However, it does not care for cases in which there is a single area of unequal spacing adjacent to several areas having equal spacing.*

William H. Watkins, Senior Naval Architect and Supervisor of the Scientific and Test Groups, Design Section, Puget Sound Navy Yard, has developed a most satisfactory method of solving such problems by a formula derived from integration of the LaGrange Interpolation Formula of 3rd Degree. The method also has the advantage that the area with unequal spacing may have its limiting ordinates spaced either greater or less than that of adjacent sections having equally-spaced ordinates; furthermore, only two such sections bounded by equally-spaced ordinates are required.

The method assumes that a third-degree curve is passed through the four points; consequently it does not apply if the actual curve has abrupt changes of curvature or points of inflexion.

EXAMPLE:



GIVEN: $y_0 = 4.0$, $y_1 = 5.6$, $y_2 = 7.8$, $y_3 = 9.0$, $h = 5.0$, $n = 0.4$

FIND: Area of shaded part A.

OUTLINE: The formula applying is

$$A = n\ h\ (K_0 y_0 + K_1 y_1 - K_2 y_2 + K_3 y_3)$$

in which the values of K_0, K_1, etc., are obtained from table on reverse side hereof, or from curve on Page 3.

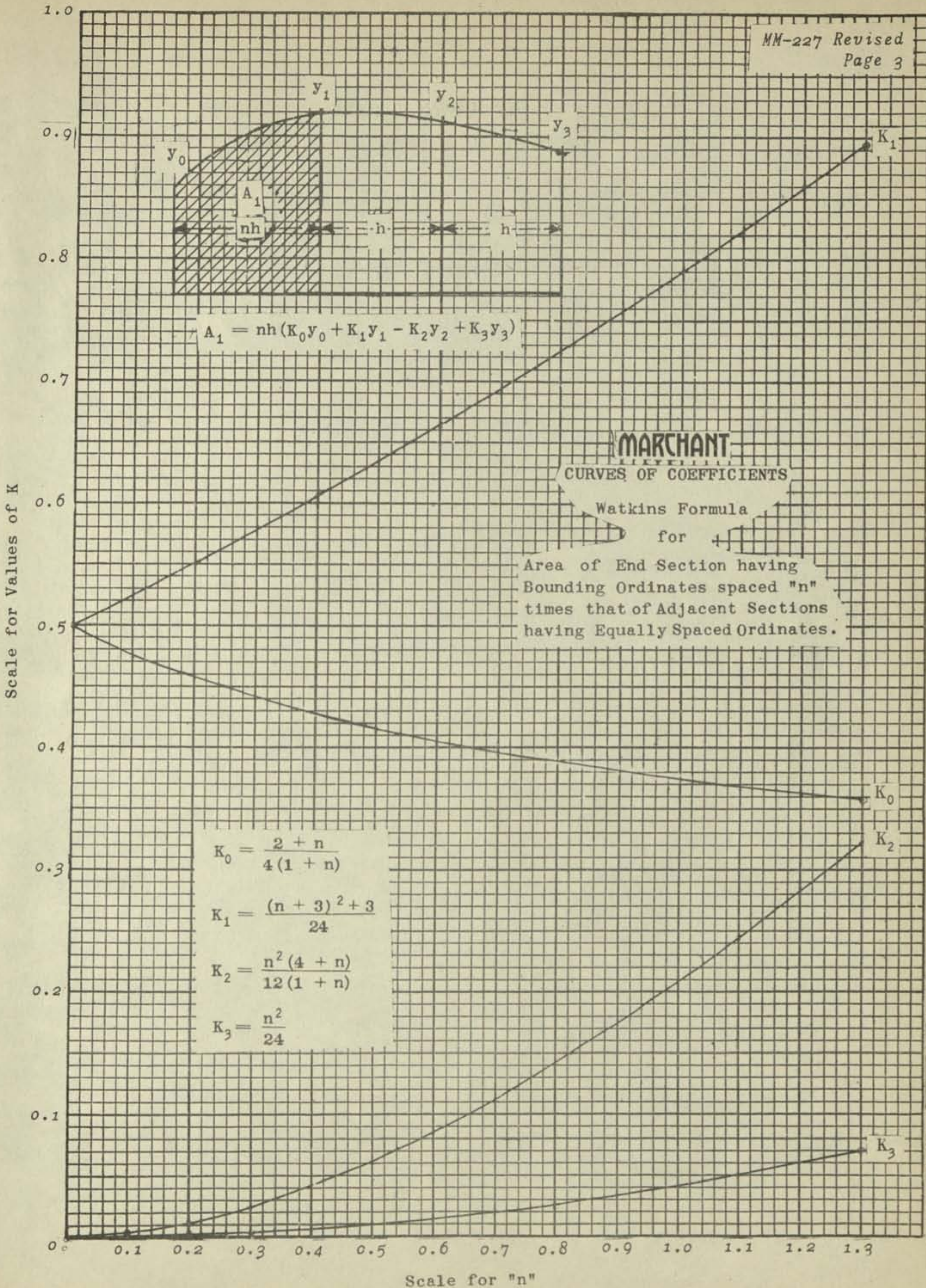
OPERATIONS: By using direct and reverse accumulative multiplication as described in Marchant Method MM-215, solution of the above formula is a continuous calculator process, requiring no copying of intermediate amounts to work sheet. In the above case, the formula with substituted values is

$$A = 0.4 \cdot 5.0\,(0.4286 \cdot 4.0 + 0.6067 \cdot 5.6 - 0.0419 \cdot 7.8 + 0.0067 \cdot 9.0) = 9.691$$

(over)

COEFFICIENTS FOR WATKINS METHOD FOR AREA BELOW CURVE WHEN END SECTION HAS DIFFERENT SPACING FROM THAT OF ADJACENT SECTIONS

n	K_0	K_1	K_2	K_3
0	.500 000	.500 000	0	0
0.05	.488 095	.512 604	-.000 803	.000 104
.10	.477 273	.525 416	.003 106	.000 417
.15	.467 391	.538 437	.006 766	.000 938
.20	.458 333	.551 667	.011 667	.001 667
.25	.450 000	.565 104	.017 708	.002 604
0.30	.442 308	.578 750	-.024 808	.003 750
.35	.435 185	.592 604	.032 893	.005 104
.40	.428 571	.606 667	.041 905	.006 667
.45	.422 413	.620 938	.051 789	.008 438
.50	.416 667	.635 416	.062 500	.010 417
0.55	.411 290	.650 104	-.073 998	.012 604
.60	.406 250	.665 000	.086 250	.015 000
.65	.401 515	.680 104	.099 223	.017 604
.70	.397 058	.695 417	.112 892	.020 417
.75	.392 857	.710 937	.127 232	.023 438
0.80	.388 889	.726 666	-.142 222	.026 667
.85	.385 135	.742 604	.157 843	.030 104
.90	.381 579	.758 750	.174 079	.033 750
.95	.378 205	.775 104	.190 913	.037 604
1.00	.375 000	.791 667	.208 333	.041 666
1.05	.371 951	.808 437	-.226 326	.045 938
.10	.369 047	.825 416	.244 880	.050 417
.15	.366 279	.842 604	.263 987	.055 104
.20	.363 636	.860 000	.283 636	.060 000
.25	.361 111	.877 604	.303 819	.065 104
1.30	.358 696	.895 417	-.324 529	.070 416
.35	.356 383	.913 437	.345 757	.075 937
.40	.354 167	.931 667	.367 500	.081 666
.45	.352 041	.950 104	.389 749	.087 604
.50	.350 000	.968 750	.412 500	.093 750
1.55	.348 039	.987 604	-.435 747	.100 104
.60	.346 154	1.006 667	.459 487	.106 666
.65	.344 340	1.025 937	.483 715	.113 438
.70	.342 593	1.045 416	.508 425	.120 416
.75	.340 909	1.065 104	.533 617	.127 604
1.80	.339 285	1.085 000	-.559 285	.135 000
.85	.337 719	1.105 104	.585 427	.142 604
.90	.336 207	1.125 416	.612 040	.150 417
.95	.334 746	1.145 937	.639 120	.158 437
2.00	.333 333	1.166 667	.666 666	.166 666


y_0
y_1
y_2
y_3
A_1
nh
h
h
$A_1 = nh(K_0y_0 + K_1y_1 - K_2y_2 + K_3y_3)$
MARCHANT
CURVES OF COEFFICIENTS
Watkins Formula
for
Area of End Section having Bounding Ordinates spaced "n" times that of Adjacent Sections having Equally Spaced Ordinates.
$K_0 = \frac{2 + n}{4(1 + n)}$
$K_1 = \frac{(n + 3)^2 + 3}{24}$
$K_2 = \frac{n^2(4 + n)}{12(1 + n)}$
$K_3 = \frac{n^2}{24}$
K_0
K_1
K_2
K_3
Scale for Values of K
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
Scale for "n"

SUMMATION OF X, XY AND XY^2

REMARKS: *In certain cases of statistical computing, it is often desired to find $\sum x$*, $\sum xy$ and $\sum xy^2$, when given a large number of pairs of factors "x" and "y". This method is suitable for cases in which the factors have no more than two digits each.*

EXAMPLE:

Given: x	y
2	7
3	12
31	43
14	17
5	22

Find:

$\sum x = 55$

$\sum xy = 1731$

$\sum xy^2 = 64315$

OPERATIONS: Decimals; Upper Dial 5 & 0, Middle Dial 10, 5 & 0, Keyboard Dial 5 & 0. Use Model ACT-10 M.

(1) Set up first "y" (7) at 5th Keyboard Dial decimal and multiply at right-hand Upper Dial decimal by first "x" (2).

(2) Decrease right-hand figure of "y" by "1" and set up all 9's in columns at right thereof (Keyboard Dial then reads 6.99999), and multiply at 5th Upper Dial decimal by "xy" (14) which appears directly below at 5th Middle Dial decimal. The Middle Dial should now show all ciphers at right of 10th decimal.

xy^2 (98) appears at 10th Middle Dial decimal and xy (14) at 5th Upper Dial decimal, but the amounts need not be separately noted.

(3) Clear Keyboard Dial only and proceed as in Steps 1 and 2 for the remaining pairs of values.

$\sum xy^2$ appears at left of Middle Dial.
$\sum xy$ " " " " Upper Dial.
$\sum x$ " " right of Upper Dial.

NOTE: If $\sum x$ at right of Upper Dial equals the sum of the x values when they are separately added, it is substantially a proof that all x values have been correctly entered as multipliers. Any error of entry of xy as multiplier in Step 2, or the improper filling-in of 9's, is signalized by Middle Dial failing to clear after Step 2.

It will thus be seen that this process provides first-run accuracy control, except for values of "y". However, the likelihood of an error in setting "y" is remote because it appears in the Keyboard Dial and it is also separately noted in that dial when its right-hand figure is reduced by 1.

(*) The symbol $\sum$ indicates "the summation of."

MARCHANT — SILENT SPEED — METHODS

HANSEN-AHLBERG METHOD FOR OBTAINING PARABOLIC TRENDS

Remarks: Statisticians who extend second-degree curves may readily do so by the procedure herein. This application to cases of statistical trends was brought to our attention by Mr. Raymond Ahlberg, Statistician, Denver, Colo. Similar procedures have been used for interpolation by integration of constant differences (see Marchant Method MM-152).

Outline: The second degree curve is characterized by having a constant second difference. Advantage is taken of this as the basis for the method. The curves may have any of several forms. Examples are

(1) $Y = a + bX + cX^2$ (2) $\text{Log } Y = a + bX + cX^2$

(3) $Y = a + b/X + c/X^2$ (4) $Y = 1/(a + bX + cX^2)$

The example herein is in the form of (1). If (2) applies, it is only necessary to obtain anti-logs of the log Y's that appear in the Marchant. If (3) applies, it is put in the form $X^2Y = aX^2 + bX + c$. The Marchant then gives the values of X^2Y, which when divided by the X^2's gives Y. If (4) applies, it is put in the form $1/Y = a + bX + cX^2$. The Marchant then gives values of $1/Y$, the reciprocals of which are the desired Y's.

Example: Given $Y = 7.2131 - .5114X + .3044X^2$, obtained from a least squares analysis. It is desired to tabulate the trend by intervals of 0.1 from $X = 2.0$

Preliminary: Compute four adjacent values of Y in the neighborhood of $X = 2.0$ and tabulate with differences, a check of correctness being the constancy of the second difference; thus,

X	Y	1st diff.	2nd diff.
1.8	7.278836		
		.061488	
1.9	7.340324		.006088
		.067576	
2.0	7.407900		.006088
		.073664	
2.1	7.481564		

Decimals: Upper Dial 1, Middle Dial 10 & 6, Keyboard Dial 10 & 6. Non-Shift Key down on any 10-column M model.

(1) By suitable means, obtain initial entries as follows: Upper Dial 2.0; Middle Dial 7.408 at 10th decimal and .073664 at 6th decimal; Keyboard Dial .074 (rounded .073664) at 10th decimal and constant 2nd difference .006088 at 6th decimal. These are starting values and are always set up in this pattern when signs of both differences are plus (see Note B).

(2) With carriage in 1st position, depress No. 1 Key of Single Row Keyboard. Y for 2.1 (7.482) appears at left of Middle Dial and the new 1st difference (.079752) appears at right. The Keyboard Dial at 10th Decimal is then changed so it reads .080 (the rounded value of .079752). See Note A.

(3) Depress No. 1 key of Single Row Keyboard. Y for 2.2 (7.562) appears at left of Middle Dial and the new 1st difference (.085840) appears at right. The Keyboard Dial at 10th Decimal is then changed so it reads .086 (the rounded value of .085840).

(4) Repeat Step 3 for succeeding values, the Upper Dial showing values of X.

NOTE A: Constant 2nd diff. should be set up as nearly exact as possible. Rounding that space limitations require should be in 1st diffs. and Y's.

NOTE B: If Y's increase but 2nd diff. is negative, set it in complementary form, bridge with 9's and proceed as herein.

If Y's decrease but 2nd diff. is positive, have Manual Counter Control toward operator, and depress Reverse Bar prior to depressing No. 1 key.

If Y's decrease and 2nd diff. is negative, invert the table; i.e., start from the smallest Y. Then, proceed exactly as outlined in the above method except have Manual Counter Control toward the operator.

Submitted by Garland McWhirter
Kansas City, Mo.

Reprinted from MATH-MECHANICS, February 1943

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MARCHANT METHODS

MILNE METHOD OF STEP-BY-STEP INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS WHEN STARTING VALUES ARE KNOWN

REMARKS: *The Milne Method is highly regarded because it uses tabular values instead of differences and because its associated Steffensen integration formulas have small repeated coefficients which readily lend themselves to the preparation of tables of factors for use in connection with any particular problem. The method also provides means for estimating the error, provided the order of differences that tend to disappear is known. The example computed herein is the same as the differential equation that was chosen for comparing processes by the Committee on Numerical Integration, National Research Council (Bulletin No. 92). The example has substantially large higher orders of difference so that use of one of the intermediate forms of the Milne Method is required. The simplest exemplification of the method, which is suitable for use when 4th differences tend to disappear, is described in the appended Explanatory Notes, which also discuss other pertinent matters.*

Though this computation appears formidable in review, it is actually extremely simple to apply. The schematic diagrams and work sheets have been developed for the purpose of reducing the computation to simple systematic procedure.

EXAMPLE: Integrate $dy/dx = -xy$ from $x = 0.5$ to $x = 1.0$, with initial value $y = 1$ when $x = 0$, and with starting values as follows:

x		y		$dy/dx = u = -xy$
0	y_{-5}	1.000 000 00	u_{-5}	-0.000 000 00
0.1	y_{-4}	0.995 012 48	u_{-4}	-0.099 501 25
0.2	y_{-3}	0.980 198 67	u_{-3}	-0.196 039 73
0.3	y_{-2}	0.955 997 48	u_{-2}	-0.286 799 24
0.4	y_{-1}	0.923 116 35	u_{-1}	-0.369 246 54
0.5	y_{0}	0.882 496 90	u_{0}	-0.441 248 45

The method of obtaining the above "starting values" or determining how many starting values are needed in any case is beyond the scope of this method. See Marchant Methods MM-260 and 261.

The computing plan for this example comprises the use of the 5-term "open-type" formula for integrating ahead and the 5-term "closed-type" formula for back checks, with a final refinement of the entire group of five values by use of the 9-term "closed-type" formula.

OPERATIONS: Decimals; Upper Dial 8, Middle Dial 16, Keyboard Dial 8. Use any 10-column "M" model with Upper Green Shift Key down.

(1) Compute the factors that are in the three right rows of the Upper and Lower arrays of Page 5 for values from and including $x = 0$ to $x = 0.5$, setting up each value of "u" from the example and multiplying it successively by 11, 14, 26, 7, 32, and 12 insofar as the arrays show that it is necessary to use the factors; for example, it is not necessary to multiply u_{-4} (0.099 501 25) by any multiplier except "11" (for the single entry in the upper array).

(over)

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COMPUTING TRIAL VALUES OF "y" AND "u"

(2) The first step in obtaining the trial value of y for x = 0.6 (y_{+1}) is to integrate u from u_{-5} to u_{+1} by using its values from u_{-4} to u_0 inclusive (not using the end values of u_{-5} and u_{+1}). This is done by summing the factors of the upper array diagonally, as indicated by the line with the arrows; thus, the sum of these "u" functions, as per formula at upper left of section "0.6" Work Sheet (Page 6) is 1.094 513 75 - 2.744 556 22 + 7.456 780 24 - 5.169 451 56 + 4.853 732 95 = 5.491 019 16, which appears in Middle Dial. It is negative as all of the u's are negative.

(3) Transfer the Middle Dial amount to Keyboard Dial, clear Middle Dial and multiply by Length Factor (.03).

Increment in y from x = 0 to x = 0.6 (0.164 730 57), which is also negative, appears in Middle Dial.

(4) Transfer the Middle Dial amount (0.164 730 57) to Keyboard Dial, clear Middle Dial, and subtract. Set up y_{-5} (1.000 000 00) and add.

Trial Value y_1 (0.835 269 43) appears in Middle Dial. Copy to Work Sheet (Page 6).

(5) Next, proceed with the calculation of dy/dx when x = 0.6 from its equation, both the y and x being known. In this case, the value of dy/dx = u is obtained by multiplying the known "y" by the known "x" (-0.6); thus, transfer Middle Dial amount (0.835 269 43) from Middle Dial to Keyboard Dial, clear Upper Dial and multiply by "0.6".

Trial Value u_1 (0.501 161 66) appears in Middle Dial. Copy to Work Sheet (Page 6).

(6) Transfer the Middle Dial amount (0.501 161 66) to Keyboard Dial and multiply by 11.

Copy Trial Value u_{-1} (0.501 161 66) to Work Sheet (Page 6) and to the Lower Array of Factor Sheet (Page 5).
Copy Middle Dial amount (3.508 131 62) to Lower Array of Factor Sheet.

COMPUTING CHECK VALUE OF "y" AND "u"

(7) The first step in obtaining the Check Value of y_{+1} is to integrate u from u_{-3} to u_{+1}, using these values as well as those that are in-between. This is done by summing the factors of the Lower Array diagonally, as indicated by the line with the arrows; thus, the sum of these "u" functions as per formula at lower left of section "0.6" of Work Sheet (Page 6) is 1.372 278 11 + 9.177 575 68 + 4.430 958 48 + 14.119 950 40 + 3.508 131 62 = 32.608 894 29, which is negative.

(8) Move Upper Dial decimal from 8 to 9 and Keyboard Dial decimal from 8 to 7, set up reciprocal of the common multiplier (225) and divide.

Increment in y from x = 0.3 to x = 0.6 inclusive (0.144 928 842) appears in Upper Dial, which enter in Work Sheet (Page 6) as negative.

(9) Move Middle Dial decimal to 17 and Keyboard Dial decimal to 8, set up y_{-3} (0.980 198 67) in Keyboard Dial, depress Add Bar, and then depress Subtract Bar.

(10) Set up "1" in 9th column of Keyboard Dial and reverse multiply by Upper Dial amount, except use rounded figure of "2" in 2nd dial instead of "19" as appears in 2nd and 1st dials.

Check value of y_{+1} (0.835 270 25) appears in Middle Dial. Upper Dial shows all ciphers, or all 9's, in every dial, except 1st dial will show effect of rounding. Copy to Work Sheet.

CORRECTING THE CHECK VALUE

(11) The Check Value is usually more nearly correct than the Trial Value, because it is obtained as an integration of only four "sections" (0.3 to 0.6), using five "ordinates." The difference δy between the Trial and Check Values is obtained and given such sign that when it is added to the Check Value their sum equals the Trial Value (see Explanatory Notes); thus,

$$0.835\ 270\ 25 - 0.835\ 269\ 43 = 0.000\ 000\ 82 \text{ recorded as negative}$$

(12) For the conditions of this example (see Explanatory Notes) the Check Value should be corrected by $\delta y/35 = -0.000\ 000\ 02$, reducing y_{+1} to 0.835 270 23.

(13) Substitution in the formula for dy/dx is then made (in this case multiplying 0.835 270 23 X - 0.6, producing the Check Value of u_{+1} - 0.501 162 14, which is entered in the Upper Array of the Factor Sheet. The value previously entered in the Lower Array is corrected, as shown, and again multiplied by 11, 14, and 26, to produce the corrected factor in the "11" column and also the values in the other columns. The Lower Array is, likewise, completed by multiplying by 7, 32, and 12.

(14) The above cycle from Steps 2 to 13 is repeated for the remaining values, dropping off values "at the top" as new ones are obtained at the bottom, all as per Work Sheet (Page 6).

In preparing this Work Sheet, it is found convenient to reproduce the formulas used, as shown, and to place the symbol number that identifies the term in the Factor Arrays directly below each term. This provides a "pattern" of calculating that serves to clarify and expedite the process.

The checked y's are tabulated below:

x	y
0.6	0.835 270 23
0.7	0.782 704 55
0.8	0.726 149 06
0.9	0.666 976 83
1.0	0.606 530 69

FINAL REFINEMENT OF VALUES

By the preceding process, there has been obtained the first group of five values beyond the starting values. Before calling the work complete, however, we may make an overall check by using the 9-term closed formula for integrating the differential "u" in the eight sections from x = 0.2 to x = 1.0 inclusive, thus

(over)

x	u	Multiplier
0.2	-0.196 039 73	989
0.3	0.286 799 24	5888
0.4	0.369 246 54	- 928
0.5	0.441 248 45	10496
0.6	0.501 162 14	-4540
0.7	0.547 893 19	10496
0.8	0.580 919 25	- 928
0.9	0.600 279 15	5888
1.0	0.606 530 69	989

$= -13241.86039 \times \frac{0.8}{28350}$

$= -0.373\ 668\ 02$
which added to y for x = 0.2 produces new y for x = 1.0 of 0.606 530 65.

This is .000 000 04 less than the previously found value, the amount being the accumulation of integrated differences that occurred during the calculation of these values. The difference of "4" should be distributed among these 5 values of y, as below, and new "u" calculated.

		Distributed	New "y"	New "u"
0.6	0.835 270 23	- .8	0.835 270 22	-0.501 162 13
.7	.782 704 55	-1.6	.782 704 53	-0.547 893 17
.8	.726 149 06	-2.4	.726 149 04	-0.580 919 23
.9	.666 976 83	-3.2	.666 976 80	-0.600 279 12
1.0	.606 530 69	-4.0	.606 530 65	-0.606 530 65

These new values of "u" should be substituted in the 9-term formula, above, but to avoid repetition, only the effect of the difference in u's, as found above, and those shown on the upper part of Page 5, is calculated, thus obtaining a final refinement of y_{+5}

x	Difference in "u"	
0.6	+.000 000 01	X -4540
.7	2	10496
.8	2	-928
.9	4	5888
1.0	4	989

$= .000\ 362\ 16$

$\times \frac{0.8}{28350} = +.000\ 000\ 01$

This increase in "y_{+5}" is, likewise, an accumulation of integrated differences throughout 5 terms, so it is distributed as follows:

x	Correction of y (rounded)	Final "y"
0.6	.000 000 00	0.835 270 22
0.7	0	.782 704 53
0.8	1	.726 149 05
0.9	1	.666 976 81
1.0	.000 000 01	.606 530 66

If additional values beyond x = 1.0 were to be obtained, a new column of values of u corresponding to the above values of y would then be obtained in the customary way.

FACTOR SHEET MILNE-STEFFENSEN 5 POINT (6 ORDINATE) FORMULA

FOR INTEGRATION BY STEP-BY-STEP METHOD

ORIGINAL CALCULATION: h = 0.1 Length Factor: 3 h/10 = .03

x	u_n	u = -xy	-- Column Multipliers -- 11	-14	26
0	u_{-5}	0.000 000 00			
0.1	u_{-4}	-0.099 501 25	1.094 513 75		
0.2	u_{-3}	-0.196 039 73	2.156 437 03	2.744 556 22	
0.3	u_{-2}	-0.286 799 24	3.154 791 64	4.015 189 36	7.456 780 24
0.4	u_{-1}	-0.369 246 54	4.061 711 94	5.169 451 56	9.600 410 04
0.5	u_0	-0.441 248 45	4.853 732 95	6.177 478 30	11.472 459 70
0.6	u_{+1}	-0.501 162 14	5.512 783 54	7.016 269 96	13.030 215 64
0.7	u_{+2}	-0.547 893 19	6.026 825 09	7.670 504 66	14.245 222 94
0.8	u_{+3}	-0.580 919 25	6.390 111 75	8.132 869 50	
0.9	u_{+4}	-0.600 279 15	6.603 070 65		
1.0	u_{+5}	-0.606 530 69			

BACK-CHECK CALCULATION: h = 0.1 Length Factor: 2 h/45 = 1/225

x	u	u = -xy	-- Column Multipliers -- 7	32	12
0.2	u_{-3}	-0.196 039 73	1.372 278 11		
0.3	u_{-2}	-0.286 799 24	2.007 594 68	9.177 575 68	
0.4	u_{-1}	-0.369 246 54	2.584 725 78	11.815 889 28	4.430 958 48
0.5	u_0	-0.441 248 45	3.088 739 15	14.119 950 40	5.294 981 40
0.6	u_{+1}	2 14 -0.501 16~~1 66~~	4 98 3.508 13~~1 62~~	16.037 188 48	6.013 945 68
0.7	u_{+2}	3 19 -0.547 89~~2 51~~	52 33 3.835 24~~7 57~~	17 532 582 08	6.574 718 28
0.8	u_{+3}	9 25 -0.580 91~~8 42~~	34 75 4.066 42~~8 94~~	18.589 416 00	6.971 031 00
0.9	u_{+4}	9.15 -0.600 27~~8 21~~	54 05 4.201 94~~7 47~~	19.208 932 80	
1.0	u_{+5}	30 69 -0.606 52~~9 74~~	4.245 708 18		

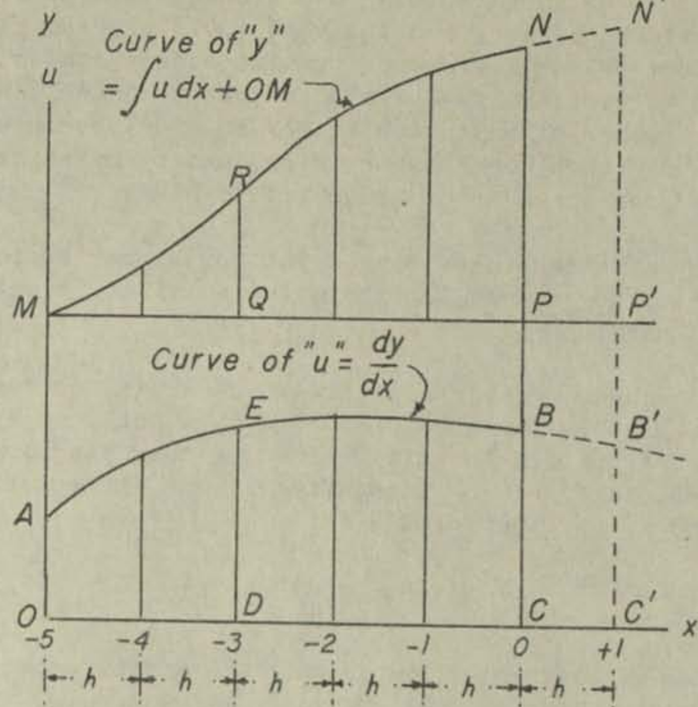
(over)

WORK SHEET MILNE-STEFFENSEN 5 POINT (6 ORDINATE) FORMULA
FOR INTEGRATION BY STEP-BY-STEP METHOD

	FORMULA		TRIAL		CHECK	δy	E
		y_{-5}	1.000 000 00	y_{-3}	0.980 198 67		(y/35)
x = 0.6	$-0.03(11u_{-4} - 14u_{-3} + 26u_{-2} - 14u_{-1} + 11u_{0})$		-0.164 730 57		-0.144 928 42		
		y_{+1}	0.835 269 43		0.835 270 25	-82	-2
			-0.6		-2		
	$-1/225(7u_{-3} + 32u_{-2} + 12u_{-1} + 32u_{0} + 7u_{+1})$	u_{+1}	-0.501 161 66	y_{+1}	0.835 270 23		
					-0.6		
				u_{+1}	-0.501 162 14		
		y_{-4}	0.995 012 48	y_{-2}	0.955 997 48		
x = 0.7	$-0.03(11u_{-3} - 14u_{-2} + 26u_{-1} - 14u_{0} + 11u_{+1})$		-0.212 308 89		-0.173 292 90		
		y_{+2}	0.782 703 59		0.782 704 58	-99	-3
			-0.7		-3		
	$-1/225(7u_{-2} + 32u_{-1} + 12u_{0} + 32u_{+1} + 7u_{+2})$	u_{+2}	-0.547 892 51	y_{+2}	0.782 704 55		
					-0.7		
				u_{+2}	-0.547 893 19		
		y_{-3}	0.980 198 67	y_{-1}	0.923 116 35		
x = 0.8	$-0.03(11u_{-2} - 14u_{-1} + 26u_{0} - 14u_{+1} + 11u_{+2})$		-0.254 050 65		-0.196 967 26		
		y_{+3}	0.726 148 02		0.726 149 09	-107	-3
			-0.8		-3		
	$-1/225(7u_{-1} + 32u_{0} + 12u_{+1} + 32u_{+2} + 7u_{+3})$	u_{+3}	-0.580 918 42	y_{+3}	0.726 149 06		
					-0.8		
				u_{+3}	-0.580 919 25		
		y_{-2}	0.955 997 48	y_{0}	0.882 496 90		
x = 0.9	$-0.03(11u_{-1} - 14u_{0} + 26u_{+1} - 14u_{+2} + 11u_{+3})$		-0.289 021 69		-0.215 520 04		
		y_{+4}	0.666 975 79		0.666 976 86	-107	-3
			-0.9		-3		
	$-1/225(7u_{0} + 32u_{+1} + 12u_{+2} + 32u_{+3} + 7u_{+4})$	u_{+4}	-0.600 278 21	y_{+4}	0.666 976 83		
					-0.9		
				u_{+4}	-0.600 279 15		
		y_{-1}	0.923 116 35	y_{+1}	0.835 270 23		
x = 1.0	$-0.03(11u_{0} - 14u_{+1} + 26u_{+2} - 14u_{+3} + 11u_{+4})$		-0.316 586 61		-0.228 739 51		
		y_{+5}	0.606 529 74		0.606 530 72	-98	-3
			-1.0		-3		
	$-1/225(7u_{+1} + 32u_{+2} + 12u_{+3} + 32u_{+4} + 7u_{+5})$	u_{+5}	-0.606 529 74	y_{+5}	0.606 530 69		
					-1.0		
				u_{+5}	-0.606 530 69		

EXPLANATORY NOTES

The diagram relates to a typical function, not to the one used in the example.



PRINCIPLE OF STEP-BY-STEP SOLUTION OF DIFFERENTIAL EQUATIONS: Let it be assumed that in the case of any given ordinary differential equation of first order $dy/dx = u = f(x,y)$, the values of y and u are known for several "starting" values of x at equally spaced intervals (x_{-5}, x_{-4} ..... x_0). Assume that these values are plotted as the solid curved lines of the diagram in which AEB represents the known values of dy/dx, and MN represents the corresponding known values of y when the initial value is OM. From the relationship shown, it is apparent that the area OABCO is a measure of the increment in y (PN) above the initial value MO, or y = NC; that is to say, any point on the curve MN corresponding to any value of x may be obtained by measuring the area under the differential curve up to that value of x and adding the result so obtained to the initial value of y, OM.

If, now, we have means of determining the area OAB'C'O, when given only the solid-line curve AEB, it is apparent that the area so obtained is likewise a measure of the increment in y, P'N', which when added to the initial value OM would give a new y corresponding to a point x_{+1}. By substituting the new y, so found, in the given differential equation, x of course being known, the new value of dy/dx = u, or the point B' becomes known.

The cycle is now complete, and by like means we can compute other values at the right of NN' and BB', thus continuing the solution to any desired point.

The crux of the solution rests, of course, in the determination of the area OAB'C'O when given only the values of u from A to B. The process is essentially one of extrapolation, and various ways exist for performing this "extension." It is obvious that errors made in computing the first extension steps are cumulative, just as if one were constructing a cantelever bridge, so any process, to be successful, must have means of reducing these errors to a minimum. Also, if possible, it must provide over-all checks to correct a series of values so that the completed structure will have dy/dx = u for every value of x and y.

TRIAL COMPUTATION OF THE MILNE METHOD: The area OAB'C'O is obtained by using only the ordinates of the curve AB at points such as x_{-4}, x_{-3}, x_{-2}, x_{-1} and x_0, for example, which values are known as they comprise all but one of the assumed starting values. If these values

(over)

of u are differenced (see Marchant Method MM-100) and it is found that fourth differences are negligible, then it would not be necessary to divide the length AB into five sections (six ordinates). It would be satisfactory to divide it into three sections (four ordinates); the sections would be larger, and the work would proceed that much faster. If it is desired to take account of 7th differences of u (corresponding to 8th differences of y), the curve AB should be sectionalized still further -- into 7 sections (8 ordinates), etc. In any case, the number of ordinates should be even, because the accuracy of the computation based upon any such even number of ordinates is substantially equal to the case if the next higher uneven number of ordinates is used.

Consideration of the above matters provides a hint for proper choice of interval (h) and of the number of ordinates to be used for the solution of any equation. The subject will not be further explored here.

In the example of this method, six starting values are known, corresponding to the solid-line ordinates of both curves. The area OAB'C'O is then obtained by applying the proper Steffenson "open-type" integration formula. In the case diagrammed, the area OAB'C'O, spanning seven ordinates, is found by using five of the known ordinates, excluding the end ordinates for x_1 and x_{-5}. The formula is:

$$(1)\quad \text{Area OAB'C'} = \frac{\text{length OC'}}{20}\,(11u_{-4} - 14u_{-3} + 26u_{-2} - 14u_{-1} + 11u_0)$$

which will be recognized as that which was used in the example, because length OC' equals 6h and 6 x 0.1/20 = 0.03.

The general Steffensen formulas for various numbers of terms are given below:

No. of Terms	Coefficients to Central Value					Divisor	Remainder
3	2	-1				3	0.000 31 $F^{(4)}$ (Xi)
5	11	-14	26			20	0.000 001 1 $F^{(6)}$ (Xi)
7	460	-954	2196	-2459		945	0.000 000 002 1 $F^{(8)}$ (Xi)
9	4045	-11690	33340	-55070	67822	9072	0.000 000 000 002 7 $F^{(10)}$ (Xi)

The use of the above coefficients and divisors will be apparent by analogy from what has been previously described. The remainder of the 3-term formula contains the expression $F^{(4)}$ (Xi), which designates the 4th derivative of u (or the 5th derivative of y) with respect to x for some value of x (unknown) that is found within the entire length OC'. This amount makes it possible to determine the maximum error if we know the maximum 4th derivative. The actual error, however, may be less than this, and usually is. The computer rarely needs to pay attention to this matter because the Milne Method offers a simplification of this point, as will be later explained.

The Milne computation of the error is also to be preferred because the above expressions for the Remainder apply to the case when the length OC' is taken as "1". If it has any other value, the above Remainder Coefficients must be multiplied by the length raised to the r th power, in which "r" equals "no. of terms of formula used, plus 2"; i.e., if the length OC' is 6 units long and the 5-term formula is used, the Remainder Coefficient (0.000 001 1) is to be multiplied by 6^7. It therefore equals 0.31.

We have now obtained the area OAB'C'O which, as stated, is a measure of the increment in y. This increment, P'N', added to the initial value OM, gives the new value y_1, corresponding to x_1. It then only remains to substitute the now known y_1 and the known x_1 in the given differential equation and thus obtain u_1, or the point B'.

The cycle is now complete, and if we are satisfied with the accuracy so far obtained, we could proceed to the next step by finding the area under the differential curve from x_{-4} to x_2 and adding the increment in y so found to the value of y_{-4}, thus producing the next value of y_2.

THE MILNE CHECK-BACK: The first refinement of the value of u and $\dot{y}$ for x_1 is obtained by re-calculating y_1 in a different manner from that previously employed. This second determination of y_1 is obtained by finding the area DEB'C'D of the differential curve between and including the values x_{-3} and x_1 and adding the increment so obtained to y_{-3}. This could be done by Simpson's Rule, if desired, but such a formula would not suffice for this example because it would not take into account 5th differences of u. The formula that was used in the example takes into account such 5th differences; thus,

$$\text{(2)}\quad \text{Area DEB'C'D} = \frac{\text{length DC'}}{90}\left(7u_{-3} + 32u_{-2} + 12u_{-1} + 32u_0 + 7u_{+1}\right)$$

Inasmuch as the length DC' in the example is 4h, the multiplier is 4 X 0.1/90 = 1/225.

The general Newton-Cotes formulas for various numbers of terms are given below:

No. of Terms	Coefficients to Central Value					Divisor	Remainder
3	1	4				6	$-0.000\ 35\ F^{(4)}$ (X1)
5	7	32	12			90	$-0.000\ 000\ 52\ F^{(6)}$ (X1)
7	41	216	27	272		840	$-0.000\ 000\ 000\ 64\ F^{(8)}$ (X1)
9	989	5888	-928	10496	-4540	28350	$-0.000\ 000\ 000\ 000\ 59\ F^{(10)}$ (X1)

The use of the above coefficients, divisors and remainders will be apparent by analogy from what has been previously described. However, the Remainder of the above Check formula, when compared with that of the Trial formula, if used in the Milne Method, is smaller than a comparison of coefficients indicates. This is because the Check formula integrates an area with a shorter base line, it being 1/2, 2/3, 3/4, etc., of that of the Trial formula for 3, 5, and 7 terms respectively; and higher powers of this ratio of intervals are involved.

Because the above formula uses the outside ordinates which close the area, it is designated a "Closed Type" formula. The previously described formula that does not use either of the end ordinates is similarly designated an "Open Type" formula.

The formula for 3 terms will be recognized as Simpson's 1/3 Rule.

THE MILNE ERROR CHECK: Inspection of the Remainders, as tabulated for the above Open and Closed-Type formulas, shows that in the case of, say, the 5-term formulas if the seventh derivative of u vanishes; that is to say, the error is only that due to the sixth derivative, and if we also assume that this sixth derivative is positive in the case of each formula, then it is evident that the integral calculated from the open-type formula will be less than its true value because the Remainder is positive. By similar reasoning, it is seen that the closed-type formula produces a value that is greater than its true value. The true value then lies between the values obtained by the two formulas.

Inasmuch as in most cases the differences of any order and the derivatives of the same order are closely proportionate, we may likewise conclude, subject to remotely unusual exceptions, that if the seventh <u>differences</u> of u vanish, the true value of the integral will, likewise, lie between its values when computed by the closed and open-type formulas (of course, the sixth differences that control this error are adjacent differences, rather than identical, because the two 5-term formulas do not use all of the identical five terms, but if these adjacent differences should be of opposite signs we would know that they would be substantially non-existent because of the assumption that seventh differences tend to vanish).

(over)

The above reasoning has been applied to the case where the difference that controls the error (in this case, the 6th) is positive. By similar reasoning, it will be seen that the true value of the integral lies between its "open type" and "closed type" values when the difference that controls the error is negative, and also that this general conclusion applies to any of the formulas, provided that the $n + 2$ difference of any n term formula is assumed to vanish.

By reference to the original expansion from which the remainders were computed, Milne develops the following conclusion for the amount of the error, the sign of which becomes known when it is remembered that the error must be such as to modify the value of the integral when obtained by the closed-type formula so the true value comes between it and the value when obtained by the open-type formula.

> Under the assumptions given, the true value differs from the value as obtained by the closed-type formula by the following fraction of the difference between the values obtained by the open-type and closed-type formulas: 3-term, 1/29; 5-term, 1/35; 7-term, 1/44; 9-term, 1/54. The sign of this difference is such as to make the true value lie between the closed-type and open-type values.

In applying the above principle, it is to be remembered that the correction is to be applied to the value obtained by the closed-term formula.

Sometimes examples are found in texts in which the true value does not lie between the open-term and closed-term values, but if these be examined, it will be found that higher orders of differences exist than permitted by the above assumption. Inasmuch as the choice of the number of starting terms of the solution and of the interval are usually such that differences of u substantially vanish as outlined above, it is seen that the Milne Error Check is as sound as it is easy to apply.

APPLICATION TO DIFFERENTIAL EQUATIONS OF HIGHER ORDER, ETC: Systems of differential equations of the first order may also be solved by this method. Each equation is solved independently for its step in y, but these y values are substituted in the simultaneous equations to give new value of u for each equation.

Differential equations of higher order or systems thereof are reducible to a system of equations of the first order which is then solvable, as above outlined. Milne has developed special means of solving second order equations in which first derivatives are absent, particularly $d^2y/dx^2 = f(x,y)$ and $d^2y/dx^2 + g(x)y = f(x)$.

REFERENCES: The following list will be of assistance to those who wish to study this subject further:

W. E. Milne, *Numerical Integration of Ordinary Differential Equations*, Am. Math. Mo. 33: 455-460 (1926).

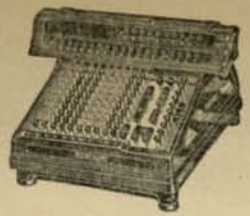
W. E. Milne, *On the Numerical Integration of Certain Differential Equations of the Second Order*, Am. Math. Mo. 40: 322-327 (1933).

National Research Council, No. 92, *Numerical Integration of Differential Equations* (Report of A. A. Bennett, W. E. Milne, H. Bateman) (1933).

J. F. Steffensen, *Interpolation*, P. 158-159, 170-177, Williams & Wilkins Co. (1927).

J. L. Scarborough, *Numerical Mathematical Analysis*, P. 280-282, The Johns Hopkins Press, Baltimore (1930).

See reverse side hereof for method of calculating 10-Figure Roots.



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100 - 197

A	Col. 1	Col. 2	Col. 3
100	300 000	139 248	646 330
102	303 987	141 098	654 920
104	307 948	142 937	663 453
106	311 883	144 763	671 932
108	315 793	146 579	680 358
110	319 681	148 383	688 731
112	323 544	150 176	697 054
114	327 384	151 959	705 328
116	331 202	153 731	713 554
118	334 999	155 493	721 732
120	338 773	157 244	729 864
122	342 527	158 987	737 952
124	346 260	160 720	745 995
126	349 974	162 443	753 995
128	353 667	164 158	761 953
130	357 342	165 863	769 869
132	360 997	167 560	777 745
134	364 635	169 248	785 581
136	368 254	170 928	793 379
138	371 855	172 600	801 138
140	375 440	174 264	808 860
142	379 007	175 919	816 545
144	382 557	177 567	824 194
146	386 091	179 208	831 808
148	389 609	180 841	839 388
150	393 111	182 466	846 933
152	396 598	184 085	854 444
154	400 069	185 696	861 923
156	403 526	187 300	869 370
158	406 967	188 897	876 784
160	410 395	190 488	884 168
162	413 807	192 072	891 520
164	417 205	193 650	898 843
166	420 591	195 221	906 136
168	423 963	196 786	913 400
170	427 321	198 345	920 634
173	432 333	200 671	931 434
176	437 317	202 985	942 171
179	442 273	205 285	952 847
182	447 200	207 572	963 466
185	452 101	209 847	974 025
188	456 976	212 109	984 525
191	461 826	214 360	994 970
194	466 648	216 598	100 536
197	471 446	218 826	101 570

200 - 375

A	Col. 1	Col. 2	Col. 3
200	476 220	221 042	102 599
203	480 971	223 247	103 622
206	485 698	225 441	104 640
209	490 402	227 624	105 654
212	495 083	229 797	106 663
215	499 743	231 960	107 666
218	504 381	234 113	108 666
221	508 998	236 256	109 660
224	513 594	238 389	110 650
227	518 170	240 513	111 636
230	522 725	242 627	112 618
233	527 260	244 733	113 595
236	531 777	246 829	114 568
240	537 769	249 610	115 859
244	543 728	252 376	117 143
248	549 654	255 127	118 419
252	555 548	257 863	119 689
256	561 412	260 584	120 952
260	567 244	263 292	122 209
264	573 047	265 985	123 459
268	578 821	268 665	124 704
272	584 567	271 332	125 941
276	590 284	273 985	127 173
280	595 973	276 626	128 399
284	601 635	279 255	129 619
288	607 272	281 870	130 833
292	612 882	284 474	132 041
296	618 466	287 067	133 245
300	624 025	289 647	134 442
304	629 560	292 216	135 634
308	635 070	294 773	136 822
312	640 557	297 320	138 004
316	646 020	299 856	139 181
320	651 460	302 381	140 353
325	658 229	305 523	141 811
330	664 963	308 649	143 262
335	671 663	311 759	144 705
340	678 329	314 853	146 142
345	684 964	317 932	147 571
350	691 566	320 996	148 993
355	698 136	324 046	150 409
360	704 676	327 082	151 818
365	711 186	330 103	153 220
370	717 666	333 111	154 616
375	724 117	336 105	156 007

380 - 703

A	Col. 1	Col. 2	Col. 3
380	730 540	339 086	157 390
386	738 208	342 646	159 042
392	745 839	346 188	160 686
398	753 431	349 712	162 322
404	760 984	353 217	163 949
410	768 500	356 706	165 568
416	775 979	360 178	167 180
422	783 423	363 633	168 783
428	790 831	367 071	170 379
434	798 205	370 494	171 968
440	805 545	373 901	173 549
446	812 851	377 292	175 124
452	820 125	380 668	176 691
458	827 367	384 030	178 251
465	835 776	387 933	180 063
472	844 143	391 817	181 865
479	852 468	395 681	183 659
486	860 753	399 527	185 448
493	868 999	403 354	187 220
500	877 206	407 163	188 988
507	885 374	410 954	190 748
514	893 505	414 728	192 500
521	901 598	418 485	194 243
528	909 656	422 225	195 979
535	917 678	425 948	197 708
542	925 666	429 656	199 429
550	934 752	433 874	201 386
558	943 794	438 071	203 334
566	952 794	442 248	205 273
574	961 751	446 405	207 203
582	970 666	450 543	209 124
590	979 541	454 663	211 036
598	988 376	458 764	212 939
606	997 171	462 846	214 834
614	100 593	466 910	216 721
622	101 465	470 957	218 599
631	102 441	475 489	220 703
640	103 413	480 000	222 796
649	104 380	484 490	224 880
658	105 343	488 959	226 954
667	106 301	493 407	229 019
676	107 254	497 835	231 075
685	108 205	502 244	233 121
694	109 151	506 634	235 159
703	110 093	511 005	237 187

712 - 1000

A	Col. 1	Col. 2	Col. 3
712	111 030	515 357	239 207
721	111 964	519 690	241 219
730	112 894	524 007	243 222
740	113 922	528 781	245 438
750	114 947	533 534	247 645
760	115 966	538 266	249 841
770	116 981	542 978	252 028
780	117 991	547 668	254 205
790	118 998	552 339	256 373
800	120 000	556 991	258 532
811	121 098	562 085	260 897
822	122 190	567 156	263 250
833	123 278	572 204	265 594
844	124 361	577 231	267 927
855	125 439	582 235	270 250
866	126 512	587 219	272 563
877	127 581	592 180	274 866
888	128 646	597 122	277 160
900	129 803	602 490	279 651
912	130 954	607 833	282 131
924	132 100	613 154	284 601
936	133 241	618 451	287 059
948	134 378	623 725	289 508
961	135 603	629 415	292 148
974	136 823	635 079	294 777
987	138 038	640 717	297 394
1000	139 248	646 330	300 000

Table intervals are so arranged that the root calculated by the method below differs from the true root by less than 5 in the sixth significant figure. Rounding of the fifth figure of true and calculated root may lead to occasional errors of "1" in the fifth figure of the calculated rounded five-figure root as compared with that of the true rounded five-figure root.

See reverse side for Extension to 10-Figure Root

MARCHANT METHOD

OUTLINE: To find the cube root of any number N, select from Col. A the number nearest to the three left significant digits of N. Add twice this number to N, lining up first significant digits. Divide this sum by the factor in Col. 1, 2, or 3 according to whether there are 1, 2, or 3 digits in the left group of digits of N after it has been pointed off in groups of three digits each way from decimal.

The root is shown as the quotient of the preceding division rounded to five figures, there being one digit of root, each way from decimal, for each group of significant digits in the number whose root is desired.

See Reverse Side for Examples and Detailed Instructions.

(SEE OVER)

1026 6436 7

MARCHANT METHOD

This method of extracting cube root by use of a table of divisors (or of multipliers, if reciprocals of divisor factors are used) is a concise, practical method of extracting cube roots accurate to any desired number of significant figures. It is based upon a well-known principle according to which an approximate root may be converted to one that is of a greater degree of accuracy. (See Barlow's Tables, 1935 Edition, Page XI, Introduction, by L. J. Comrie, M.A., Ph.D.)

The table on the reverse side was first published in April 1940 and is so far as known the first and only published table of its type. The method and table represent original work of the staff of Marchant Calculating Machine Company.

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MARCHANT OPERATIONS—FIVE-FIGURE CUBE ROOT

EXAMPLE: Find the cube root of 65324.74.

Use any model of "Silent-Speed" Marchant Calculator.

(1) With carriage in 6th position, set up at extreme left of Keyboard Dial the significant figures of the number whose root is desired (6532474), and add.

(2) Set up at extreme left of Keyboard Dial the number in Col. A (649) which is nearest to the left-hand three digits of the number whose root is desired, and multiply by 2.

The Middle Dial shows a number that is approximately three times the number whose root is desired.

(3) Point off into groups of three digits the number whose root is desired, beginning from decimal point (65'324.740'), from which it is noted that the left-hand group (65) contains two* figures. Set up at extreme left of Keyboard Dial the number that appears in Col. 2* at right of 649 (484490) and divide.

Upper Dial shows 402742 which rounds to 40274 as five significant figures of the root. As there is one figure of the root each way from decimal for each group of three figures each way from decimal in the number whose root is desired, the cube root is 40.274 which is correct to five significant figures; i.e., the error in the 6th figure is less than 5.

Or, arithmetically

$$\begin{array}{r} 6532474 \\ 649 \\ \underline{649} \\ 484490\,\overline{)19512474} \\ 402742554 \end{array}$$

which rounds to five figures as 40.274.

CONVERTING FIVE-FIGURE CUBE ROOT TO ONE OF TEN FIGURES

Rule: Divide the number whose ten-figure root is desired by the square of its five-figure cube root, noting the quotient to ten figures. Add twice the five-figure cube root to the left-hand five figures of this quotient. One-third of this sum will be the desired cube root accurate to ten significant figures; i.e., the error of the 11th digit will not exceed 5.

EXAMPLE: Extend the above five-figure cube root to one of ten figures.

The square of the five-figure root is $40.274^2 = 1621.995076$

$$\begin{array}{r} 65324.74/1621.995076 = 40.27431462 \\ 40.274 \\ \underline{40.274} \\ 3\,\overline{)120.82231462} \\ 40.27410487 \end{array}$$

*If the left-hand group contained one digit, such as if the number whose root is desired were 6'532.48, divisor would have been taken from Col. 1, as 104380. If it contained three digits, such as if the number were 653'247.400' divisor would have been taken from Col. 3, as 224880.

MARCHANT CALCULATING MACHINE COMPANY

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AREA BELOW CURVE IN THE CASE OF FRACTIONALLY-SPACED ORDINATES

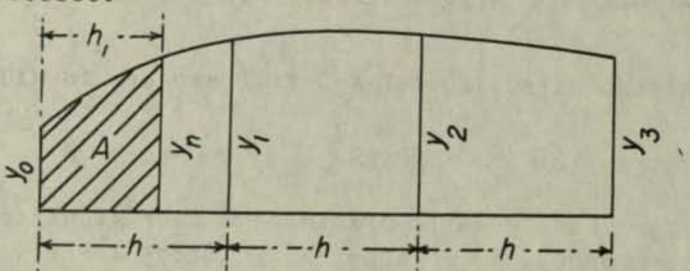
(Also Suitable for Integration of Equations of 3rd Degree, or Less)

REMARKS: *The usual rules for approximate integration (Simpson, Weddle, etc.) require that the base-line be divided into sections by equidistant ordinates. When the heights of these ordinates are known, the area of any or all of the individual sections may be readily computed within limits of error which vary according to the formula used (in Simpson's 1/3 Rule, 3rd differences are taken into account). See Note C on reverse side hereof for limitations.*

It is often desired to know the area of a fractional part of one of the sections bounded by such equidistant ordinates. This problem frequently occurs in computing areas and volumes of irregular shapes in ship construction and elsewhere. The problem is one of approximate integration of $y = f(x)$ when the upper limit of integration is a fraction.

The method herein takes into account 3rd differences, yet it does not require the computation of tables of differences of the ordinates. So far as we are able to determine, the method is an original contribution of the Marchant Calculating Machine Company, though it has its foundation in well-known processes.

OUTLINE:



$$n = \frac{h_1}{h}$$

The shaded area of the above diagram is obtained by the following formula, assuming that 4th differences may be disregarded:

$$\text{Area} = \int_{x_0}^{x_0+nh} y(dx) = h\,(K_0y_0 + K_1y_1 - K_2y_2 + K_3y_3),$$

in which the values of the "K" constants are taken from table below at various values of n. The table shows constants to five decimals. Exact values will be supplied upon application.

n	K_0	K_1	K_2	K_3
0.1	.09116 +	.01418 -	.00685 -	.00151 -
0.2	.16593 +	.05353 +	.02486 +	.0054
0.3	.22616 +	.11351 +	.05051 +	.01084 -
0.4	.2736	.18987 -	.08053 +	.01706 +
0.5	.30990 -	.27864 +	.11198 -	.02344 -
0.6	.3366	.3762	.1422	.0294
0.7	.35516 +	.47918 +	.16885	.03451 -
0.8	.36693 +	.58453 +	.18986 +	.0384
0.9	.37316 +	.68952 -	.20352	.04084 -
1.0	.375	.79167 -	.20833 +	.04166 +

(over)

Values of K's on preceding page that are not followed by + or - are exact.

The attached curve of these coefficients enables their values to be approximately determined at intermediate values of "n".

EXAMPLE: In the diagram on reverse side, the values of y at points 4 ft. apart are $y_0 = 6.342$, $y_1 = 8.502$, $y_2 = 8.680$, and $y_3 = 7.128$. Find area A when y_n is 2.8 ft. from left-hand ordinate ($h_1 = 2.8$, or $n = 2.8/4 = 0.7$).

OPERATIONS: Decimals; Upper Dial 4, Middle Dial 9, Keyboard Dial 5. Any Marchant model. Upper Green Shift Key should be down if M models are used.

(1) Set up in Keyboard Dial K_0 for $n = 0.7$ (.35516) and multiply by y_0 (6.342).

(2) Clear Upper and Keyboard Dials only, set up K_1 (.47918) in Keyboard Dial and multiply by y_1 (8.502).

(3) Clear Upper and Keyboard Dials only, move Manual Counter Control toward operator, and similarly reverse multiply K_2 (.16885) by y_2 (8.680).

(4) Clear Upper and Keyboard Dials only, move Manual Counter Control away from operator, and multiply K_3 (.03451) by y_3 (7.128).

Area /h (5.10678) appears in Middle Dial.

(5) Clear Upper and Keyboard Dials only, shift to 5th position, transfer Middle Dial amount (5.10678) to Keyboard Dial, clear Middle Dial, and multiply by h (4.0).

Area desired (20.427 sq. ft.) appears in Middle Dial.

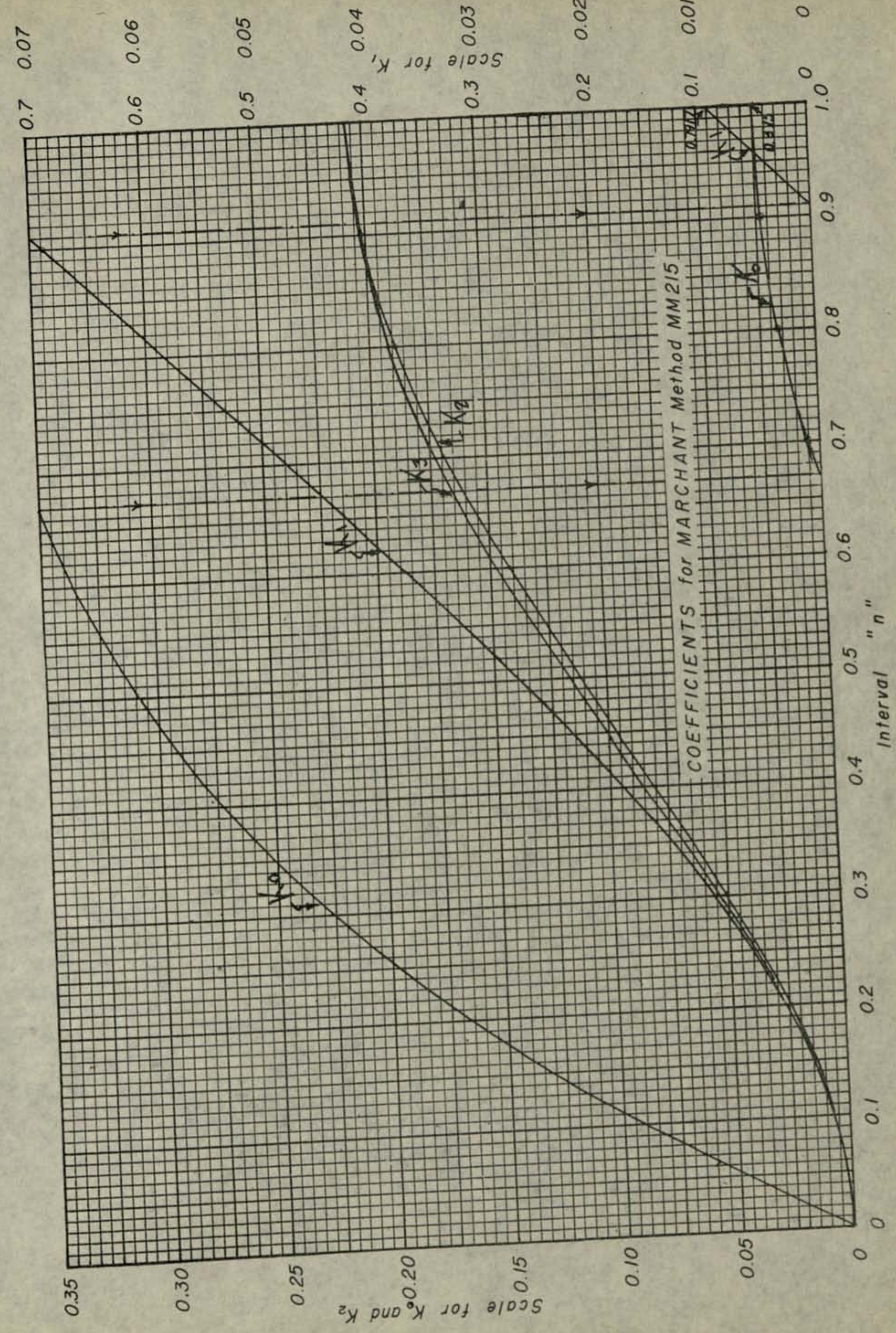
NOTES

A: *The ordinates in the above example are taken as four significant figures from the equation $y = .000656\ X^3 - .06981\ X^2 + .80875\ X + 6.342$, the integration of which, between limits of $X = 0$ to $X = 2.8$, is 20.427, the same as found by the above process. This illustrates that the method may be used for integrating 3rd degree equations, or any of lesser degree, and of course for the approximate integration of equations of higher degree.*

B: *If only the ordinates y_0, y_1 and y_2 are available, and it is not convenient to divide the distance between y_0 and y_2 into three equal spaces, the best procedure is to draw a free-hand curve to extend it to a hypothetical y_3, which is then approximately measured.*

C: *This method obtains the area under a smooth curve of minimum continuous curvature that connects the four points. If the curve has waves in any of the equal sections, or angularities showing abrupt changes of curvature, the method does not apply. Mathematically, this process integrates a continuous single-valued function having constant third derivative.*

D: *The rounding of the exact coefficients to five figures has been slightly adjusted in order that the algebraic sum of the coefficients of any row will equal "n".*


COEFFICIENTS for MARCHANT Method MM215
Interval "n"
Scale for K₃
Scale for K₁
Scale for K₀ and K₂
K₀
K₁
K₂
K₃
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
0 0.05 0.10 0.15 0.20 0.25 0.30 0.35
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
0 0.01 0.02 0.03 0.04 0.05 0.06 0.07

MARCHANT METHODS

FIFTH ROOT

REMARKS: The table and its explanation submitted herein represent the work of Mr. Charles S. Larkey who has followed the principles exemplified in Marchant Tables 56 and 68 relating to square and cube root, respectively, to extend the method to the extraction of fifth roots. Mr. Larkey's manuscript is reproduced herewith without checking of the tabular amounts by us, but knowing as we do of his skill as a mathematician, we have complete confidence that his table will produce the desired results within the limit of accuracy set forth in the explanation.

OUTLINE

The method described below enables one to obtain directly, by means of division, the fifth root of any number, correct to five significant figures.

TO OBTAIN FIFTH ROOT CORRECT TO 5 SIGNIFICANT FIGURES

Selection of Proper Number in Column of Table:

Column N contains a sequence of numbers from 10,000 to 100,000. In finding the fifth root of a number, the number in Column N nearest the given number is selected, it being understood that the decimal point may be placed in any desired position in the tabular value in order to conform to the number whose root is sought.

For example: The fifth root of 302.456 is desired. In Column N, 30,129 is the nearest number since it may be written 301.29 by moving the decimal point, and this value is nearer 302.456 than 305.10, the next greater value in the table.

Finding Column for Divisor:

First separate the number whose fifth root is to be found into periods of five digits each, beginning at the decimal point. In general, the number of digits in the first period is also the number of the Column for the divisor; but in case of a number entirely decimal, subtract the number of zeros immediately following the decimal point from five or a multiple of five such that the remainder is equal to five or less; this remainder is the Column number of the divisor.

Example: Thus if the given number is 12,453,575, dividing it into periods it becomes 124'53575, and Column 3 is to be used; whereas, if the number is 0.012453575, Column 4 is to be used.

Determining the Fifth Root to 5 Significant Figures:

Set the number whose root is to be found in the Middle Dials of the calculating machine with the carriage to the right for division. Set in Keyboard the nearest number from Column N as found above with the corresponding digits opposite those in the Middle Dial and multiply by four; then divide by the number appearing in Column 1, 2, 3, 4 or 5, as the case may be, from the selection as explained above. The quotient will be the fifth root of the number, correct to five significant figures.

(over)

EXTENSION OF CALCULATION TO NINE SIGNIFICANT FIGURES

Divide the number whose nine-figure root is desired by the fourth power of the five-figure root already obtained, carrying the quotient to ten places. Add four times the five-figure root to this quotient and divide by five. The result will be the fifth root correct to nine significant figures.

FORMULAS FOR CALCULATIONS

The operations described above may be expressed in formulas as follows:

$$R_5 = \sqrt[5]{X} = \frac{X + 4N}{D}$$

correct to five significant figures, and

$$R_9 = \sqrt[5]{X} = \frac{1}{5}\left(\frac{X}{R_5{}^4} + 4R_5\right)$$

correct to nine significant figures, where X is the number whose root is sought, N is the nearest basic tabular number, and D is the indicated divisor from the table.

EXAMPLES

Example 1:

Find the fifth root of 12,453,575.

Writing this number in periods of five digits, it becomes 124'53575. The nearest corresponding value under N is 12,497, and since there are three digits in the first period, the divisor 237.911 from Column 3 is to be used. After adding four times 12497000 to 12453575, the Middle Dial reads 62441575. Dividing by 237.911, the root is found to be 26.246, the number of periods to the left of the decimal in the number determining the number of digits to the left of the decimal in the root.

To determine the root to nine places, square 26.246 by multiplying it by itself, and then square the result, thus obtaining the fourth power of the five-figure root. This gives 474,517.7888 which is contained in 12,453,575, 26.24469576 times, carrying to the full capacity of a 10-column machine. Set this quotient in the Middle Dial and add four times 26.24600000, causing the Middle Dial to read, 131.22869576. Divide by five, giving 26.2457392 as the value of the root correct to nine figures.

Example 2:

Find the fifth root of 0.000125.

The nearest value of N is 12,497. Since there are three zeros after the decimal point, subtracting from five this leaves two for the number of the column for the divisor which is found to be 37.7064. Proceeding as in Example 1, the root is found to be 0.16572.

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
10 000	5.00000	31.5479	199.054	1255.94	7924.47
10 142	5.05672	31.9058	201.312	1270.19	8014.36
10 285	5.11368	32.2651	203.579	1284.50	8104.63
10 430	5.17127	32.6285	205.872	1298.96	8195.91
10 577	5.22950	32.9959	208.190	1313.59	8288.20
10 726	5.28835	33.3672	210.533	1328.37	8381.47
10 877	5.34783	33.7425	212.901	1343.31	8475.73
11 030	5.40792	34.1217	215.293	1358.41	8570.98
11 185	5.46863	34.5047	217.710	1373.66	8667.20
11 342	5.52996	34.8917	220.152	1389.06	8764.39
11 501	5.59189	35.2824	222.617	1404.62	8862.55
11 662	5.65442	35.6770	225.107	1420.33	8961.66
11 825	5.71756	36.0754	227.620	1436.19	9061.72
11 990	5.78130	36.4775	230.158	1452.20	9162.74
12 157	5.84563	36.8834	232.719	1468.36	9264.69
12 326	5.91055	37.2930	235.303	1484.66	9367.59
12 497	5.97605	37.7064	237.911	1501.12	9471.41
12 670	6.04215	38.1234	240.542	1517.72	9576.16
12 845	6.10882	38.5440	243.196	1534.47	9681.83
13 022	6.17607	38.9684	245.874	1551.36	9788.41
13 201	6.24389	39.3963	248.574	1568.39	9895.90
13 383	6.31267	39.8302	251.312	1585.67	10004.9
13 567	6.38200	40.2677	254.072	1603.09	10114.8
13 753	6.45190	40.7088	256.855	1620.65	10225.6
13 941	6.52237	41.1533	259.660	1638.34	10337.3
14 132	6.59376	41.6038	262.502	1656.28	10450.4
14 325	6.66570	42.0577	265.366	1674.35	10564.4
14 520	6.73819	42.5151	268.252	1692.56	10679.3
14 718	6.81160	42.9783	271.175	1711.00	10795.7
14 918	6.88555	43.4449	274.119	1729.57	10912.9
15.121	6.96040	43.9172	277.099	1748.37	11031.5

1

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
15 326	7.03579	44.3928	280.100	1767.31	11151.0
15 534	7.11208	44.8742	283.137	1786.47	11271.9
15 744	7.18889	45.3589	286.195	1805.77	11393.6
15 957	7.26659	45.8491	289.288	1825.29	11516.8
16 172	7.34482	46.3427	292.402	1844.93	11640.8
16 390	7.42392	46.8418	295.551	1864.80	11766.1
16 610	7.50353	47.3441	298.721	1884.80	11892.3
16 833	7.58402	47.8519	301.925	1905.02	12019.9
17 059	7.66537	48.3652	305.164	1925.45	12148.8
17 289	7.74722	48.8816	308.422	1946.01	12278.5
17 518	7.82993	49.4035	311.715	1966.79	12409.6
17 752	7.91349	49.9307	315.042	1987.78	12542.0
17 989	7.99789	50.4633	318.402	2008.98	12675.8
18 229	8.08314	51.0012	321.796	2030.39	12810.9
18 471	8.16888	51.5421	325.209	2051.93	12946.8
18 716	8.25544	52.0883	328.655	2073.67	13084.0
18 964	8.34284	52.6398	332.134	2095.63	13222.5
19 215	8.43106	53.1964	335.647	2117.79	13362.3
19 469	8.52010	53.7582	339.192	2140.15	13503.5
19 726	8.60996	54.3252	342.769	2162.72	13645.9
19 986	8.70063	54.8973	346.378	2185.50	13789.6
20 249	8.79211	55.4744	350.020	2208.48	13934.5
20 515	8.88438	56.0567	353.694	2231.66	14080.8
20 785	8.97780	56.6461	357.413	2255.12	14228.9
21 058	9.07201	57.2405	361.163	2278.79	14378.2
21 334	9.16701	57.8399	364.944	2302.65	14528.7
21 613	9.26280	58.4443	368.759	2326.71	14680.5
21 895	9.35935	59.0535	372.603	2350.96	14833.6
22 181	9.45703	59.6698	376.491	2375.50	14988.4
22 470	9.55548	60.2910	380.411	2400.23	15144.4
22 762	9.65469	60.9170	384.360	2425.15	15301.7

2

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
23 058	9.75500	61.5499	388.354	2450.35	15460.6
23 357	9.85607	62.1876	392.377	2475.73	15620.8
23 660	9.95822	62.8321	396.444	2501.39	15782.7
23 966	10.0611	63.4814	400.541	2527.24	15945.8
24 276	10.1651	64.1375	404.680	2553.36	16110.6
24 590	10.2701	64.8003	408.862	2579.75	16277.1
24 907	10.3759	65.4677	413.073	2606.32	16444.7
25 228	10.4828	66.1418	417.327	2633.15	16614.1
25 553	10.5907	66.8226	421.622	2660.26	16785.1
25 881	10.6993	67.5080	425.946	2687.54	16957.2
26 213	10.8089	68.1998	430.312	2715.08	17131.0
26 549	10.9196	68.8983	434.719	2742.89	17306.5
26 889	11.0314	69.6033	439.167	2770.96	17483.6
27 233	11.1441	70.3147	443.656	2799.28	17662.3
27 581	11.2579	71.0326	448.186	2827.86	17842.6
27 933	11.3727	71.7570	452.756	2856.70	18024.5
28 289	11.4885	72.4877	457.366	2885.79	18208.1
28 649	11.6053	73.2247	462.017	2915.13	18393.2
29 013	11.7231	73.9680	466.707	2944.72	18579.9
29 381	11.8420	74.7177	471.437	2974.56	18768.2
29 753	11.9617	75.4735	476.206	3004.65	18958.1
30 129	12.0825	76.2356	481.014	3034.99	19149.5
30 510	12.2046	77.0059	485.874	3065.66	19343.0
30 895	12.3277	77.7823	490.773	3096.57	19538.0
31 284	12.4517	78.5648	495.710	3127.72	19734.6
31 678	12.5770	79.3553	500.698	3159.19	19933.2
32 076	12.7032	80.1520	505.725	3190.91	20133.3
32 479	12.8308	80.9566	510.801	3222.94	20335.4
32 886	12.9592	81.7671	515.916	3255.21	20539.0
33 298	13.0889	82.5856	521.080	3287.79	20744.6
33 714	13.2196	83.4100	526.282	3320.61	20951.6

3

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
34 135	13.3515	82.2422	531.533	3353.74	21160.7
34 561	13.4846	85.0823	536.833	3387.19	21371.7
34 992	13.6190	85.9300	542.182	3420.94	21584.6
35 427	13.7543	86.7836	547.567	3454.92	21799.0
35 867	13.8908	87.6448	553.001	3489.20	22015.4
36 312	14.0285	88.5136	558.483	3523.79	22233.1
36 762	14.1674	89.3901	564.013	3558.68	22453.8
37 217	14.3075	90.2741	569.591	3593.88	22675.8
37 677	14.4488	91.1656	575.216	3629.37	22899.8
38 142	14.5913	92.0646	580.889	3665.16	23125.6
38 612	14.7349	92.9711	586.608	3701.24	23353.3
39 088	14.8801	93.8868	592.386	3737.70	23583.3
39 569	15.0264	94.8100	598.211	3774.45	23815.2
40 055	15.1738	95.7404	604.081	3811.50	24048.9
40 547	15.3228	96.6801	640.010	3848.90	24284.9
41 044	15.4728	97.6270	615.985	3886.60	24522.8
41 546	15.6240	98.5810	622.004	3924.58	24762.4
42 054	15.7767	99.5442	628.081	3962.92	25004.4
42 567	15.9305	100.514	634.203	4001.55	25248.1
43 086	16.0857	101.494	640.382	4040.54	25494.1
43 611	16.2423	102.482	646.617	4079.87	25742.3
44 141	16.4000	103.477	652.896	4119.49	25992.2
44 677	16.5591	104.481	659.230	4159.46	26244.4
45 219	16.7196	105.494	665.621	4199.78	26498.8
45 767	16.8815	106.515	672.066	4240.45	26755.4
46 321	17.0448	107.546	678.566	4281.46	27014.2
46 881	17.2095	108.584	685.121	4322.82	27275.2
47 447	17.3755	109.632	691.731	4364.53	27538.3
48 019	17.5429	110.688	698.394	4406.57	27803.6
48 597	17.7116	111.753	705.111	4448.95	28071.0
49 182	17.8820	112.828	711.893	4491.74	28341.0

4

FIFTH ROOT DIVISCRS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
49 773	18.0537	113.911	718.729	4534.87	28613.1
50 370	18.2267	115.003	725.617	4578.34	28887.3
50 974	18.4013	116.104	732.570	4622.20	29164.1
51 584	18.5773	117.215	739.575	4666.40	29443.0
52 201	18.7548	118.335	746.643	4711.00	29724.4
52 825	18.9340	119.465	753.775	4756.00	30008.3
53 455	19.1144	120.604	760.958	4801.32	30294.3
54 092	19.2964	121.752	768.204	4847.04	30582.7
54 736	19.4800	122.910	775.512	4893.15	30873.7
55 387	19.6651	124.078	782.882	4939.65	31167.1
56 045	19.8518	125.256	790.314	4986.54	31463.0
56 710	20.0400	126.444	797.807	5033.82	31761.3
57 382	20.2298	127.641	805.361	5081.48	32062.0
58 061	20.4210	128.848	812.976	5129.53	32365.1
58 747	20.6138	130.064	820.651	5177.96	32670.7
59 440	20.8081	131.290	828.387	5226.77	32978.7
60 141	21.0042	132.528	836.193	5276.02	33289.4
60 849	21.2018	133.774	844.059	5325.65	33602.6
61 565	21.4011	135.032	851.995	5375.73	33918.5
62 288	21.6020	136.299	859.990	5426.17	34236.8
63 019	21.8046	137.577	868.055	5477.06	34557.9
63 758	22.0089	138.867	876.189	5528.38	34881.7
64 504	22.2146	140.165	884.381	5580.07	35207.8
65 258	22.4221	141.474	892.641	5632.19	35536.7
66 020	22.6313	142.794	900.970	5684.74	35868.3
66 790	22.8423	144.125	909.367	5737.72	36202.5
67 568	23.0549	145.466	917.831	5791.12	36539.5
68 354	23.2692	146.819	926.363	5844.95	36879.2
69 149	23.4854	148.183	934.972	5899.28	37221.9
69 952	23.7034	149.558	943.648	5954.02	37567.3
70 763	23.9230	150.944	952.390	6009.18	37915.3

FIFTH ROOT DIVISORS

N	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
71 583	24.1445	152.341	961.209	6064.82	38266.4
72 411	24.3676	153.749	970.094	6120.88	38620.1
73 248	24.5927	155.170	979.054	6177.41	38976.8
74 094	24.8197	156.602	988.090	6234.42	39336.6
74 948	25.0483	158.044	997.190	6291.85	39698.9
75 811	25.2788	159.498	1006.37	6349.74	40064.1
76 683	25.5111	160.964	1015.62	6408.10	40432.4
77 564	25.7453	162.442	1024.94	6466.93	40803.6
78 454	25.9814	163.931	1034.34	6526.23	41177.7
79 353	26.2193	165.432	1043.81	6585.98	41554.8
80 262	26.4593	166.947	1053.36	6646.27	41935.1
81 180	26.7011	168.473	1062.99	6707.01	42318.4
82 107	26.9448	170.010	1072.69	6768.22	42704.6
83 044	27.1905	171.560	1082.47	6829.94	43094.0
83 991	27.4382	173.124	1092.34	6892.17	43486.7
84 947	27.6878	174.698	1102.27	6954.86	43882.2
85 913	27.9394	176.286	1112.29	7018.06	44281.0
86 889	28.1930	177.886	1122.38	7081.77	44682.9
87 875	28.4487	179.499	1132.56	7145.99	45088.1
88 871	28.7064	181.125	1142.82	7210.71	45496.5
89 877	28.9660	182.763	1153.16	7275.94	45908.1
90 893	29.2277	184.414	1163.57	7341.66	46322.7
91 920	29.4916	186.079	1174.08	7407.95	46741.0
92 957	29.7575	187.757	1184.67	7474.73	47162.4
94 005	30.0255	189.448	1195.34	7542.08	47587.3
95 063	30.2956	191.152	1206.09	7609.91	48015.3
96 132	30.5678	192.870	1216.93	7678.29	48446.7
97 212	30.8422	194.601	1227.85	7747.22	48881.7
98 303	31.1188	196.347	1238.86	7816.70	49320.0
99 405	31.3976	198.106	1249.96	7886.72	49761.9
100 000	31.5479	199.054	1255.94	7924.47	50000.0

MARCHANT METHODS

DIVISION -- WHEN FACTORS EXCEED CAPACITY OF CALCULATOR

REMARKS: *If the dividend has more digits than the number of columns of the Marchant, division is accomplished by splitting the dividend into two parts, dividing each by the divisor and adding the respective quotients; thus,*

$$\frac{12345678912345}{123456} = \frac{12345678910000}{123456} + \frac{2345}{123456}$$

If the divisor has more digits than the number of columns of the Marchant, the problem is solved by an expansion of series; thus,

$$\frac{B}{A+s} = \frac{B}{A}\left(1 - \frac{s}{A} + \frac{s^2}{A^2} - \ldots\ldots\right)$$

When s is small, compared with A, the final terms may be eliminated, reducing to

$$\frac{B}{A}\left(1 - \frac{s}{A}\right)^{*}$$

EXAMPLE: Solve $\frac{.01875438}{10.285\ 714\ 285\ 71}$ to 13 significant figures, assuming that only a 10-column model is available.

Substituting in the above,

$$B = .018\ 754\ 38$$
$$A = 10.285\ 714\ 28$$
$$s = .000\ 000\ 005\ 71$$

Then $\frac{B}{A} = \frac{.018\ 754\ 38}{10.285\ 714\ 28} = .001\ 823\ 342\ 501\ \frac{1334}{102857}$

$= .001\ 823\ 342\ 501\ 013\ 0$

$\frac{s}{A} = \frac{.000\ 000\ 005\ 71}{10.285\ 714\ 28} = .000\ 000\ 000\ 555\ 1$

Now, $\frac{B}{A}\left(1 - \frac{s}{A}\right) = \frac{B}{A} - \left(\frac{B}{A} \cdot \frac{s}{A}\right)$ and it will be noted that $\left(\frac{B}{A} \cdot \frac{s}{A}\right)$ need be computed to four figures, at most; thus,

$.001\ 823\ 34 \times .000\ 000\ 000\ 555\ 1 = .000\ 000\ 000\ 001\ 012\ 2$

which, being subtracted from B/A, shows final quotient of

.001 823 342 500 001 (to 13 significant figures).

A Marchant Method that short-cuts the above work is on reverse side hereof.

(*) *Whether higher terms of the series may be disregarded depends upon accuracy desired. In this case, obviously, s^2 will not affect 13th place.*

(See other side for Marchant application.)

MARCHANT APPLICATION FOR EXAMPLE ON REVERSE SIDE

OPERATIONS: As it is desired to utilize the full 10-figure capacity, decimals will be set by inspection of the results. Use 10-column Model M with Upper Green Shift Key down.

(1) With carriage in 10th position, set up the significant figures of B (1875438) at extreme left of Keyboard Dial and add.

(2) Similarly, Set up figures of A (1028571428) and divide.

Significant figures of A/B' (1823342501) appear in Upper Dial. Copy to report in relation to decimal. Remainder (13338..) appears in Middle Dial.

(3) Clear Upper and Keyboard Dials, set up at extreme right of Keyboard Dial the leftmost five significant figures of A (10286), shift to 5th position, and divide.

The figures to be affixed to A/B appear in Upper Dial. It is noted that the first figure produced as a result of the division, though a cipher, is significant. The 14-figure quotient is thus .001 823 342 501 013 0.

(4) Clear all dials and, by division in the manner of step (1), obtain s/A as .000 000 000 555 1, as it need not be computed to more than an additional place of "s." Divisor need not be set up to more than six significant figures.

(5) Clear all dials and by suitable Keyboard-Dial entries, carriage shifts, and Add-Bar depressions, enter the 14 significant figures of B/A at extreme left of Middle Dial, and point off in triads by placing decimal markers where commas would appear in the amount as written.

(6) It is noted from a rough estimate that B/A x s/A is approximately .000 000 000 001. Set up in Keyboard Dial in columns 5 to 1 (extreme right) the leftmost significant figures of B/A (18233), move Manual Counter Control toward operator and, noting that the leftmost figure of s/A is "5," shift carriage so that reverse multiplication by a "5" will cause approximately "1" to be subtracted from the "1" of the triad "501" that appears in Middle Dial. By this rule the reverse multiplication by s/A should be started with carriage in 5th position, whence reverse multiplying by the significant figures of s/A (5551) causes the complete final quotient to appear in Middle Dial to 13 significant figures, as

.001 823 342 500 001

1 026 6436 7

MARCHANT METHODS

CURVILINEAR INTERPOLATION WITH UNEQUAL INTERVALS OF THE ARGUMENT
THE METHOD OF DIVIDED DIFFERENCES

REMARKS: *When the independent variable (argument) is tabulated at unequal intervals, interpolation in the column of the corresponding dependent variable (function) may be performed by the method of divided differences. Instead of using actual differences (and differences of differences), as in difference-interpolation when arguments are tabulated at equal intervals (see Marchant Method MM-189), the calculation is based on the average difference per unit of argument in the range of the tabular values. These are the first-order divided differences. Similarly, the average difference per unit of argument of these first-order divided differences becomes a second-order divided difference, and so on.*

The method is equally suitable for direct or inverse interpolation because inasmuch as the argument intervals are unequal, the problem does not change character if the columns of independent and dependent variables are interchanged. In such a case, however, the number of terms required to obtain a result within a given limit of error may differ materially in cases of direct and inverse interpolation in the same portion of the table.

OUTLINE: The nomenclature is as follows:

Argument	Function	Orders of Divided Differences 1st	2nd	3rd	4th
x_0	$y_0 = f(x_0)$				
		$f(x_1x_0)$			
x_1	$y_1 = f(x_1)$		$f(x_2x_1x_0)$		
		$f(x_2x_1)$		$f(x_3x_2x_1x_0)$	
x_2	$y_2 = f(x_2)$		$f(x_3x_2x_1)$		$f(x_4x_3x_2x_1x_0)$
		$f(x_3x_2)$		$f(x_4x_3x_2x_1)$	
x_3	$y_3 = f(x_3)$		$f(x_4x_3x_2)$		
		$f(x_4x_3)$			
x_4	$y_4 = f(x_4)$				

and so forth, in which

$$\text{(1)}\quad f(x_1x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}\ ;\ f(x_2x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}\ ;\ \text{etc.}$$

$$\text{(2)}\quad f(x_2x_1x_0) = \frac{f(x_2x_1) - f(x_1x_0)}{x_2 - x_0}\ ;\ f(x_3x_2x_1) = \frac{f(x_3x_2) - f(x_2x_1)}{x_3 - x_1}\ ;\ \text{etc.}$$

$$\text{(3)}\quad f(x_3x_2x_1x_0) = \frac{f(x_3x_2x_1) - f(x_2x_1x_0)}{x_3 - x_0}\ ;\ f(x_4x_3x_2x_1) = \frac{f(x_4x_3x_2) - f(x_3x_2x_1)}{x_4 - x_1}\ ;\ \text{etc.}$$

$$\text{(4)}\quad f(x_4x_3x_2x_1x_0) = \frac{f(x_4x_3x_2x_1) - f(x_3x_2x_1x_0)}{x_4 - x_0}\ ;\ \text{etc.}$$

(over)

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Then, for any unknown x, say u,

$$
\begin{aligned}
(5)\quad f(u) = f(x_0) &+ (u - x_0) f(x_1 x_0) + (u - x_0)(u - x_1) f(x_2 x_1 x_0) \\
&+ (u - x_0)(u - x_1)(u - x_2) f(x_3 x_2 x_1 x_0) \\
&+ (u - x_0)(u - x_1)(u - x_2)(u - x_3) f(x_4 x_3 x_2 x_1 x_0) \\
&\text{etc.}
\end{aligned}
$$

For greatest precision, the array of divided differences should as closely as possible be symmetric with respect to u; i.e., the array shown on the preceding page, provided none of the divided differences vanish, is well adapted to obtaining values of u in the vicinity of x_2. This is not possible, however, if there are insufficient values as when interpolating near the end of a table.

If the divided differences of order "n" are rigorously constant, the function is exactly representable by a polynomial of degree "n", for in such a case, (5) is a polynomial in u. This has a bearing on the choice of the portion of any array to be used for calculating f(x) when x = u. For example, if in the above array, it should be noticed that the divided differences of 2nd order are constant (in which case, those of higher order vanish), and it is desired to obtain f(u) in the vicinity of x_2, the most suitable values of the array to be used in the calculation would be

$$
\begin{array}{ccc}
f(x_1) & & \\
 & f(x_2 x_1) & \\
f(x_2) & & f(x_3 x_2 x_1) \\
 & f(x_3 x_2) & \\
f(x_3) & &
\end{array}
$$

which are symmetric with respect to x_2.

In this case, (5) may be applied by a change of subscripts; i.e., designating x_1 as x_0, x_2 as x_1, etc., or it may be written in a form to suit the altered conditions; thus

$$
(5a)\quad f(u) = f(x_1) + (u - x_1)\, f(x_2 x_1) + (u - x_2)(u - x_1)\, f(x_3 x_2 x_1)
$$

In practical work, the computing scheme becomes memorized and little attention is paid to identifying the values except by their position in the array.

It is not often that the divided differences of any order are rigorously constant, but for the usual functions that exhibit a gradual increase or decrease in value, it often will be found that not many orders of divided differences have to be computed before one is found that is so small when substituted in (5) that the value of f(u) is not affected to the number of places it is desired to retain.

One should be certain when applying this principle that the divided difference that appears to be so small that its effect may be disregarded is not one that is close to zero merely because of approaching change of sign of vertically adjacent values. In such a case, the next higher order of divided difference may significantly affect the result. Calculation of a few adjacent values of the divided differences of the order that it is proposed to disregard will usually disclose this condition.

(continued on next page)

EXAMPLE: Find f (96.94) when given the values of f(x) at right that correspond to the values of x at left. The columns of x and f(x) are separated to permit showing the array of divisors, $x_1 - x_0$, etc., $x_2 - x_0$, etc., $x_3 - x_0$, etc.

	x	$x_1 - x_0$ etc.	$x_2 - x_0$ etc.	$x_3 - x_0$ etc.	$x_4 - x_0$ etc.	$x_5 - x_0$ etc.		f(x)
x_0	96.55						$f(x_0)$	1.4452 112
		.15						
x_1	96.70		.35				$f(x_1)$	1.4498 542
		.20		.47				
x_2	96.90		.32		.70		$f(x_2)$	1.4560 764
		.12		.55		.85		
x_3	97.02		.35		.70		$f(x_3)$	1.4598 272
		.23		.50		.81		
x_4	97.25		.38		.61		$f(x_4)$	1.4670 530
		.15		.49				
x_5	97.40		.26				$f(x_5)$	1.4717 918
		.11						
x_6	97.51						$f(x_6)$	1.4752 802

As 96.94 lies between x_2 and x_3, (1) and (2) are applied for the range x_1 to x_4 (see Outline, Page 2), as below:

		$f(x_2)-f(x_1)$ etc.	$f(x_2x_1)$ etc.	$f(x_3x_2)-f(x_2x_1)$ etc.	$f(x_3x_2x_1)$ etc.
$f(x_1)$	1.4498542				
		.0062222	.0311110		
$f(x_2)$	1.4560764			.0001457	.0004553
		.0037508	.0312567		
$f(x_3)$	1.4598272			.0001598	.0004566
		.0072258	.0314165		
$f(x_4)$	1.4670530				

At this point, it is advisable to test the effect of considering the second-order divided differences as the mean of those shown (.0004560). The error in the divided differences is .0000007, which affects f(96.94) by applying (5a) as follows:

$$(u - x_1)(u - x_2)\left[\text{ error in assumed } f(xxx)\right] = \text{error in } f(u)$$

$$.24 \qquad .04 \qquad .0000007 \qquad .000000007$$

This affects 8th place by 1, and as it is desired to retain 7 places, it may reasonably be concluded, unless the function is exceptional, that it is satisfactory to use the mean divided difference and disregard the effect of divided differences of a higher order. As a check, in this case $f(x_4x_3x_2x_1)$ is .0000024. When applying it in (5a), it is multiplied by .24 x .04 x -.08, so it barely affects 9th place.

The desired f(96.94) is then computed as follows:

$f(x_1)$	1.4498542
$(u - x_1) f(x_2x_1) = .24 \times .0311110$	.00746664
$(u - x_1)(u - x_2)\left[f(xxx) \text{ assumed constant}\right]$ $.24 \times 04 \times .0004560$	.00000438
To 7 places	1.4573252

Effect of Rounding, etc.: The calculation assumes that values of x and f(x) are exact, whereas normally f(x) at least would have final figure rounded. Analysis of the effect of this is possible, but not overly helpful. The practical way of taking it into account is to note whether the column of any order of divided differences shows alternate changes of sign yet the average of an even number of adjacent values is close to zero. In such a case, the fluctuation may be assumed as due to rounding, or lack of adequate smoothing of the original data. The order of divided differences in which this occurs is then considered to vanish. In the example, the fourth order of divided differences contains values -.0000109, +.0000114, -.0000090, so may be assumed to vanish. Furthermore, as has been seen, the effect of the third-order divided difference does not significantly alter the desired value.

*

Application of the Marchant is obvious in cases when only a single value is desired. However, if a series of values is to be had, such as when converting smoothed observations to tabular form at equal intervals of the argument, computing work may be reduced by following the suggestions outlined below.

The divided-difference method should be used only to obtain pivotal points of the desired tabulation. The in-between values may then be readily had by sub-tabulating within these pivotal values by the more rapid direct-interpolation methods of Marchant Method MM-189 or MM-228.

At the start, a few widely separated values should be obtained by the method of divided differences as previously outlined for single values. This is for the purpose of determining the order of divided differences that may be assumed to vanish in the region of the selected points and by inference to vanish also at other parts of the table, provided the function is of the usual type.

Example: Assuming that the work done previously indicates that the third order of divided differences may be assumed to vanish, find f(x) for x = 97.00 as a pivotal point in an extended table of which the values are as shown at left of the array below. The entries at the right of the columns of x and f(x) are developed by the methods described in the following sections.

x	f(x)	first-order $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$, etc.	second-order $\frac{f(x_2) - f(x_0)}{x_2 - x_0}$, etc.	3rd-order divisors
96.55	1.4452 112			
		$\frac{.0046430}{.15} = .0309533$		
96.70	1.4498 542		$\frac{.0001577}{.35} = .0004506$	
		$\frac{.0062222}{.20} = .0311110$		.47
96.90	1.4560 764		$\frac{.0001457}{.32} = .0004553$	
		$\frac{.0037508}{.12} = .0312567$		.55
97.02	1.4598 272		$\frac{.0001598}{.35} = .0004566$	
		$\frac{.0072258}{.23} = .0314165$		.50
97.25	1.4670 530		$\frac{.0001755}{.38} = .0004618$	
		$\frac{.0047388}{.15} = .0315920$		.49
97.40	1.4717 918		$\frac{.0001207}{.26} = .0004642$	
		$\frac{.0034884}{.11} = .0317127$		
97.51	1.4752 802			

(continued on next page)

A) OBTAINING THE DENOMINATOR-PORTION OF THE ARRAY:

Operations: Decimals: Upper Dial 0, Middle Dial 2, Keyboard Dial 2

(1) With carriage at extreme left, set up largest "x" (97.51) in Keyboard Dial and depress Add Bar.

(2) Set up the next smaller argument (97.40) in Keyboard Dial, reverse-multiply by 1. Copy difference (.11).

(3) Clear Middle Dial and depress Add Bar.

(4) Repeat Steps 1, 2, and 3 for the remaining adjacent and decreasing values, thus completing the first column of divisors.

(5) The general method of obtaining a column of divisors from the column at left is to add the diagonally opposite amounts of the column next-to-the-left and subtract the directly opposite amount of the second column of divisors to the left of the one being computed. In computing the second column of divisors, the "second column of divisors to the left" is nonexistent, so the second column is obtained as follows: Set up .15 and add; set up .20 and multiply by 1. Copy .35. Clear Middle Dial and add; set up .12 and multiply by 1. Copy .32. Clear Middle Dial and add; set up .23 and multiply by 1. Copy .35, and so forth.

(6) For this example, there is no need for additional columns of divisors, so the work thus far would be checked as follows: To 96.55, add the second column of divisors and subtract the interior amounts of the next column of divisors to the left; i.e., 96.55 plus (.35 to .26 incl) minus (.20 to .15 incl) equals 97.51.

Note: If additional columns of divisors are required, they are obtained by applying the principle of Step (5), slightly modified to permit continuous operation. At the extreme right of the array will be noted a column of divisors for third-order divided differences (not needed in this computation). They are obtained as follows:

Set up .35 and add; set up .20 and subtract; set up .32 and multiply by 1. Copy .47. Clear Middle Dial and add; set up .12 and subtract; set up .35 and multiply by 1. Copy .55. Clear Middle Dial and add; set up .23 and subtract; set up .38 and multiply by 1. Copy .50. Continue in this manner for a considerable section of the column.

The check of any column is made in a similar manner to Step (6); i.e., to the starting value of x (96.55) is added the right-hand column of divisors (.47, .55, .50, .49), from which is subtracted the interior values of the next column of divisors at left (.32, .35, .38). The remainder (97.51) checks with the final value of x (97.51).

B) OBTAINING THE NUMERATOR-PORTION OF THE COLUMN OF FIRST-ORDER DIVIDED DIFFERENCES:

Though it is possible to obtain each divided difference by the continuous calculator-operation of finding the difference of the proper values of f(x, etc.) and dividing by the corresponding divisor, obtained mentally or by aid of another calculator, the most time-saving routine when large amounts of work are to be done appears to be as follows:

Operations: Decimals: Upper Dial 9, Middle Dial 17, Keyboard Dial 8. Start with Non-Shift Key down and with Division-Clear Lever toward operator on 10-column M-series Marchant.

(1) With carriage in 10th position, set up largest f(x) (1.4752802) in Keyboard Dial and depress Add Bar.

(over)

(2) Similarly set up next adjacent smaller amount (1.4717918) and reverse-multiply by 1. Copy corresponding Numerator (.0034884).

(3) Clear Middle Dial and depress Add Bar. Change Keyboard Dial to read the next adjacent smaller amount (1.4670530) and reverse-multiply by 1. Copy corresponding Numerator (.0047388), and continue in this manner until a considerable section of the column has been completed.

(4) Check the work thus far by adding the end f(x) (1.4452112) to the column of differences (.004630 to .0034884, incl) to give the other end f(x) (1.4752802).

(5) Continue as in Step 3 to obtain the remaining values of the column. Progressing downward, increases of positive amounts implies positive differences (as in this case.) Similar increases of the absolute value of negative amounts implies negative differences. If the function has peaks, start at the largest value and progress each way. Obvious alteration of method is required to accomodate these possibilities.

(6) Upon completion of a considerable section of each column of differences, check the work thus far by adding to the end value of the function the amounts in the column of differences, which should check with the other end value of the function, as in Step 4.

C) OBTAINING THE FIRST-ORDER DIVIDED DIFFERENCES:

This is done by dividing the respective numerators by the previously found divisors. The decimal-setting for B is suitable for this work. Copy quotient to one more significant figure than appears in the numerator.

D) OBTAINING ADDITIONAL ORDERS OF DIVIDED DIFFERENCES:

Second-order divided differences, as well as those of higher orders, are obtained in a similar manner to the first-order divided differences, except that the quotients are rounded to not more significant figures than the numerators, thus keeping the work within the approximate degree of precision of the starting data. Even this assumption is based upon the fact that the divisors are exact, which may not always be the case.

E) FIND F(X) FOR A PARTICULAR INTERMEDIATE VALUE OF X, SAY U:

That portion of the array is used that centrally surrounds the desired value and equation (5) is applied. In the example, u = 97.00, so the array extending from 96.90 to 97.25, inclusive, would be used, as follows:

f(96.90)
(97.00 - 96.90) .0312567
(97.00 - 96.90) (97.00 - 97.02) ·.0004566

f(97.00) =
1.4560764
.00312567
-.00000091
1.4592012

When there is a volume of this work, the coefficients should be listed separately, from which, applying accumulative multiplication, the desired value is obtained without recording the components. The factors of the components are most easily obtained, when not obvious mentally, by applying the principle of subtracting from a constant when u is the greater, and subtracting a constant when u is the smaller.

In the example, there are not enough factors of the coefficients to take advantage of these principles.

− +

x_1	x_2	x_3	x_4		check
a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}
a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}
a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}

x_1	x_2	x_3	x_4		Check
b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
b_{21}	b_{22}	b_{23}	b_{24}	b_{25}	b_{26}
b_{31}	b_{32}	b_{33}	b_{34}	b_{35}	b_{36}
b_{41}	b_{42}	b_{43}	b_{44}	b_{45}	b_{46}

		Check
x_1	c_{11}	c_{12}
x_2	c_{21}	c_{22}
x_3	c_{31}	c_{32}
x_4	c_{41}	c_{42}

$b_{11} = a_{11}$ $b_{21} = a_{21}$ $b_{31} = a_{31}$ . . . . $b_{n1} = a_{n1}$

$b_{12} = a_{12} \div b_{11}$; $b_{13} = a_{13} \div b_{11}$ $b_{1n} = a_{1n} \div b_{11}$

$b_{22} = a_{22} - b_{21} \times b_{12}$; $b_{32} = a_{32} - b_{31} \times b_{12}$ . . . . $b_{n2} = a_{n2} - b_{n1} \times b_{12}$

$b_{23} = (a_{23} - b_{21} \times b_{13}) \div b_{22}$; $b_{24} = (a_{24} - b_{21} \times b_{14}) \div b_{22}$; . . . $b_{2n} = (a_{2n} - b_{21} \times b_{1n}) \div b_{22}$

$b_{33} = a_{33} - b_{31} \times b_{13} - b_{32} \times b_{23}$; $b_{43} = a_{43} - b_{41} \times b_{13} - b_{42} \times b_{23}$; . . . $b_{n3} = a_{n3} - b_{n1} \times b_{13} - b_{n2} \times b_{23}$

$b_{34} = (a_{34} - b_{31} \times b_{14} - b_{32} \times b_{24}) \div b_{33}$; $b_{35} = (a_{35} - b_{31} \times b_{15} - b_{32} \times b_{25}) \div b_{33}$ . . . $b_{3n} = (a_{3n} - b_{31} \times b_{1n} - b_{32} \times b_{2n}) \div b_{33}$

$b_{44} = a_{44} - b_{41} \times b_{14} - b_{42} \times b_{24} - b_{43} \times b_{34}$; $b_{54} =$ . . . . $b_{n4} = a_{n4} - b_{n1} \times b_{14} - b_{n2} \times b_{24} - b_{n3} \times b_{3n}$

$b_{45} = (a_{45} - b_{41} \times b_{15} - b_{42} \times b_{25} - b_{43} \times b_{35}) \div b_{44}$ $b_{4n} = (a_{4n} - b_{41} \times b_{1n} - b_{42} \times b_{2n} - b_{43} \times b_{3n}) \div b_{44}$

Check: $b_{16} = 1 + b_{12} + b_{13} + b_{14} + b_{15}$

$b_{26} = 1 + b_{23} + b_{24} + b_{25}$

$b_{36} = 1 + b_{34} + b_{35}$

$b_{46} = 1 + b_{45}$

$c_{41} = b_{45}$

$c_{31} = b_{35} - b_{34} \times c_{41}$

$c_{21} = b_{25} - b_{23} \times c_{31} - b_{24} \times c_{41}$

$c_{11} = b_{15} - b_{12} \times c_{21} - b_{13} \times c_{31} - b_{14} \times c_{41}$

$c_{42} = b_{46}$

[illegible]

$c_{22} =$ [illegible]